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Generalized Rayleigh wave propagation in a covered half-space with liquid upper layer

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Abstract. Propagation of the generalized Rayleigh waves in an initially stressed elastic half-space covered by an elastic layer is investigated. It is assumed that the initial stresses are caused by the uniformly distributed normal compressional forces acting on the face surface of the covering layer. Two different cases where the compressional forces are "dead" and "follower" forces are considered. Three-dimensional linearized theory of elastic waves in initially stressed bodies in plane-strain state is employed and the elasticity relations of the materials of the constituents are described through the Murnaghan potential where the influence of the third order elastic constants is taken into consideration. The dispersion equation is derived and an algorithm is developed for numerical solution to this equation. Numerical results for the dispersion of the generalized Rayleigh waves on the influence of the initial stresses and on the influence of the character of the external compressional forces are presented and discussed. These investigations provide some theoretical foundations for study of the near-surface waves propagating in layered mechanical systems with a liquid upper layer, study of the structure of the soil of the bottom of the oceans or of the seas and study of the behavior of seismic surface waves propagating under the bottom of the oceans.

Keywords: generalized Rayleigh wave; initial stresses; dead forces; follower forces; wave dispersion; third order elasticity constants

1. Introduction

Near-surface waves are ubiquitous in various branches of engineering and also in some natural sciences such as geophysics. Non-destructive testing of structural or mechanical elements, defects and cracks detection in civil engineering infrastructures, material characterization using acoustic surface waves or some geophysical applications in the study of fault dynamics and earthquakes are typical examples. In many practical applications though, initial stresses are present due to the temperature variations or through the manufacturing or assembling processes. Moreover, the stresses which appeared under the action of the exploitation load in the members of the constructions can also be taken as initial or residual stresses with respect to the additional loading. At the same time, in the Earth's crustal layer initial stresses might occur under the action of geostatic and geodynamic forces. These initial stresses significantly affect the dynamical behavior

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of the systems under consideration. Therefore several studies have been presented so far on the influence of the initial stresses in the elements of the constructions as well as in the layered medium on the dispersion of the propagated waves.

Theoretical studies of wave propagation in bodies with initial stresses has been made by many researchers and some systematic analysis and summary of those studies can be found in the monographs by Biot (1965), Eringen and Suhubi (1975a, b), Guz (2004). Reviews of the certain part of those investigations carried out before the year 2007 are given in the papers by Guz (2002, 2005), Akbarov (2007) as well. However, more recent related investigations can be found in papers by Akbarov (2012), Akbarov and Ipek (2010, 2012), Akbarov *et al.* (2011). Many works have been done on the near-surface wave propagation in initially stresses layered half-spaces so far, yet, here we will present a few of those numerous studies which were carried out in the last twenty years.

Dowaikh and Ogden (1991) studied the propagation of interfacial waves (Stoneley waves) along the boundary between two half-spaces of pre-stressed incompressible isotropic elastic material and they obtained the equation for the wave speed propagation along a principal axis in respect of general strain-energy functions. In particular they showed that when an interfacial wave exists its speed is greater than that of the least of the Rayleigh wave. The propagation of elastic interfacial waves along the plane boundary separating two pre-strained compressible half-spaces has also been studied by Sotiropoulos (1998) assuming that the half-spaces were subjected to pure homogeneous finite strains. Rogerson and Fu (1995) carried out an asymptotic analysis of dispersion relations for wave propagation in a pre-strained incompressible elastic plate and obtained an asymptotic expansions for the wave speed as a function of wave number and pre-stress. Generalized Rayleigh wave propagation in a pre-stressed stratified half-plane was investigated by Akbarov and Ozisik (2003). It was assumed that complete contact conditions between the layer and half-plane were satisfied. Moreover, it was assumed that the initial strains were small and the strains and stresses corresponding to the initial state were determined within the scope of the classical linear theory of elasticity, corresponding dispersion equation was obtained and the dispersion curves which were constructed from the solution to this equation were analyzed. Wijeyewickrema *et al.* (2008) investigated the time-harmonic wave propagation in a pre-strained and constrained homogeneous compressible high-elastic layer and the influence of the degree of this constraint on the dispersion relations. Ogden and Singh (2011) in the presence of initial stresses derived the general constitutive equation for a transversely isotropic hyperelastic solid based on the theory of invariants to examine the propagation of both homogeneous plane waves and Rayleigh surface waves. Akbarov *et al.* (2011) investigated the extensional and flexural Lamb waves in a sandwich plate with finite initial strains made from compressible highly elastic materials. It was assumed that the initial strains were caused by the uniformly distributed normal compression forces acting on the face planes of the plate and the cases where the compression forces are dead and follower were considered. Gupta *et al.* (2012) studied the propagation of torsional surface wave in an initially stressed non-homogeneous layer over a non-homogeneous half-space and they showed that the inhomogeneity parameter and the initial stress play an important role for the propagation of torsional surface waves. Zhang and Yu (2013) based on the mechanics of incremental deformations investigated the guided wave propagation in unidirectional plates under gravity and initial stresses. Shams and Ogden (2014) by applying the theory of the superposition of infinitesimal deformations on finite deformations in a hyperelastic material studied the propagation of Rayleigh waves in an initially stressed incompressible half-space subjected to a pure homogeneous deformation. Zhang *et al.* (2014) using quasistatic approximation

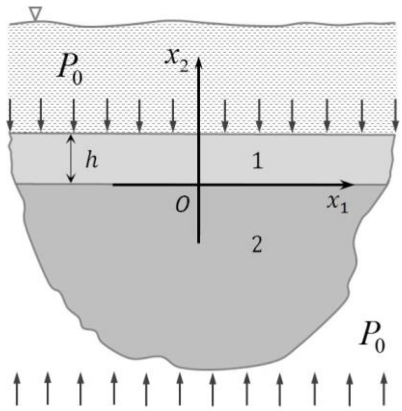


Fig. 1 Geometry of the considered mechanical system

and linearity assumption, investigated the propagation of Rayleigh waves in a magneto-electroelastic half-space with initial stress and obtained the wave propagation velocity for four types of electromagnetic boundary.

In the present paper, the study of Negin *et al.* (2014), Akbarov and Negin (2015) on propagation of the generalized Rayleigh waves in an initially stressed elastic half-space covered by an elastic layer is extended for the case where the initial stresses are caused by the uniformly distributed compression forces acting on the face surface of the layered half-space. The three-dimensional linearized theory of elastic waves in initially stressed bodies (TLTEWISB) is utilized and the plane-strain state is considered. Elasticity relations of the materials are described through the Murnaghan potential where the influence of the third order elastic constants is taken into consideration. The dispersion equation for this system is derived and a computer algorithm is developed for numerical solution to this equation. Numerical results for the dispersion of the generalized Rayleigh waves on the influence of the initial stresses and on the influence of the character of the external compressional forces are presented and discussed. Two cases are considered:

Case 1: it is assumed that the external forces are "dead" forces. Therefore, in this case the external forces only cause the initial stresses and do not constrain the wave propagation in the layered half-space.

Case 2: it is assumed that the external forces are "follower" forces. Consequently, in this case the external forces not only cause the initial stresses, but also constrain the wave propagation in the system. These distributed external forces, can be caused, for instance, by the weight of the liquid on the layered half-space if the system represents as a model for the soil of the bottom of the oceans.

2. Formulation of the problem

We consider an elastic half-space covered by an elastic layer with thickness h. Fig. 1 shows the geometry of the problem. The layer and the half-plane occupy the regions $\{-\infty<x_1<+\infty,\ 0\leq x_2\leq h,\ -\infty<x_3<+\infty\}$ and $\{-\infty<x_1<+\infty,\ -\infty\leq x_2\leq 0,\ -\infty<x_3<+\infty\}$, respectively. Note that the values related to the layer and half-space are denoted by upper indices (1) and (2), respectively. Furthermore, the values relating to the initial state are denoted by the additional upper index 0. We determine the positions of the points by the Lagrange coordinates in the Cartesian system of coordinates $Ox_1x_2x_3$. A plane-strain state in the Ox_1x_2 plane is considered, thus the displacement components along Ox_1 and Ox_2 directions, u_1 and u_2 are non-zero while displacement component u_3 along Ox_3 direction is zero. We assume that the Rayleigh waves propagate in the positive direction of Ox_1 axis.

It is assumed that the considered system is compressed with the uniformly distributed normal forces with the intensity P_0 along its thickness. The uniformly distributed normal force P_0, as mentioned before, can be caused for instance, by the weight of the fluid on the stratified half-space if the system corresponds as a model for the soil of the bottom of the ocean or of the sea or it can also represents as a model of the weight of the bodies which are located on the stratified half-space under consideration.

According to Guz (2004), the equations of the TLTEWISB are obtained from the corresponding geometrical non-linear equations of motion by their linearization with respect to the perturbations of the stresses, strains and displacements

$$\frac{\partial \sigma_{11}^{(m)}}{\partial x_1}+\frac{\partial \sigma_{12}^{(m)}}{\partial x_2}+P_0\frac{\partial^2 u_1^{(m)}}{\partial x_2^2}=\rho^{(m)}\frac{\partial^2 u_1^{(m)}}{\partial t^2},$$

$$\frac{\partial \sigma_{12}^{(m)}}{\partial x_1}+\frac{\partial \sigma_{22}^{(m)}}{\partial x_2}+P_0\frac{\partial^2 u_2^{(m)}}{\partial x_2^2}=\rho^{(m)}\frac{\partial^2 u_2^{(m)}}{\partial t^2}. \tag{1}$$

Now we formulate the boundary conditions. We assume that the following complete contact conditions between the layer and half-space are satisfied

$$\sigma_{12}^{(1)}\big|_{x_2=0}=\sigma_{12}^{(2)}\big|_{x_2=0},\quad \sigma_{22}^{(1)}\big|_{x_2=0}=\sigma_{22}^{(2)}\big|_{x_2=0},$$

$$u_1^{(1)}\big|_{x_2=0}=u_1^{(2)}\big|_{x_2=0},\quad u_2^{(1)}\big|_{x_2=0}=u_2^{(2)}\big|_{x_2=0}. \tag{2}$$

As has been noted in the previous section we consider two cases with respect to the boundary conditions:

Case 1: we assume that the external compression forces with intensity P_0 are “dead” forces, i.e., any magnitude in either direction does not change in the perturbation state. Therefore, in this case the boundary conditions are as follow

$$\sigma_{12}^{(1)}\Big|_{x_2=h}=0,\qquad \sigma_{22}^{(1)}\Big|_{x_2=h}=0. \tag{3}$$

Case 2: in this case we assume that the aforementioned forces are “follower” forces. The load is said to be “follower” if it is applied normally to the surface of a body and does not change its direction and magnitude during deformation, i.e., if it acts normally on the surface in the deformed state too. According to this definition of the “follower” forces, we have the boundary conditions written below:

$$\sigma_{12}^{(1)}\Big|_{x_2=h}=-P_0\frac{\partial u_2^{(1)}}{\partial x_1}\Big|_{x_2=h},$$

$$\left.\sigma_{22}^{(1)}\right|_{x_2=h} = P_0 \left.\frac{\partial u_1^{(1)}}{\partial x_1}\right|_{x_2=h} . \tag{4}$$

The expression for the calculation of the “follower” forces acting on the surface, which is also used for writing the conditions (4), is given in the monograph by Guz (1999). Moreover, we assume that the following decay conditions also are satisfied

$$\left.\sigma_{ij}^{(2)}\right|_{x_2\to-\infty} \to 0, \quad \left.u_i^{(2)}\right|_{x_2\to-\infty} \to 0, \quad i=1,2. \tag{5}$$

As stated above, we assume that the constitutive relations of the materials of the constituents are given by the Murnaghan potential. In fact, experimental data detailed in the monograph by Guz (2004) and other references listed therein, show that the absolute values of the influence of the initial stretching and the initial compressing stresses on the wave propagation velocity in the pre-stressed bodies fabricated from the materials such as those which are chosen in this study, differ from each other in the quantitative sense. Consequently, in a theoretical sense such experimental results can be described only by employing the strain energy potential containing not only the second and the square of the first algebraic invariants of Green’s strain tensor, but also the cube of the first invariant, the multiplication of the first and second invariants and the third invariant with corresponding coefficients, i.e., with the third order elastic constants. The Murnaghan potential can be taken as an example of such potential. These third order elastic constants enter into the relations of the TDLTEWISB and through these constants the aforementioned effect of the difference between the absolute values of the influence of the initial stretching and the initial compressing stresses on the wave propagation velocity in the pre-stressed bodies is described and estimated. Therefore, in the present investigation we use the Murnaghan potential which is given as follow, Guz and Makhort (2000)

$$\Phi^{(m)} = \frac{1}{2}\lambda^{(m)}\left(A_1^{(m)}\right)^2 + \mu^{(m)}A_2^{(m)} + \frac{a^{(m)}}{3}\left(A_1^{(m)}\right)^3 + b^{(m)}A_1^{(m)}A_2^{(m)} + \frac{c^{(m)}}{3}A_3^{(m)}, \tag{6}$$

where $\lambda^{(m)}$ and $\mu^{(m)}$ are Lame’s and $a^{(m)}$, $b^{(m)}$ and $c^{(m)}$ are the aforementioned third order elasticity constants. Here $A_1^{(m)}$, $A_2^{(m)}$ and $A_3^{(m)}$ are the first, second and the third algebraic invariants of Green’s strain tensor, respectively. For the case under consideration, the expressions of these invariants are

$$A_1^{(m)} = \varepsilon_{11}^{(m)\prime} + \varepsilon_{22}^{(m)\prime}, \quad A_2^{(m)} = \left(\varepsilon_{11}^{(m)\prime}\right)^2 + 2\left(\varepsilon_{12}^{(m)\prime}\right)^2 + \left(\varepsilon_{22}^{(m)\prime}\right)^2,$$

$$A_3^{(m)} = \left(\varepsilon_{11}^{(m)\prime}\right)^3 + 3\left(\varepsilon_{12}^{(m)\prime}\right)^2\left(\varepsilon_{11}^{(m)\prime} + \varepsilon_{22}^{(m)\prime}\right) + \left(\varepsilon_{22}^{(m)\prime}\right)^3 \tag{7}$$

where

$$\varepsilon_{ij}^{(m)\prime} = \frac{1}{2}\left(\frac{\partial u_i^{(m)\prime}}{\partial x_j} + \frac{\partial u_j^{(m)\prime}}{\partial x_i} + \frac{\partial u_n^{(m)\prime}}{\partial x_j}\frac{\partial u_n^{(m)\prime}}{\partial x_j}\right),$$

$$\sigma_{ij}^{(m)\prime} = \frac{1}{2}\left(\frac{\partial}{\partial \varepsilon_{ij}^{(m)\prime}} + \frac{\partial}{\partial \varepsilon_{ji}^{(m)\prime}}\right)\Phi^{(m)} . \tag{8}$$

Note that in Eqs. (6)-(8) the upper prime on the symbols $u_i^{(m)\,\prime} = (u_i^{(m),0} + u_i^{(m)})$, $\varepsilon_{ij}^{(m)\,\prime} = (\varepsilon_{ij}^{(m),0} + \varepsilon_{ij}^{(m)})$ and $\sigma_{ij}^{(m)\,\prime} = (\sigma_{ij}^{(m),0} + \sigma_{ij}^{(m)})$ denote the total values of the displacements, strains and stresses, respectively. Consequently, in the case under consideration by linearization of the non-linear relations (6)-(8) with respect to the perturbations, i.e. with respect to $u_i^{(m)}$, $\varepsilon_{ij}^{(m)}$ and $\sigma_{ij}^{(m)}$, the following linearized constitutive relations for the layer and the half-space materials are obtained.

$$\sigma_{11}^{(m)} = A_{11}^{(m)}\varepsilon_{11}^{(m)} + A_{12}^{(m)}\varepsilon_{22}^{(m)}, \quad \sigma_{22}^{(m)} = A_{12}^{(m)}\varepsilon_{11}^{(m)} + A_{22}^{(m)}\varepsilon_{22}^{(m)}, \quad \sigma_{12}^{(m)} = 2\mu_{12}^{(m)}\varepsilon_{12}^{(m)} \tag{9}$$

where

$$A_{11}^{(m)} = \lambda^{(m)} + 2\mu^{(m)} + \frac{2P_0}{3K_0^{(m)}}\left[\left(a^{(m)} + b^{(m)}\right) - \left(2b^{(m)} + c^{(m)}\right)\frac{\lambda^{(m)}}{2\mu^{(m)}}\right],$$

$$A_{22}^{(m)} = \lambda^{(m)} + 2\mu^{(m)} + \frac{P_0}{\mu^{(m)}}\left(2b^{(m)} + c^{(m)}\right) + \frac{2P_0}{3K_0^{(m)}}\left[\left(a^{(m)} + b^{(m)}\right) - \left(2b^{(m)} + c^{(m)}\right)\frac{\lambda^{(m)}}{2\mu^{(m)}}\right],$$

$$A_{12}^{(m)} = \lambda^{(m)} + \frac{b^{(m)}}{\mu^{(m)}}P_0 + \frac{2P_0}{3K_0^{(m)}}\left[a^{(m)} - b^{(m)}\frac{\lambda^{(m)}}{\mu^{(m)}}\right],$$

$$\mu_{12}^{(m)} = \mu^{(m)} + \frac{b^{(m)}}{3K_0^{(m)}}P_0 + \frac{c^{(m)}P_0}{4\mu^{(m)}}\left[\frac{\lambda^{(m)} + 2\mu^{(m)}}{3K_0^{(m)}}\right],$$

$$K_0^{(m)} = \lambda^{(m)} + \frac{2\mu^{(m)}}{3}, \qquad \varepsilon_{ij}^{(m)} = \frac{1}{2}\left(\frac{\partial u_i^{(m)}}{\partial x_j} + \frac{\partial u_j^{(m)}}{\partial x_i}\right). \tag{10}$$

In the case where P_0=0, this formulation transforms to the corresponding one made within the scope of the classical linear theory of elastodynamics.

3. Solution procedure

Each displacements component of the considered system are represent as follows

$$u_1^{(m)} = \varphi_1^{(m)}(x_2)\sin(kx_1 - \omega t), \quad u_2^{(m)} = \varphi_2^{(m)}(x_2)\cos(kx_1 - \omega t). \tag{11}$$

Substituting presentation (11) into the relations (10) and (9) we obtain the following equations for the $\varphi_1^{(m)}(x_2)$ and $\varphi_2^{(m)}(x_2)$ from the equation of motion (1)

$$\frac{\mathrm{d}^2\varphi_1^{(m)}}{\mathrm{d}(kx_2)^2} + b_{21}^{(m)}\varphi_1^{(m)} + c_{21}^{(m)}\frac{\mathrm{d}\varphi_2^{(m)}}{\mathrm{d}(kx_2)} = 0,$$

$$\frac{\mathrm{d}^2\varphi_2^{(m)}}{\mathrm{d}(kx_2)^2} + b_{22}^{(m)}\varphi_2^{(m)} + c_{22}^{(m)}\frac{\mathrm{d}\varphi_1^{(m)}}{\mathrm{d}(kx_2)} = 0, \tag{12}$$

where

$$b_{21}^{(m)} = -\frac{A_{11}^{(m)}}{\mu_{12}^{(m)} + P_0} + \frac{\rho^{(m)}\omega^2}{\left(\mu_{12}^{(m)} + P_0\right)k^2}, \qquad c_{21}^{(m)} = \frac{-A_{12}^{(m)} - \mu_{12}^{(m)}}{\mu_{12}^{(m)} + P_0},$$

$$b_{22}^{(m)} = -\frac{\mu_{12}^{(m)}}{A_{22}^{(m)} + P_0} + \frac{\rho^{(m)}\omega^2}{\left(A_{22}^{(m)} + P_0\right)k^2}, \qquad c_{22}^{(m)} = \frac{\mu_{12}^{(m)} + A_{12}^{(m)}}{A_{22}^{(m)} + P_0}. \tag{13}$$

After some mathematical procedures, we derive the following equation for $\varphi_2^{(m)}\left(x_2\right)$

$$\frac{\mathrm{d}^4\varphi_2^{(m)}}{\mathrm{d}\left(kx_2\right)^4} + B_2^{(m)}\frac{\mathrm{d}^2\varphi_2^{(m)}}{\mathrm{d}\left(kx_2\right)^2} + C_2^{(m)}\varphi_2^{(m)} = 0,$$

$$B_2^{(m)} = b_{22}^{(m)} + b_{21}^{(m)} - c_{21}^{(m)}c_{22}^{(m)}, \qquad C_2^{(m)} = b_{21}^{(m)}b_{22}^{(m)}. \tag{14}$$

We determine the solution to the Eq. (14) as follows

$$\varphi_2^{(1)}\left(x_2\right) = Z_1^{(1)}\exp\left(R_1^{(1)}kx_2\right) + Z_2^{(1)}\exp\left(-R_1^{(1)}kx_2\right) + Z_3^{(1)}\exp\left(R_2^{(1)}kx_2\right) + Z_4^{(1)}\exp\left(-R_2^{(1)}kx_2\right),$$

$$\varphi_2^{(2)}\left(x_2\right) = Z_1^{(2)}\exp\left(R_1^{(2)}kx_2\right) + Z_3^{(2)}\exp\left(R_2^{(2)}kx_2\right), \tag{15}$$

where

$$R_1^{(m)} = \sqrt{-\frac{B_2^{(m)}}{2} + \sqrt{\frac{\left(B_2^{(m)}\right)^2}{4} - C_2^{(m)}}}, \qquad R_2^{(m)} = \sqrt{-\frac{B_2^{(m)}}{2} - \sqrt{\frac{\left(B_2^{(m)}\right)^2}{4} - C_2^{(m)}}}. \tag{16}$$

Substituting the expression for $\varphi_2^{(m)}\left(x_2\right)$ from (15) into the second equation in (12) we obtain an equation with respect to $\varphi_1^{(m)}\left(x_2\right)$ from which we can determine the similar expression for $\varphi_1^{(m)}\left(x_2\right)$. Then using the relations (11), (10) and (9) we can get the expressions of stresses, strains and displacements of each constituents of the system. Finally, we can obtain the dispersion equation considering the conditions (2)-(5). This dispersion equation after some mathematical manipulations can be expressed formally as follows

$$\det\left\|\alpha_{ij}\left(c, kh, P_0, a^{(1)}, b^{(1)}, c^{(1)}, a^{(2)}, b^{(2)}, c^{(2)}\right)\right\| = 0, \tag{17}$$

where $i; j = 1, 2, ..., 6$ and

$$c = \frac{\omega}{k}. \tag{18}$$

In the present paper we study the generalized Rayleigh waves dispersion in the system under consideration and as in the works by Tolstoy and Usdin (1953), Eringen and Suhubi (1975b), the propagation velocity of the generalized Rayleigh waves is determined within the scope of the following assumptions

$$\mathrm{Re}(R_1^{(1)})=\mathrm{Re}(R_2^{(1)})=0,\quad R_1^{(2)}>0,\quad R_2^{(2)}>0. \tag{19}$$

Considering (14) and (17) this is concise way of saying that in order to satisfy the conditions (19) the following relations must hold

$$B_2^{(1)}>0,\qquad C_2^{(1)}>0,\qquad B_2^{(2)}<0,\qquad C_2^{(2)}<0. \tag{20}$$

The relations (20) hold when

$$\max\left(\tilde{c}_1^{(1)},\tilde{c}_2^{(1)},\tilde{c}_3^{(1)}\right)<c<\min\left(\tilde{c}_1^{(2)},\tilde{c}_2^{(2)},\tilde{c}_3^{(2)}\right) \tag{21}$$

where

$$\tilde{c}_1^{(m)}=\sqrt{\frac{A_{11}^{(m)}}{\rho^{(m)}}},\qquad \tilde{c}_2^{(m)}=\sqrt{\frac{\mu_{12}^{(m)}}{\rho^{(m)}}},$$

$$\tilde{c}_3^{(m)}=\sqrt{\frac{A_{11}^{(m)}}{\rho^{(m)}}\left(\frac{A_{22}^{(m)}+P_0}{\mu_{12}^{(m)}+A_{22}^{(m)}+2P_0}+\frac{\mu_{12}^{(m)}}{A_{11}^{(m)}}\frac{\mu_{12}^{(m)}+P_0}{\mu_{12}^{(m)}+A_{22}^{(m)}+2P_0}-\frac{\left(\mu_{12}^{(m)}+A_{12}^{(m)}\right)^2}{A_{11}^{(m)}\left(\mu_{12}^{(m)}+A_{22}^{(m)}+2P_0\right)}\right)}. \tag{22}$$

The relation (21) guarantee that $R_1^{(2)}$, $R_2^{(2)}$ always are real and positive, and $R_1^{(1)}$, $R_2^{(1)}$ always are pure imaginary. Note that the other cases under which the relations (19)-(22) are violated, in the present work do not considered. Thus, within the scope of the assumptions (19)-(21) we obtain the following expressions for the α_{ij} in Eq. (17)

$$\alpha_{11}=-\frac{R_1^{(1)}}{c_{22}^{(1)}}-\frac{b_{22}^{(1)}}{R_1^{(1)}c_{22}^{(1)}},\qquad \alpha_{12}=\frac{R_1^{(1)}}{c_{22}^{(1)}}+\frac{b_{22}^{(1)}}{R_1^{(1)}c_{22}^{(1)}},\quad \alpha_{13}=-\frac{R_2^{(1)}}{c_{22}^{(1)}}-\frac{b_{22}^{(1)}}{R_2^{(1)}c_{22}^{(1)}},$$

$$\alpha_{14}=\frac{R_2^{(1)}}{c_{22}^{(1)}}+\frac{b_{22}^{(1)}}{R_2^{(1)}c_{22}^{(1)}},$$

$$\alpha_{15}=\frac{R_1^{(2)}}{c_{22}^{(2)}}+\frac{b_{22}^{(2)}}{R_1^{(2)}c_{22}^{(2)}},\qquad \alpha_{16}=\frac{R_2^{(2)}}{c_{22}^{(2)}}+\frac{b_{22}^{(2)}}{R_2^{(2)}c_{22}^{(2)}},$$

$$\alpha_{21}=1,\quad \alpha_{22}=1,\quad \alpha_{23}=1,\quad \alpha_{24}=1,\quad \alpha_{25}=-1,\quad \alpha_{26}=-1,$$

$$\alpha_{31}=-\mu_{12}^{(1)}\left(\frac{\left(R_1^{(1)}\right)^2+b_{22}^{(1)}}{c_{22}^{(1)}}+1\right),\qquad \alpha_{32}=-\mu_{12}^{(1)}\left(\frac{\left(R_1^{(1)}\right)^2+b_{22}^{(1)}}{c_{22}^{(1)}}+1\right),$$

$$\alpha_{33}=-\mu_{12}^{(1)}\left(\frac{\left(R_2^{(1)}\right)^2+b_{22}^{(1)}}{c_{22}^{(1)}}+1\right),\qquad \alpha_{34}=-\mu_{12}^{(1)}\left(\frac{\left(R_2^{(1)}\right)^2+b_{22}^{(1)}}{c_{22}^{(1)}}+1\right),$$

$$\alpha_{35}=\mu_{12}^{(2)}\left(\frac{\left(R_1^{(2)}\right)^2+b_{22}^{(2)}}{c_{22}^{(2)}}+1\right),\qquad \alpha_{36}=\mu_{12}^{(2)}\left(\frac{\left(R_2^{(2)}\right)^2+b_{22}^{(2)}}{c_{22}^{(2)}}+1\right),$$

$$\alpha_{41}=A_{22}^{(1)}R_1^{(1)}-A_{12}^{(1)}\left(\frac{R_1^{(1)}}{c_{22}^{(1)}}+\frac{b_{22}^{(1)}}{R_1^{(1)}c_{22}^{(1)}}\right),\qquad \alpha_{42}=-A_{22}^{(1)}R_1^{(1)}+A_{12}^{(1)}\left(\frac{R_1^{(1)}}{c_{22}^{(1)}}+\frac{b_{22}^{(1)}}{R_1^{(1)}c_{22}^{(1)}}\right),$$

$$\alpha_{43}=A_{22}^{(1)}R_2^{(1)}-A_{12}^{(1)}\left(\frac{R_2^{(1)}}{c_{22}^{(1)}}+\frac{b_{22}^{(1)}}{R_2^{(1)}c_{22}^{(1)}}\right),\qquad \alpha_{44}=-A_{22}^{(1)}R_2^{(1)}+A_{12}^{(1)}\left(\frac{R_2^{(1)}}{c_{22}^{(1)}}+\frac{b_{22}^{(1)}}{R_2^{(1)}c_{22}^{(1)}}\right),$$

$$\alpha_{45}=-A_{22}^{(2)}R_1^{(2)}+A_{12}^{(2)}\left(\frac{R_1^{(2)}}{c_{22}^{(2)}}+\frac{b_{22}^{(2)}}{R_1^{(2)}c_{22}^{(2)}}\right),\qquad \alpha_{46}=-A_{22}^{(2)}R_2^{(2)}+A_{12}^{(2)}\left(\frac{R_2^{(2)}}{c_{22}^{(2)}}+\frac{b_{22}^{(2)}}{R_2^{(2)}c_{22}^{(2)}}\right),$$

$$\alpha_{51}=-e^{iR_1^{(1)}kh}\left(1+\frac{\left(R_1^{(1)}\right)^2}{c_{22}^{(1)}}+\frac{b_{22}^{(1)}}{c_{22}^{(1)}}\right),\qquad \alpha_{52}=-e^{-iR_1^{(1)}kh}\left(1+\frac{\left(R_1^{(1)}\right)^2}{c_{22}^{(1)}}+\frac{b_{22}^{(1)}}{c_{22}^{(1)}}\right),$$

$$\alpha_{53}=-e^{iR_2^{(1)}kh}\left(1+\frac{\left(R_2^{(1)}\right)^2}{c_{22}^{(1)}}+\frac{b_{22}^{(1)}}{c_{22}^{(1)}}\right),\qquad \alpha_{54}=-e^{-iR_2^{(1)}kh}\left(1+\frac{\left(R_2^{(1)}\right)^2}{c_{22}^{(1)}}+\frac{b_{22}^{(1)}}{c_{22}^{(1)}}\right),\quad \alpha_{55}=0,\quad \alpha_{56}=0,$$

$$\alpha_{61}=e^{iR_1^{(1)}kh}\left(A_{22}^{(1)}R_1^{(1)}-A_{12}^{(1)}\left(\frac{R_1^{(1)}}{c_{22}^{(1)}}+\frac{b_{22}^{(1)}}{R_1^{(1)}c_{22}^{(1)}}\right)\right),$$

$$\alpha_{62}=e^{-iR_1^{(1)}kh}\left(A_{12}^{(1)}\left(\frac{R_1^{(1)}}{c_{22}^{(1)}}+\frac{b_{22}^{(1)}}{R_1^{(1)}c_{22}^{(1)}}\right)-A_{22}^{(1)}R_1^{(1)}\right),$$

$$\alpha_{63}=e^{iR_2^{(1)}kh}\left(A_{22}^{(1)}R_2^{(1)}-A_{12}^{(1)}\left(\frac{R_2^{(1)}}{c_{22}^{(1)}}+\frac{b_{22}^{(1)}}{R_2^{(1)}c_{22}^{(1)}}\right)\right),$$

$$\alpha_{64}=e^{-iR_2^{(1)}kh}\left(A_{12}^{(1)}\left(\frac{R_2^{(1)}}{c_{22}^{(1)}}+\frac{b_{22}^{(1)}}{R_2^{(1)}c_{22}^{(1)}}\right)-A_{22}^{(1)}R_2^{(1)}\right),\quad \alpha_{65}=0,\quad \alpha_{66}=0.$$

4. Numerical results and discussion

In the cases where the assumptions (19)-(22) are satisfied the solution (15) corresponds to such a wave propagation in the layered half-space that the layer undergoes an oscillatory motion in the Ox_2 direction propagating in the Ox_1 direction with velocity c. The disturbances in the layer decay exponentially with depth in the half-space and therefore the wave can be considered as a generalized Rayleigh wave confined to the pre-stressed covered layer. The dispersion Eq. (17) has infinitely many modes unlike ordinary Rayleigh waves, which can propagate only in one mode. By ordinary Rayleigh wave we mean the Rayleigh wave that propagated in the homogeneous, isotropic, elastic half-space media. Velocity of propagation of this ordinary Rayleigh wave only depends on the mechanical properties of the medium and not on the wavenumber of the propagated waves, thus this wave is not dispersive and can propagated only in one mode. Moreover, the dispersion curves related to each mode has two branches which were denoted by

Table 1 Values of the elastic constants of selected materials (according to Guz, 2004)

Materials	Density	Lame constants		Third order elastic constants		
	ρ (g/cm^3)	$\lambda\times10^{-4}$ (MPa)	$\mu\times10^{-4}$ (MPa)	$a\times10^{-5}$ (MPa)	$b\times10^{-5}$ (MPa)	$c\times10^{-5}$ (MPa)
Steel 3	7.795	9.26	7.75	−2.35	−2.75	−4.90
Bronze	7.20	8.16	3.84	1.20	−3.10	4.80
Brass 59–1	7.20	9.49	4.47	−0.70	2.70	−3.40
Brass 62	7.20	9.49	4.47	−2.80	−2.10	−3.20
Plexiglas	1.16	0.404	0.19	2.68×10^{-3}	-3.12×10^{-2}	-6.77×10^{-2}

M_{1n} and M_{2n} respectively for the n-th mode. For the first M_{1n} branches the displacement of the layer circumscribes the ellipse similar to the ordinary Rayleigh waves, but for the second M_{2n} branches leads to an opposite type of motion. Moreover, according to the restriction (20), it must be $c/c_2^{(2)} < 1$ and $c/c_1^{(1)} > 1$, i.e., the near-surface wave propagated in the system under consideration is subsonic in the half-space, but it is supersonic in the covering layer, Tolstoy and Usdin (1953).

In this study we consider two real material pairs, values of the mechanical constants of which are given in Table 1 (i.e., the values of the mechanical constants which enter the expression (6) of the Murnaghan potential). We select four pairs from these materials. For the *I*, *II*, *III* and *IV* pairs, the material of the covering layer we take as *bronze*, *brass* 59-1, *brass* 62 and *Plexiglas*, respectively, but for all the pairs the material of the half-space we take as *steel*.

For estimation of the magnitude of the initial stresses we introduce the parameter

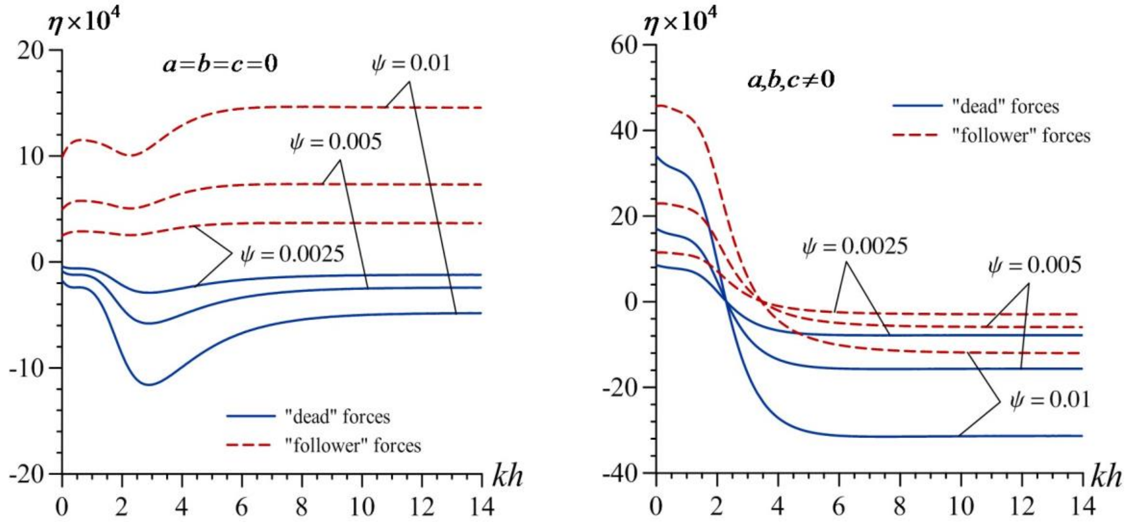
$$\psi = P_0 / \mu^{(2)} . \tag{23}$$

Moreover, we introduce the notation

$$\eta = \frac{c|_{\psi\neq 0} - c|_{\psi=0}}{c|_{\psi=0}} , \tag{24}$$

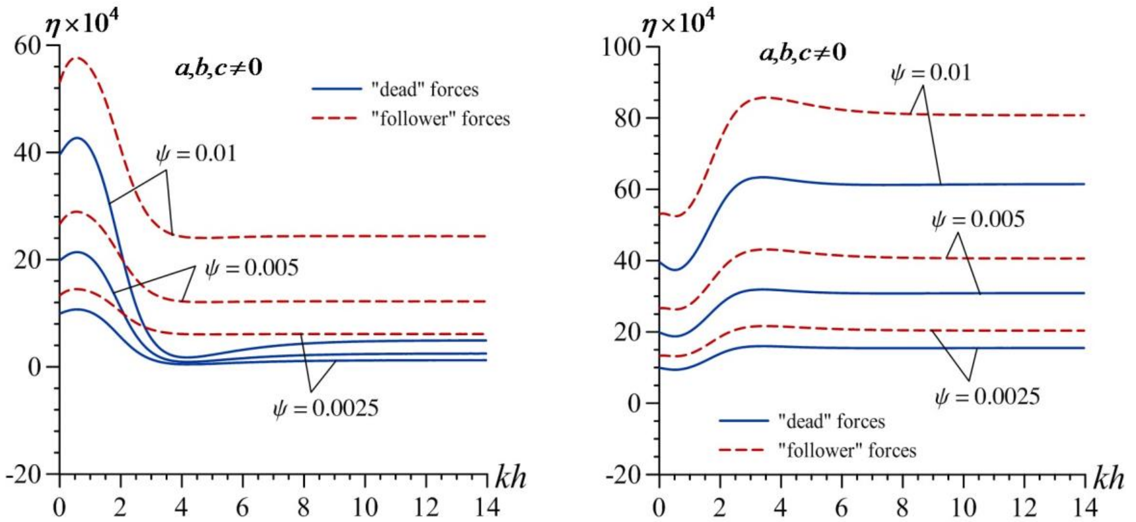
for estimation of the influence of the initial stresses in the constituents, i.e., the influence of the parameters ψ on the wave propagation velocity. Thus, through the graphs of the dependencies between η (24) and kh constructed for various values of the parameters ψ (23) we analyze the effect of the initial stresses in the constituents on the wave propagation velocity.

Fig. 2 shows the dispersion curves obtained for the first branch of the first mode of the generalized Rayleigh wave for the *I* pair of the materials for both dead and follower forces. Fig. 2(a) shows the results for the case when the third order elasticity constants are not considered. This figure shows that for all values of the *kh*, under the action of the dead forces the wave propagation velocity *c* decreases with increasing ψ, i.e., with increasing absolute values of the compressional forces that cause the initial stresses. However, under the action of the follower forces the behavior of the influence of the ψ on the values of *c* changes completely in the opposite direction, i.e., under the action of the follower forces wave propagation velocity increases with increasing ψ. Fig. 2(b) shows the results for the case when the third order elasticity constants are not zero. It follows from these graphs that considering the effect of third order elasticity constants into account, for both dead and follower cases, first of all the behavior of dispersion curves will be similar for dead and



(a) $a=b=c=0$ (b) $a,b,c\neq0$

Fig. 2 Influence of the compressional "dead" and "following" forces to the dispersion of the generalized Rayleigh wave for the *I* pair of the materials for the first branch of the first mode

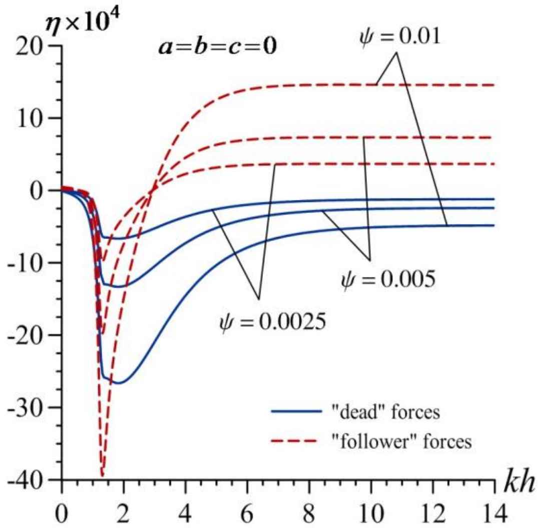


(a) *II* pair (b) *III* pair

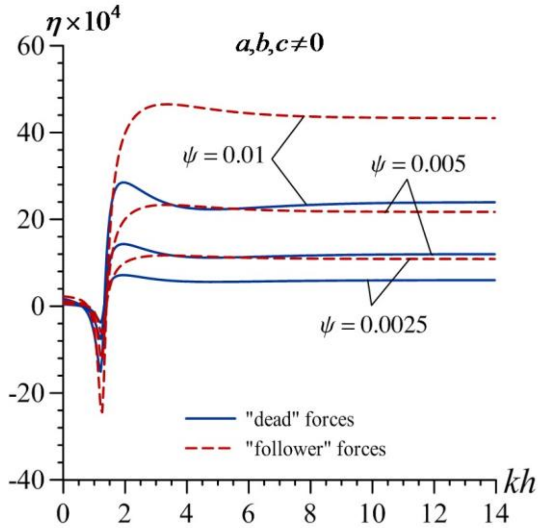
Fig. 3 Influence of the compressional "dead" and "following" forces to the dispersion of the generalized Rayleigh wave for the first branch of the first mode when $a,b,c\neq0$

follower forces and the second, not only the effect of initial compressional forces increases dramatically in this cases, but the behavior of the dispersion curves changes completely. For example in the case of dead forces (follower forces) wave propagation velocity increases with increasing ψ before some value of the wavenumber, say $kh\approx2.3$ ($kh\approx3.4$ in the case of follower forces) and decreases after that value with increasing ψ.

Fig. 3 shows the dispersion curves obtained for the first branch of the first mode for the *II* and



(a) a=b=c=0



(b) $a,b,c\neq0$

Fig. 4 Influence of the compressional "dead" and "following" forces to the dispersion of the generalized Rayleigh wave for the *IV* pair of the materials for the first branch of the first mode

III pairs of the materials for both dead and follower forces for the case when the third order elasticity constants are not zero. As can be seen from these figures for both these material pairs and for both cases of the initial stresses the wave propagation velocity increases as the absolute values of the compressional forces increase.

Fig. 4 illustrates the dispersion curves of the first branch of the first mode for the *IV* pair of the materials and influence of the dead and follower forces on the wave dispersion curves for the cases when the third order elasticity constants are taken to be zero (a) and when they are considered to not to be zero (b). It follows from these graphs that as in the case of the *I* pair of the materials considering the effect of the third order elasticity constants into account the influence of initial compressional forces increases dramatically. Furthermore, behavior of the wave propagation curves depends on the wavenumbers, for example in the case of follower forces Fig. 4(a) wave propagation velocity decreases with increasing ψ before some value of the wavenumber (say $kh\approx3$) and increases after that value with increasing ψ. Finally, for this material case the low wavenumber limit values of the wave propagation velocity as $kh\to0$ do not depend on the character of the initial compressional forces, i.e. dead or follower forces.

Dispersion curves related to the second branch of the first mode for all four material pairs have been shown in Fig. 5 for the case where the third order elasticity constants are considered in the analysis. Thus, it follows from Fig. 5 that the dimensionless wavenumber kh has cut off values for the second branch of the first mode. Furthermore, low and high wavenumber limit values of the wave propagation velocity as $kh\to0$ and $kh\to\infty$, respectively, do not depend on the character of the initial compressional forces, i.e., on the dead or follower forces. It follows from Figs. 5(b)-(c) that in the case of the *II* and the *III* pairs of the materials the wave propagation velocity increase monotonically as the values of initial compressional forces, i.e. ψ increase. However, in the case of the *I* and the *IV* pairs of the materials, Figs. 5(a)-(d) respectively, the behavior of the wave propagation curves depend on the wavenumbers. For example, in Fig. 5(a) the wave propagation

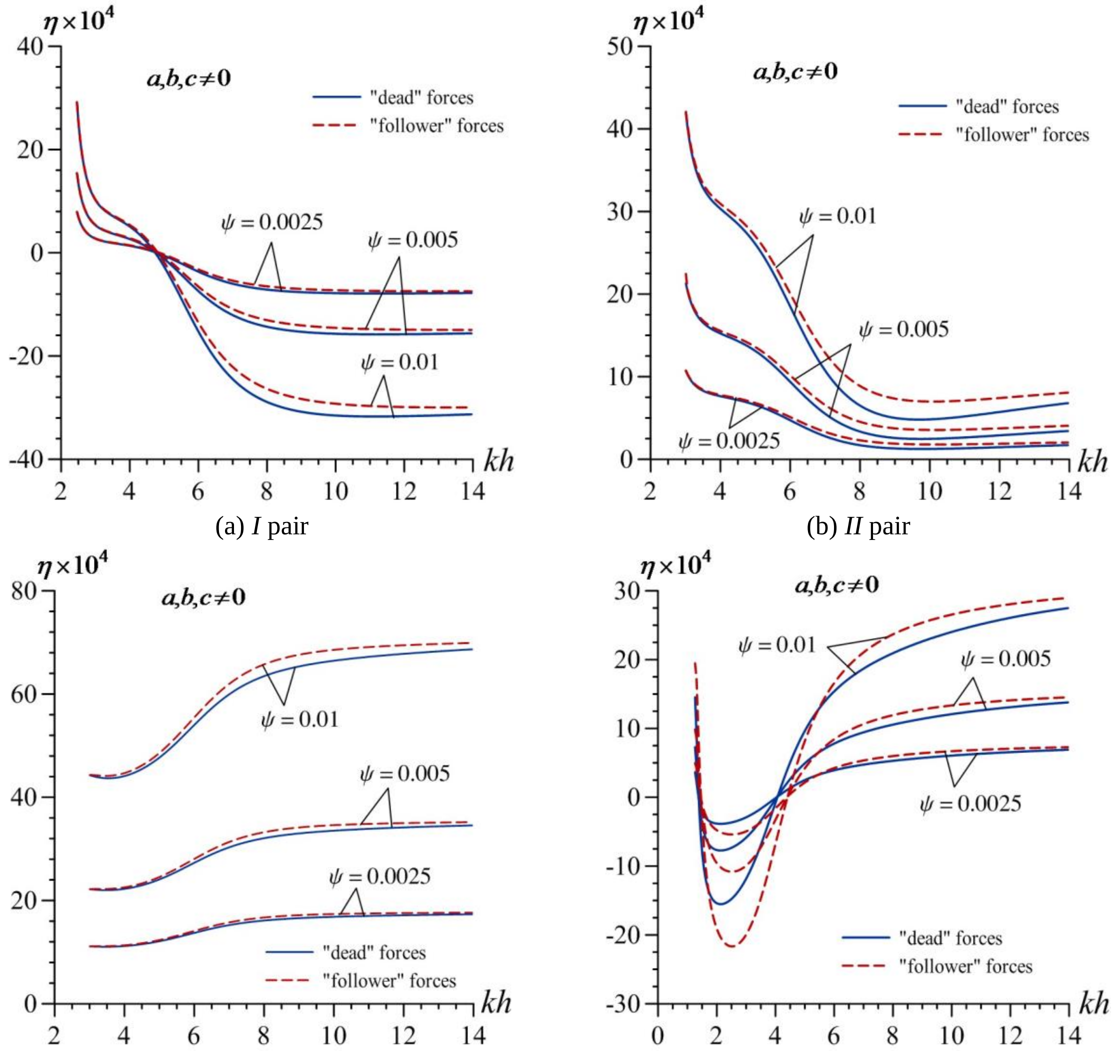
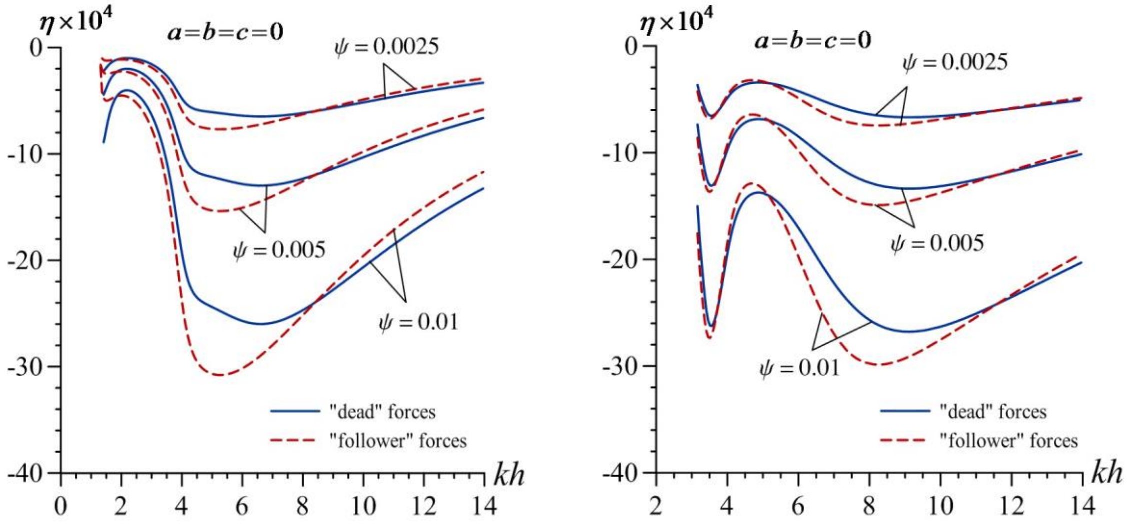


Fig. 5 The influence compressional “dead” and “following” forces to the dispersion of the generalized Rayleigh wave for the second branch of the first mode when $a,b,c\neq0$

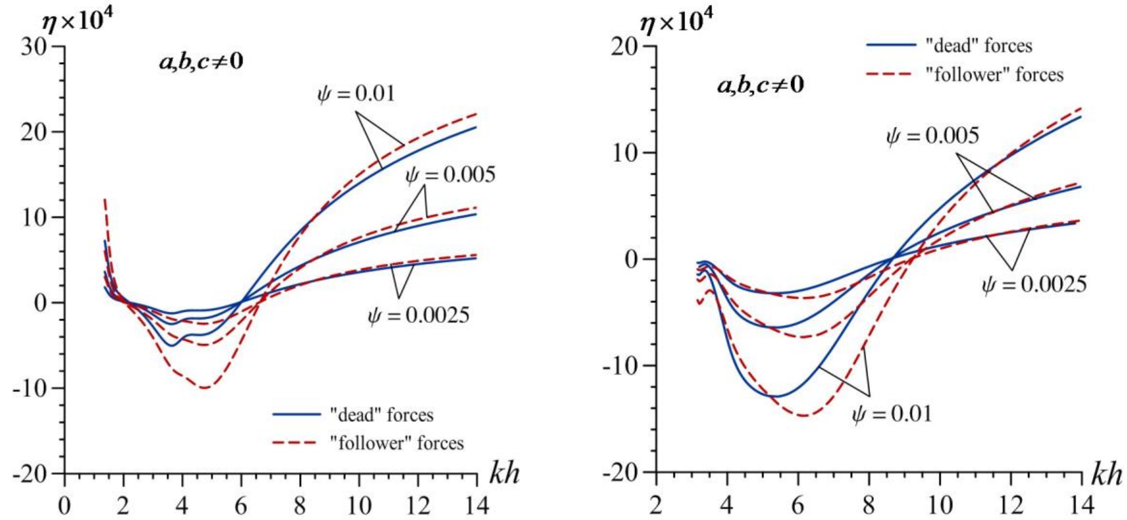
velocity decreases with increasing the absolute values of ψ after some value of the wavenumber (say $kh\approx4.8$) and increases after that value with increasing ψ, or in Fig. 5(d) the wave propagation velocity decreases with increasing the absolute values of ψ after some value of the wavenumber (say $kh\approx4.3$) and increases after that value with increasing ψ.

Finally, we consider the graphs given in Figs. 6-7 which illustrate the dependence between η and kh constructed for the first and second branches of the second mode for the *IV* pair of the materials. Similar results as previous ones can be obtained here. First of all, it follows from these figures that for the first and second branch of the second mode the dimensionless wavenumber kh also has cut off values. Fig. 6 shows the results for the case when the third order elasticity constants are taken to be zero. This figure shows that for all values of the kh, under the action of



(a) first branch of the second mode (b) second branch of the second mode

Fig. 6 Influence of the compressional “dead” and “following” forces to the dispersion of the generalized Rayleigh wave for the *IV* pair of the materials when $a=b=c=0$



(a) first branch of the second mode (b) second branch of the second mode

Fig. 7 Influence of the compressional “dead” and “following” forces to the dispersion of the generalized Rayleigh wave for the *IV* pair of the materials when $a,b,c \neq 0$

the dead forces the wave propagation velocity c decreases with increasing the absolute value of the compressional forces that cause the initial stresses. Fig. 7 shows the results for the case when the third order elasticity constants are not zero. It follows from these graphs that considering the third order elasticity constants into account, has similar effects on both dead and follower cases, and the character of the influence of the ψ on the values of wave propagation velocity c changes completely. For example in Fig. 7(a) wave propagation velocity decreases with increasing ψ

before some value of the wavenumber (say $kh\approx6.7$) and increases after that value with increasing ψ for the first branch of the second mode or in Fig. 7(b) wave propagation velocity behavior changes after some value of the wavenumber (say $kh\approx9.3$) for the second branch of the second mode.

5. Conclusions

In the present paper within the scope of the piecewise homogeneous body model with the use of the 3D linearized theory of elastic waves in initially stressed bodies, dispersion of the generalized Rayleigh waves in an initially stressed elastic half-space covered by an elastic layer is investigated. It is assumed that the initial stresses are caused by the uniformly distributed normal compressional forces acting on the face surface of the covering layer. Two cases are considered: in Case 1 it is supposed that the compressional forces are "dead" forces, while in Case 2 it is assumed that the they are "follower" ones. The dispersion equations for each case are obtained and an algorithm was developed to do numerical investigations. The basic numerical results for the low and high wave number limit values of the wave propagation velocity are presented and discussed. Moreover, the numerical results related to different branches of the first and second modes are presented and the effect of third order elasticity constants are considered yielding the following main conclusions:

• In the case where the initial stresses in the considered system are caused by the compressional "dead" forces which act on the face surface of the covering layer, for all values of kh the propagation velocity of the generalized Rayleigh wave decreases with the absolute values of the compressional forces (i.e., with decreasing ψ).

• In the case where the above mentioned forces are the "follower" ones, the behavior of the influence of the initial stresses on the wave propagation velocity is more complicated and in general depends on the values of kh, (i.e., the velocity of wave propagation before/after some values of kh decreases/increases).

• Taking the third order elastic constants into account the behavior of the influence of the initial stresses on the velocity of wave propagation vary for different material pairs and before/after some values of kh decreases/increases the velocity of wave propagation.

These results can be used for study and for characterization of the structure of the soil of the bottom of the oceans, study of the dispersion of seismic surface waves propagating under the bottom of the oceans and study of the near-surface wave propagation in layered mechanical systems with a liquid upper layer.

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