Structural Engineering and Mechanics, *Vol. 56, No. 3 (2015) 461-472*
DOI: http://dx.doi.org/10.12989/sem.2015.56.3.461

Brief and accurate analytical approximations to nonlinear static response of curled cantilever micro beams

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(Received April 25, 2014, Revised October 25, 2015, Accepted October 28, 2015)

Abstract. In this paper, the nonlinear static response of curled cantilever beam actuators subjected to the one-sided electrostatic field is focused on. By assuming the deflection function of electrostatically actuated beam, analytical approximate solutions are established via using Galerkin method to solve the equilibrium equation. The Pull-In voltages which determine the stability of the curled beam actuators are also obtained. These approximate solutions show excellent agreements with numerical solutions obtained by the shooting method and the experimental data for a wide range of beam length. Expressions of these analytical approximate solutions are brief and could easily be used to derive the effects of various physical parameters on MEMS structures.

Keywords: MEMS; Galerkin method; large deformation; analytical approximation

1. Introduction

Micro-electromechanical systems (MEMS) have widely been applied as switches, inductors and variable capacitors for high radio frequency circuits (Gupta 1997, Senturia 2001, Rebeiz 2003, Al-Sadder 2006, Zhang and Zhao 2006, Zamanian *et al.* 2010, Zamanian and Hosseini 2012, Abbasnejad *et al.* 2013, Mobki *et al.* 2013) as well as bio-MEMS (Wang and Soper 2007, Hess *et al.* 2011, Miyashita *et al.* 2014). Analyzing behavior of electrically actuated MEMS is crucial for their designs. Besides, the material parameters of thin films, such as Young's modulus and residual stresses can be extracted from the Pull-In voltages of test structures (Elata and Abu-Salih 2005).

The finite element method (FEM) is often used for modeling the nonlinear response of MEMS structures and has been implemented in various commercial MEMS simulation software (Chen *et al.* 2008, Chuang *et al.* 2010, Kazama *et al.* 2013). Compared with the finite element method (FEM), the analytical model providing explicit solutions could give more intuitive expression of the physical characteristics of MEMS structures (Gabbay and Senturia 2000, Younis *et al.* 2003, Krylov 2007). Continuum-based modeling of MEMS for the description of the nonlinear static behavior is often presented as a boundary value problem of the microbeam structures. There have

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http://www.techno-press.org/?journal=sem&subpage=8 ISSN: 1225-4568 (Print), 1598-6217 (Online)

been numerous works on analytical modeling of the nonlinear behavior for beams or diaphragms.

An efficient model for describing the nonlinear static (and dynamic) behavior of a clamped-clamped microbeam structure with large deflections and the nonlinear electromechanical coupling force has been proposed (Younis *et al.* 2003, Krylov 2007). A reduced-order model commonly based on the Galerkin decomposition with undamped linear eigenmodes as base functions is constructed (Gabbay and Senturia 2000, Nayfeh *et al.* 2005) to investigate the behavior of MEMS. However, for a continued analytical description of the system's response, the integral of the electromechanical forcing term has to be solved in closed form. To deal with this problem, one approach is to solve the integral actuation term numerically (Krylov 2007) and to set further analytical investigations aside. Another approach is to expand this term into a Taylor series, but poor accuracy has been reported, even more higher-order terms were included (Younis *et al.* 2003, Nayfeh *et al.* 2005). By multiplying the equation of motion by the denominator of the electrostatic force before applying the discretization technique, Younis *et al.* (2003) have introduced a different method. In contrast, a single-mode approximation is sufficient enough to predict also large displacement equilibria (including Pull-In) if the boundary value problem is discretized following the Galerkin method without premultiplication of the denominator of the forcing term (Gutschmidt 2010). However, the determination of the coefficient in such a reduced-order model deduced from the discretization without premultiplication of the denominator requires solving integral equation by the numerical method. Challenges could be faced by such a method, in finding solutions whenever the system approaches a singularity, like in the cases of primary and secondary Pull-In instabilities (Gutschmidt 2010).

A simple lumped model, which consists of a single parallel plate capacitor suspended by an ideal linear spring, was often used formerly to simulate the electrostatically actuated microstructures (Pamidighantam *et al.* 2002, Cheng *et al.* 2004, Chowdhery *et al.* 2005), with the effect of the fringing field and the distributed structural deformation as well as non-ideal boundary conditions. Based on well-known beam and plate theories, the distributed model could be established, but whose solution could only be obtained via numerically solving the highly nonlinear governing differential equations (Osterberg 1995, Gupta 1997, Yu *et al.* 2012). Some other analytical models (Petersen 1978, Lee and Kim 2000, Wu *et al*. 2013) give the Pull-In voltage of MEMS beams by assuming the deflection functions of electrostatically actuated beams as polynomial or Trigonometric functions.

Among the above-mentioned analytical models, a few literatures (Gupta 1997, Hu 2006) consider the curled beam induced by residual stress releasing. Based on the Euler-Bernoulli beam theory and Taylor's series expansion, Hu derives three analytical models of a curled cantilever micro beam subjected to electrostatic loads, namely the full-order, the fourth-order and the third-order models, which are then solved by the energy method to obtain the corresponding closed form solutions for the pull-in voltages of curled micro beam subjected to electrostatic loads (Hu, 2006). The accuracy of the present models is verified through comparing with former analytical models and the experimentally measured data conducted in former works. Recently, considering the fringing fields, Hu and Wei (2007) derive a high precise analytical solution to determine the Pull-In voltages of a curled beam subjected to electrostatic loads.

This paper is concerned with presenting an alternative approach to solve the nonlinear static response of curled cantilever micro beam actuator for the one-sided electrode configuration. The model is based on the nonlinear differential governing equation for micro beam structure. The analytical approximate solutions are established by choosing a shape of deflection of MEMS beam, and then employing the Galerkin method. These approximate solutions are brief and explicit

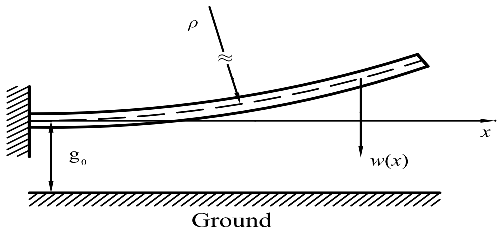

Fig. 1 Sketch of the curled cantilever micro beam for the one-sided electrode ground

in expression and show excellent agreement with respect to the numerical solutions obtained by the shooting method (Yu and Sun 2012, Yu *et al.* 2012) and the experimental data measured by Gupta (1997), and they are valid for small as well as large free-end deflection of the beam.

2. Analytical model

A curled cantilever beam subjected to a uniform electrostatic field is considered. A typical sketch of the microbeam system for the one-sided electrode configuration is shown in Fig. 1. The micro beam is made of elastic material and electrostatically actuated by the electrode. It is assumed that the initial gap compared with the length of the beam is very small. As known, fringing field effects begin to influence for a very narrow width of the beam, which is the case of beam-width smaller than the gap between the beam and the ground (Pamidighantam *et al.* 2002). Since the MEMS structures considered in the present paper are generally outside of the case of very narrow beam-width, therefore, throughout the present paper, fringing fields are ignored. However, omitting fringing fields does not affect the nature of the considered electromechanical response. Based on the assumption of the Euler-Bernoulli beam and a uniform electrical field, the static equilibrium equation and boundary conditions of a cantilever micro beam could be presented as (Gupta 1997, Lee and Kim 2000, Hu 2006)

$$EI\frac{\mathrm{d}^4 w}{\mathrm{d}x^4}-\frac{\varepsilon_0\varepsilon_{\mathrm{r}} bV^2}{2\left(\overline{G}-w\right)^2}=0 \tag{1}$$

$$w(0)=\frac{\mathrm{d}w}{\mathrm{d}x}(0)=0,\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}(L)=\frac{\mathrm{d}^3 w}{\mathrm{d}x^3}(L)=0 \tag{2}$$

where L, E and $I=bh^3/12$ represent the length of the beam, Young's modulus and the second moment of cross-section, respectively, and b is the width and h is the thickness**. Note that** w is the deflection difference between the present state and the initial curled status of the beam (Hu 2006), as a function of the position x, please see Fig. 1. Moreover, V^2, ε_0, and ε_r represent the applied voltage, the permittivity of free space, the dielectric constant of the dielectric medium between the beam and the ground, respectively. The initial gap between the beam and the ground plane is presented by $\overline{G}$. Because of the residual stress releasing, the cantilever beam would be curled, and the initial gap between the beam and the ground plane can be expressed as

$$\bar{G} = g_0 + \bar{\rho}\left(1 - \cos\frac{x}{\bar{\rho}}\right) \tag{3}$$

where g_0 is the gap between the fixed end of the cantilever beam and the ground plane, $\bar{\rho}$ represents the initial radius of curvature of the curled cantilever beam. For details of derivation in this section, we refer the readers to Hu (2006).

For simple expression, the dimensionless nonlinear governing equation with boundary conditions in equilibrium can be expressed

$$W^{(4)} - \frac{\Theta^2}{(G-W)^2} = 0 \tag{4}$$

$$W(0) = W'(0) = 0, W''(\frac{\pi}{2}) = W'''(\frac{\pi}{2}) = 0 \tag{5}$$

where

$$W = \frac{w}{g_0}, s = \frac{\pi x}{2L},\ \Theta^2 = \frac{96\varepsilon_0\varepsilon_r L^4 V^2}{\pi^4 E g_0{}^3 h^3}, G = \frac{\bar{G}}{g_0} = 1 + \rho\left(1 - \cos\Omega s\right), \rho = \frac{\bar{\rho}}{g_0}, \Omega = \frac{2L}{\pi\bar{\rho}} \tag{6}$$

and $' = \frac{\mathrm{d}}{\mathrm{d}s}$.

The exact solution for the electrostatic-actuated beam is very difficult to obtain, since the nonlinear electrostatic force is coupled with the structural deflection. This paper is concerned with analytical approximate solutions to the nonlinear static response. These analytical approximate solutions are derived by using Galerkin method and the assumed shape of the micro beam deflection (Fang and Wickert 1994, Elata and Abu-Salih 2005, Gutschmidt 2010, Yu *et al.* 2012).

3. Solution methodology

In this section, Galerkin method (Shames and Dym 1985) is applied to derive the analytical approximate solutions. By the use of the Galerkin method, the deflection function $w(s)$ is expressed as

$$W(s) = c \cdot \phi(s) \tag{7}$$

where $\phi(s)$ is the assumed deflection shape function satisfying the boundary conditions in Eq. (5) and the coefficient c is the amplitude of the associated shape. Furthermore, let c be the normalized free end deflection of the beam $W(s)|_{s=\pi/2}=c$ (i.e., $w(x)|_{x=L}=\zeta=cg_0$). A reasonable and simple deflection shape function satisfying the conditions in Eq. (5) and $W(s)|_{s=\pi/2}=c$ can be taken as

$$\phi(s) = \frac{27}{14}\left(\frac{1-\cos s}{2} + \frac{1-\cos 3s}{54}\right), s \in \left[0, \pi/2\right] \tag{8}$$

Substituting Eqs. (7)-(8) into Eq. (4), multiplying Eq. (4) by $(G-W)^2$ and $\phi(s)$, and then integrating with respect to s from 0 to $\pi/2$ yield

$$\left[D_1\cdot\cos(\Omega\pi)+D_2\cdot\cos\left(\frac{\Omega\pi}{2}\right)+D_3\cdot\sin(\Omega\pi)+D_4\cdot\sin\left(\frac{\Omega\pi}{2}\right)+D_5\right]\Big/Z \\ -D_0\cdot\Theta^2=0 \tag{9}$$

From Eq. (9), the normalized approximate voltage is obtained

$$\Theta=\sqrt{\left[D_1\cdot\cos(\Omega\pi)+D_2\cdot\cos\left(\frac{\Omega\pi}{2}\right)+D_3\cdot\sin(\Omega\pi)+D_4\cdot\sin\left(\frac{\Omega\pi}{2}\right)+D_5\right]\Big/(Z\cdot D_0)} \tag{10}$$

where the expressions for D_0, D_1, D_2, D_3, D_4, D_5, and Z are presented in Appendix.

The applied voltage could be obtained by using Eqs. (6) and (10)

$$V=\sqrt{\frac{\pi^4Eg^3h^3\left[D_1\cdot\cos(\Omega\pi)+D_2\cdot\cos\left(\frac{\Omega\pi}{2}\right)+D_3\cdot\sin(\Omega\pi)+D_4\cdot\sin\left(\frac{\Omega\pi}{2}\right)+D_5\right]}{96\varepsilon_0\varepsilon_rL^4Z\cdot D_0}} \tag{11}$$

Where

$c=\zeta/g_0$

In the next section, it will be shown that Eq. (11) is able to predict excellent analytical approximations to the nonlinear deformation for an example of MEMS.

The normalized approximate Pull-In voltages and Pull-In free-end deflections could be obtained in terms of g_0, by solving c from equation $\frac{d\Theta(c)}{dc}=0$. Likely, the actual approximate Pull-In voltages V^P and Pull-In free-end deflections ζ^P can be obtained in terms of g_0, by solving g_0 from equation $\frac{dV(\zeta)}{d\zeta}=0$.

4. Results and discussion

In this section, accuracy of the proposed analytical approximation is illustrated by comparison with numerical solution, as well as with experimentally measured result of Gupta (1997). The corresponding numerical solutions are obtained from Eqs. (1)-(2) by constructing extended systems and applying the shooting method (Yu *et al.* 2012). Note that the numerical approach above can be easily implemented for many electrostatically actuated microstructures described by non-linear ordinary differential equations.

Consider a MEMS beam with the geometric and material parameters given in Table 1, which are the experimental samples conducted by Gupta (1997). Note that Gupta's experimental beam samples are made of polysilicon and manufactured by the standard MUMPs 5 die process.

For various beam length, Pull-In voltages of the present analytical approximate solution $V_{\mathrm{a}}^{\mathrm{P}}$, the present numerical one $V_{\mathrm{e}}^{\mathrm{P}}$ obtained by shooting method, Hu's result $V_{\mathrm{H}}^{\mathrm{P}}$ (Hu 2006), and the experimental result $V_{\mathrm{G}}^{\mathrm{P}}$ by Gupta (1997), are showed in Fig. 2. As seen in Fig. 2, the present numerical and analytical approximate results agree very well with Gupta's measured results, no

Table 1 Material and geometrical parameters of a curled beam (Gupta 1997, Hu 2006)

Variables	Values
Young's modulus, E (GPa)	153
Permittivity of free space, ε_0 (F m^{-1})	8.85×10^{-12}
Dielectric constant between the beam and the ground, ε_r	1.2046
Beam length, L (μm)	100-500
Beam width, b (μm)	40
Beam thickness, h (μm)	2.1
Initial radius of curvature, $\bar{\rho}$ (μm)	40000
Initial gap, g_0 (μm)	2.4

Note. The samples are made of polysilicon and manufactured by the standard MUMPs 5 die process.

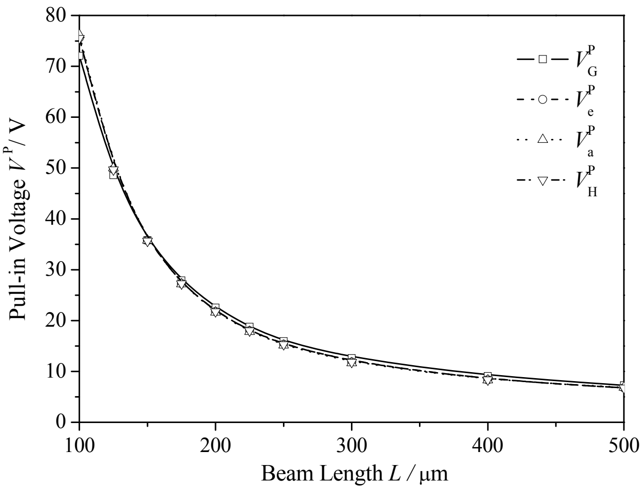

Fig. 2 Comparison of the present model with the former works and measured results (The material and geometrical parameters are given in Table 1)

matter for longer or shorter beams. So the present theoretical model is sufficiently accurate, compared with experimental results. In the following discussion, only agreements of present approximate results with numerical ones are listed in the figures. Note that once the geometric and material parameters of the curled cantilever micro beams are given in Table 1(L is given special value, i.e., L=100 h=210 μm), Eq. (11) could be simply rewritten as follows

$$V = \sqrt{4.2594133\times10^{8}\zeta - 2.0699083\times10^{14}\zeta^{2} + 2.5226319\times10^{19}\zeta^{3}} \tag{12}$$

It is easy to investigate the relation of applied voltage and the free-end deflection of beam via Eq. (12), and Pull-In voltage V^P and Pull-In free-end deflection ζ^P could also be obtained. A comparison of the numerical applied voltage V_e and the analytical approximate voltage V_a in terms of the free-end deflection of beam ζ is shown in Fig. 3. Here, the stable and unstable solutions are represented by thick solid lines and thick dashed lines, respectively. As observed from this figure, Eq. (12) shows excellent agreement with the numerical voltages V_e.

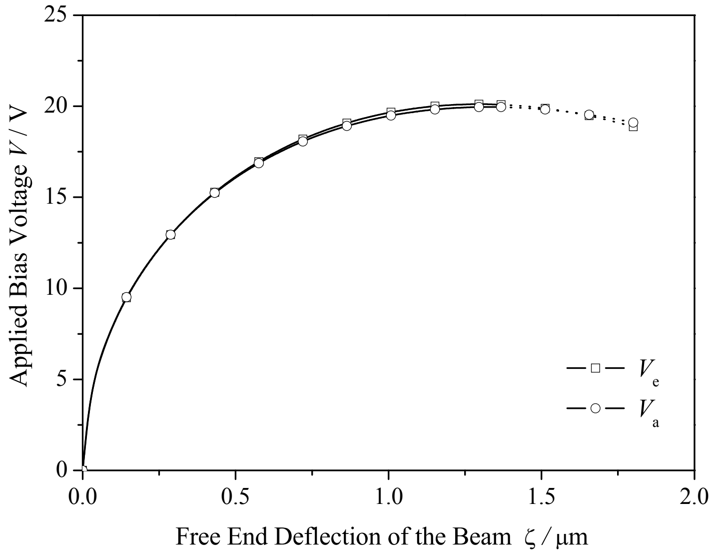

Fig. 3 Variation of numerical and approximate applied voltages V with respect to the free-end beam deflection ζ (L=210 μm and other parameters given in Table 1)

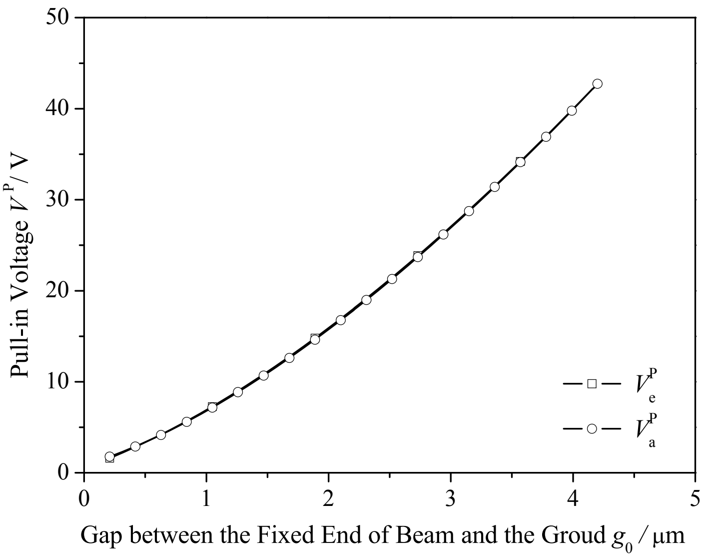

Fig. 4 Variation of numerical and approximate Pull-In voltages V^P with respect to the gap g_0 (L=210 μm and other parameters given in Table 1)

With geometric and material parameters given in Table 1 (L=100 h=210 μm and here, g_0 not given), variations of the numerical and approximate Pull-In voltage V^P and Pull-In free-end deflection ζ^P with respect to g_0 are illustrated in Figs. 4-5, respectively. Excellent agreements are observed in these figures, Eq. (12) could thus be used to achieve very accurate approximations to the numerical Pull-In voltages and Pull-In free-end deflections for small as well as large nominal gap g_0.

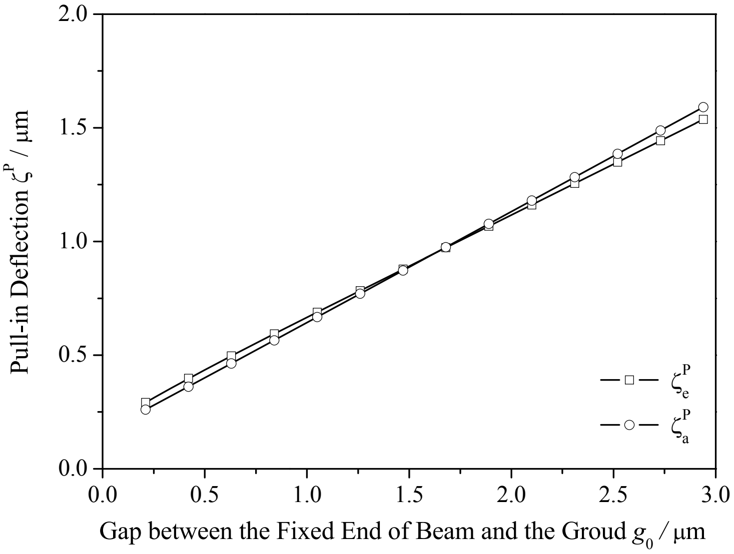

Fig. 5 Variation of numerical and approximate Pull-In defections ζ^P with respect to the gap g_0 (L=210 μm and other parameters given in Table 1)

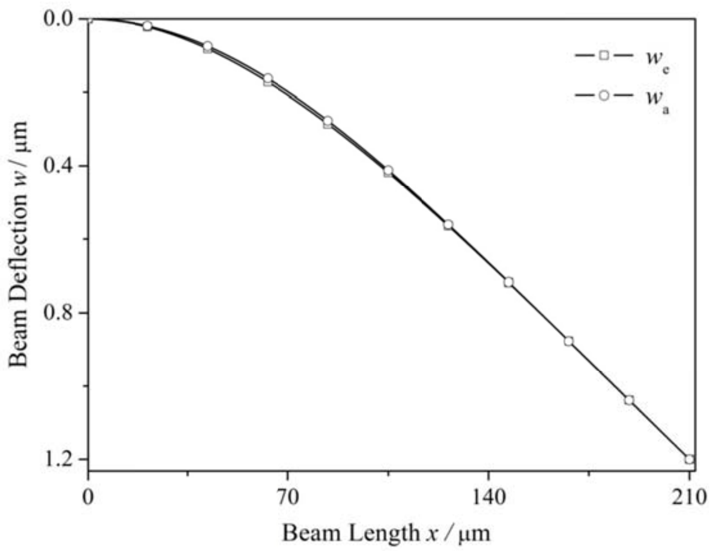

Fig. 6 Comparison of numerical and approximate beam deflections w (L=210 μm, ζ=1.2 μm, and other parameters given in Table 1)

For the deformation of the MEMS beam with L=210 h=210 μm and $\zeta=c\cdot g_0$=0.5×2.4 μm=1.2 μm, other parameters given in Table 1, the numerical solution w_e and approximate solution w_a given by Eqs. (7)-(8) are displayed in Fig. 6. We can find that Eqs. (7)-(8) show very good agreements with numerical solutions.

Fig. 7 depicts variation of the analytical approximate applied voltage V_a with the free-end beam deflection ζ for L=100 h=210 μm, various values of the gap between the fixed end of the cantilever

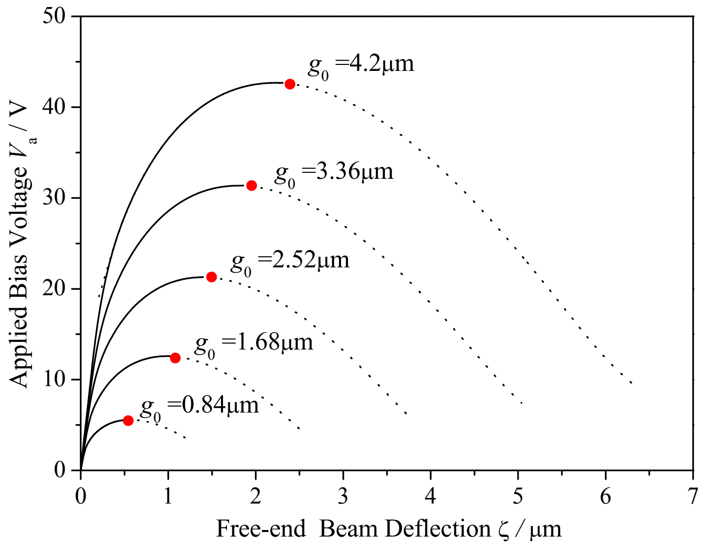

Fig. 7 Variation of the approximate applied voltage V_a with free-end beam deflection ζ (L=210 μm, various values of the gap g_0, and other parameters given in Table 1)

beam and the ground plane g_0, and other parameters given in Table 1. The stable and unstable solutions are represented by thick solid lines and thick dashed lines, respectively. It is observed that the deflection and voltage at the Pull-In (limit) point increase with increasing g_0, as indicated by the red dots in these figures.

The prediction of reliable Pull-In voltages and displacements is an important goal of MEMS/NEMS modeling. Using the brief and explicit expressions of post buckling voltage V, Pull-In parameters V^P and ζ^P can easily be obtained for small as well as large beam-center deflection. Moreover, the effect of various parameters, such as gap g_0, and thickness h of beam, to the post buckling voltage and Pull-In parameters, could also be expediently established by employing these analytical approximate solutions. Therefore, the present method and results could improve MEMS/NEMS understanding, reduce times of experiment for their designers and guide application of these devices.

In this paper, the construction of the proposed analytical approximate solutions depends on the cantilever condition of the micro-beam. For micro-beams with other boundary conditions, such as the clamped-clamped boundary condition, the proper shape functions should be chosen.

5. Conclusions

In this paper, the analytical approximate solutions to the nonlinear static behavior of a curled cantilever actuator, modeled as a beam, subjected to the one-sided electrostatic field have been established, via choosing a proper deflection shape function and using Galerkin method to solve the equilibrium equation. Excellent agreement of these analytical approximate solutions with respect to numerical solution obtained by the shooting method and the experimental data measured by Gupta has been demonstrated. These approximate solutions are valid, no matter for longer or shorter beams. Their expressions are brief explicit functions for the applied voltage and are accurate enough. Thus, they are very convenient and accurate enough for implementation in MEMS design.

Acknowledgments

The work described in this paper is supported by SinoProbe-09-05 (Grant No. 201011082), the National Natural Science Foundation of China (Grant No. 11402095), and International S&T Cooperation Program of China (Grant No. 2013DFR70490).

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Appendix

$$D_0 = \frac{1}{42}(21\pi - 40)$$

$$D_1 = 379330560c\rho^2\Omega^3\left(99225 - 7219\Omega^2 + 155\Omega^4 - \Omega^6\right)$$
$$\times\left(-20736 + 30096\Omega^2 - 10648\Omega^4 + 1353\Omega^6 - 66\Omega^8 + \Omega^{10}\right)$$

$$D_2 = 7741440c\rho\Omega\left(-20736 + 99216\Omega^2 - 68728\Omega^4 + 14793\Omega^6 - 936\Omega^8 + 16\Omega^{10}\right)$$
$$\times\left[49\Omega^2(1+\rho)\left(-99225 + 7219\Omega^2 - 155\Omega^4 + \Omega^6\right) + c\left(6055560 + 5680449\Omega^2\right.\right.$$
$$\left.\left.-382171\Omega^4 + 7811\Omega^6 - 49\Omega^8\right)\right]$$

$$D_3 = 2540160c\rho^2\left(576 - 52\Omega^2 + \Omega^4\right)\left(-45 - 72\Omega^2 + 8\Omega^4\right)\left(9 - 40\Omega^2 + 16\Omega^4\right)$$
$$\times\left(-99225 + 7219\Omega^2 - 155\Omega^4 + \Omega^6\right)$$

$$D_4 = 650280960c\rho(1 - 2c + \rho)\left(9 - 40\Omega^2 + 16\Omega^4\right)\left(99225 - 7219\Omega^2 + 155\Omega^4 - \Omega^6\right)$$
$$\times\left(-810 + 576\Omega^2 + 279\Omega^4 - 46\Omega^6 + \Omega^8\right)$$

$$D_5 = 9c\Omega\left[c^2(3118635\pi - 8749056) + 256c(22784 - 11025\pi)(1+\rho)\right.$$
$$\left.+352800\pi\left(2 + 4\rho + 3\rho^2\right)\right]\left(576 - 52\Omega^2 + \Omega^4\right)\left(9 - 40\Omega^2 + 16\Omega^4\right)$$
$$\times\left(-99225 + 7219\Omega^2 - 155\Omega^4 + \Omega^6\right)\left(-36 + 49\Omega^2 - 14\Omega^4 + \Omega^6\right)$$

$$Z = 24586240\Omega\left(\Omega^2 - 81\right)\left(\Omega^2 - 49\right)\left(\Omega^2 - 36\right)\left(\Omega^2 - 25\right)\left(\Omega^2 - 16\right)\left(\Omega^2 - 9\right)\left(\Omega^2 - 4\right)$$
$$\times\left(\Omega^2 - 1\right)\left(4\Omega^2 - 9\right)\left(4\Omega^2 - 1\right)$$