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A fuzzy residual strength based fatigue life prediction method

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Abstract. The fatigue damage problems are frequently encountered in the design of civil engineering structures. A realistic and accurate fatigue life prediction is quite essential to ensure the safety of engineering design. However, constructing a reliable fatigue life prediction model can be quite challenging. The use of traditional deterministic approach in predicting the fatigue life is sometimes too dangerous in the real practical designs as the method itself contains a wide range of uncertain factors. In this paper, a new fatigue life prediction method is going to be proposed where the residual strength is been utilized. Several cumulative damage models, capable of predicting the fatigue life of a structural element, are considered. Based on Miner's rule, a randomized approach is developed from a deterministic equation. The residual strength is used in a one to one transformation methodology which is used for the derivation of the fatigue life. To arrive at more robust results, fuzzy sets are introduced to model the parameter uncertainties. This leads to a convoluted fuzzy based fatigue life prediction model. The developed model is illustrated in an example analysis. The calculated results are compared with real experimental data. The applicability of this approach for a required reliability level is also discussed.

Keywords: fatigue life; residual strength; fuzzy model; reliability

1. Introduction

The consideration of fatigue failure is one of the most important issues in engineering structural design. It is widely believed that more than 80% of the steel structural failures are related to fatigue fractures (Zhu *et al*. 2012). As fatigue reliability is usually required for the safe operation of mechanical structures, an unexpected failure could lead to high consequences. For example, common failures in welds and pipelines can be found in Costa *et al*. (2012). Therefore, an accurate prediction of the fatigue life of structural components are particular critical in the engineering structure designs (Dai *et al*. 2013).

Fatigue damage is different from common brittle or ductile fractures. The damage increases with the load cycles in a cumulative manner which finally result to fracture. With the effect from the cumulative damage, the remaining life becomes much more limited under certain further loading. Many studies on the cumulative fatigue damage have been conducted in the past. Generally, the basic damage theories can be classified into two categories: linear damage cumulative theories and nonlinear damage cumulative theories. The linear damage cumulative rule

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(LDR) was initially proposed by Miner (1945). Because of its ease of use, it receives wide usage among the engineering designs (Zhang and Wang 2010). However, there are some drawbacks with the linear cumulative damage model. The main shortcoming of Miner's method is that the damage accumulation process does not account the effects from the load conditions. More specifically, the load sequence, interactions with the material properties as well as imperfections are not considered in the fundamental rule (Luo *et al*. 2014). Therefore, a plenty number of new techniques have been developed in the fatigue life predictions. These include, for example, damage curve based approaches (Manson and Halford 1981), mechanic model considering load interaction effects (Skorupa 1999), damage model based on energy entropy (Kim *et al*. 2013), continuum mechanic damage model (Fatemi and Yang 1998), a fuzzy set based Miner's method (Zhu *et al*. 2011) and ductility based methods (Cheng and Plumtree 1998). More comprehensive review on the fatigue life prediction methods can be found in Cunha *et al*. (2014). Nevertheless, it should be pointed out, the cumulative fatigue damage prediction problem is still a classic but not resolved case.

The stochastic nature of fatigue damage process makes the formulation of a deterministic approach usually impossible. Various sources of uncertainties are associated with the fatigue damage process, such as external loading, materials properties, defects and elemental geometries. Generally, most of the stochastic variables in the fatigue problems can be classified as three categories (Sankararaman 2011, Sankararaman and Mahadevan 2013): (I) model uncertainty, or sometimes known as epistemic uncertainty, that is mainly due to the imperfect knowledge or errors in the realization of the mathematical model or uncertainty in the choices of statistical models for the variables, (II) inherent uncertainty, which is referring to the natural randomness of a quantity and (III) statistical uncertainty, one that is related to errors caused from statistical inferences, e.g., estimations from limited observed data. Therefore, based on these concerned parameters, many stochastic mathematical formulations have been proposed by former researchers for the fatigue damage process (Zhu *et al*. 2013a). A general methodology for predicting the stochastic fatigue life by utilizing nonlinear accumulation damage theory is proposed by Liu and Mahadevan (2007). Wu and Huang (1993) developed a fatigue life prediction model for the structural elements to undertake variable loadings which undergoes a Gaussian random process. The probabilistic modeling of the fatigue damage accumulation with single stress and multi-level stress loadings can be found in Rathod *et al*. (2011). Ductility based methods are also developed in recent years (Zhu *et al*. 2013b). However, most of these works require extensive data information for the establishing of the fatigue model. The detailed information of crack growth law, crack geometry, materials properties and other mechanisms could hardly be fully available. A robust fatigue life prediction model is therefore required to provide a proper treatment of associated imprecise information and knowledge. Based on these concerns, alternatively, fuzzy model might be implemented in the fatigue model to represent the imperfect information. As such, the significance of this work can be justified.

In this paper, the fuzzy set theory is introduced in a fatigue life prediction method which utilizes the residual strength as an input parameter. The basic aim is to explore a way of handling the parameter uncertainties and thus improve the expected performance of the original method through the fuzzy concept. Recognizing that, the paper is organized as follows. The basic review of fatigue damage theory is introduced first. Then a simple approach which utilizes one to one transformation technique to model the relationship between the fatigue life and the residual strength is proposed. After that, a linear fuzzy model is introduced to offset the model uncertainties in the developed model. To facilitate the accuracy of this fuzzy model, a validation study is performed based on experimental data. The model is then tested in an example analysis to predict the fatigue life of the structural elements which are then compared with the experimental data. Finally, the concluding remarks are provided.

2. Basic theory of fatigue life and damage

According to most published fatigue damage accumulation theories, Miner's rule is most widely applied in engineering applications (Miner 1945). The rule utilizes a damage function (D) to represent the accumulated damage of the structural component. It is assumed the rate of damage accumulation remains unchanged over the whole loading process and damage only occurs when loading exceeds the fatigue limit. The failure will occur when the cumulative damage, the value of D, reaches the unity. The value of fatigue damage function is usually related to a number of factors. A simplified form which includes the most significant parameters can be expressed as

$$D = f\left(n, C, m, r, S, \omega, T, M\right), \tag{1}$$

where n is the number of loading cycles, C and m denote the material properties, r is the stress ratio, S represents the cyclic loading intensity, ω is the loading frequency, T is the temperature and M is the moisture which represent the current environment condition. Most commonly, the environment factors are assumed to be constant in fatigue analysis. This leads to a degenerated form as

$$D = f\left(n, C, m, r, S, \omega\right). \tag{2}$$

The damage function represents the level of damage which is directly linked to the residual strength of a structure or specimen. The residual strength is initially the same as static strength and decreases with the increasing of loading cycles. Under the fatigue loading, the residual strength is referring to the maximum static stress to cause the ultimate failure in the post fatigue condition. Therefore, the residual strength can be regarded as a variable which is dependent on the damage function. Based on the results in Xiong and Shenoi (2011), the change rate of the residual strength $R(n)$ in terms of the number of loading cycle n can be expressed as

$$\frac{dR(n)}{dn} = \frac{f'(C, m, r, S, \omega)}{R^{(b-1)}(n-1)}, \tag{3}$$

where $R(n)$ is the residual strength after n loading cycles, $f'(C, m, r, S, \omega)$ represents the fatigue strength which is a function of C, m, r, S and ω. The value of $R(n)$ can be measured using the X-ray diffraction (Lu 1996). One should note the use of Eq. (3) must have the assumption that fatigue damage under multi-level stress loading conditions cumulates linearly.

This reveals a fact that the change of the residual strength is inversely proportional to some power (b-1) of the residual strength itself. Thus, the number of loading cycles can be derived from Eq. (3) if the information of residual strength is known. By an integration of Eq. (3) from n_0=0 to the current state value n, the following strength degradation equation can be obtained

$$n = f'(C, m, r, S, \omega)\left[R_0 - R(n)\right]^b, \tag{4}$$

where R_0 is the initial fatigue strength which is commonly estimated as some fraction of ultimate tensile strength that is specific to a material type (for example, 35% is used for austenitic stainless steels). Usually, for simplicity, the value of the stress ratio r and loading frequency ω are assumed to be constant. Therefore, the damage function $f'(C, m, r, S, \omega)$ can be simplified to $f'(C, m, S)$ and Eq. (4) can be written as

$$n = f'(C,m,S)\left[R_0 - R(n)\right]^b . \tag{5}$$

Thus, Eq. (5) identifies the relationships among the residual strength $R(n)$, fatigue loading S and the number of loading cycles n.

The determination of the fatigue strength function $f'(C, m, S)$ relies on quite a number of factors. Many forms of equations have been proposed to approach the exact value of this function (Kwofie and Rahbar 2013). In this paper, since the developed model does not require too many material properties, the well known *S*-*N* curve, which is used to express the relationship between fatigue life N and stress S, is used to approximate the fatigue strength. The *S*-*N* curve generally models the relationship between the fatigue life and stress with a nonlinear power function expression as follow

$$N = \frac{C}{S^m} , \tag{6}$$

where C is the fatigue strength constant and m represents the slope of the *S*-*N* curve which is related to various material properties of the structural component, e.g., specimen configuration, material strength and etc. The values of Eq. (6) are usually determined from experiments. By combining Eqs. (5) and (6), the strength degradation equation can be further expressed as

$$n = \frac{C}{S^m}\left[R_0 - R(n)\right]^b . \tag{7}$$

This equation is frequently used in the prediction of fatigue life while the residual strength is known. In case that only initial strength R_0 is available, Eq. (7) can be rewritten to calculate the residual strength based on the available information. This is derived as

$$R(n) = R_0 - \left[\frac{S^m n}{C}\right]^{\frac{1}{b}} . \tag{8}$$

Normally the value of initial strength R_0 is determined based on a large number of test data. The variability of this value is represented by a probabilistic model. In the literature, Weibull distribution has been proved to be adequate in modeling the observed failure data of many different types of phenomenon and components. It has been used for various engineering applications, such as lifetime analysis and fatigue reliability analysis. Most works have adopted a two parameter Weibull model for modeling the initial strength of the material (Doudard *et al*. 2005).

$$F_{R_0}(x) = \mathrm{Prob}\left[R_0 \le x\right] = 1 - \exp\left[-\left(\frac{x}{\beta}\right)^{\alpha}\right] , \tag{9}$$

where α is the shape parameter and β is the scale parameter. The theoretical mean and variance of this Weibull model can be estimated from the followings

$$\mu_{R_0} = \beta \cdot \Gamma\left(1+\frac{1}{\alpha}\right), \tag{10}$$

$$\sigma_{R_0} = \beta \cdot \left\{\Gamma\left(1+\frac{2}{\alpha}\right) - \left[\Gamma\left(1+\frac{1}{\alpha}\right)\right]^2\right\}^{\frac{1}{2}}, \tag{11}$$

where Γ is the gamma function. It is usually difficult to have a direct estimation of the Weibull model parameters. Therefore, more commonly, the calculated sample mean and variance from a set of tested experimental data are utilized to estimate the scale and shape parameters based on Eq. (10) and Eq. (11).

By substituting Eq. (9) into Eq. (8), the cumulative distribution function (CDF) of the residual strength $R(n)$ can be derived as following

$$\begin{aligned} F_{R(n)}(x) &= \text{Prob}[R(n) \le x] = \text{Prob}\left[R_0 - \left[\frac{S^m n}{C}\right]^{\frac{1}{b}} \le x\right] \\ &= \text{Prob}\left[R_0 \le x + \left[\frac{S^m n}{C}\right]^{\frac{1}{b}}\right] = 1 - \exp\left[-\left(\frac{x + \left[\frac{S^m n}{C}\right]^{\frac{1}{b}}}{\beta}\right)^{\alpha}\right]. \end{aligned} \tag{12}$$

Therefore, it could be observed from Eq. (12), the residual strength is also following a Weibull distribution.

If we assume the fatigue failure occurs at the time when residual strength equals the cyclic stress level (e.g., $R(n)=S$), then the fatigue life could be predicted from Eq. (7) and Eq. (12). For instance, the fatigue life would then be estimated as

$$N = \frac{C}{S^m}\left[R_0 - S\right]^b, \tag{13}$$

where N represents the estimated total fatigue life (the number of cycles) of the specimen under the cyclic stress S. Following the same way of deriving the distribution of residual strength, the CDF of the fatigue life can be obtained as

$$\begin{aligned} F_N(x) &= \text{Prob}[N \le x] = \text{Prob}[\frac{C}{S^m}\left[R_0 - S\right]^b \le x] \\ &= \text{Prob}\left[R_0 \le \left(\frac{xS^m}{C}\right)^{\frac{1}{b}} + S\right] = 1 - \exp\left[-\left(\frac{\left(\frac{xS^m}{C}\right)^{\frac{1}{b}} + S}{\beta}\right)^{\alpha}\right]. \end{aligned} \tag{14}$$

Thus, it could be seen the fatigue life N is following a three parameter Weibull distribution. However, it is clear that in reality, the fatigue damage can be accumulated from different stress levels. In order to assess the remaining life of the specimen under uncertain loadings, the distribution of the total damage value D is of great interest. According to Miner's rule (1945), the damage accumulation with a single stress level S is given by

$$D = \frac{n}{N}, \tag{15}$$

where n and N have the same meanings as explained in Eq. (7) and Eq. (13). Based on the linear damage accumulation rule, for a load spectrum which includes stress levels $S_1,...,S_k$, with the corresponding number of load cycles $n_1,...,n_k$, the fatigue life prediction under constant block loading can be computed as

$$D = \sum_{i=1}^{k} \frac{n_i}{N_i}, \tag{16}$$

where $N_1,...,N_i$ represent the cycles to failure under each stress level which could be estimated from Eq. (13). With this help of Miner's rule, by combining the S-N curve model and the linear damage accumulation model, the relationship between the accumulated fatigue damage and the number of usage cycles at any given stress level can be derived as followings.

For single stress level

$$D = \frac{n_i}{N_i} = \frac{n_i}{\frac{C}{S_i^m}\left[R_0 - S_i\right]^b} = \frac{S_i^m n_i}{C\left[R_0 - S_i\right]^b} = \frac{S_i^m n_i}{A_i}. \tag{17}$$

For multiple stress levels

$$D = \sum_{i=1}^{k} \frac{n_i}{N_i} = \sum_{i=1}^{k} \frac{S_i^m n_i}{C\left[R_0 - S_i\right]^b} = \sum_{i=1}^{k} \frac{S_i^m n_i}{A_i}. \tag{18}$$

Since $C[R_0 - S_i]^b$ is independent of the usage cycles n_i, it is convoluted into one term, e.g. $A_i = C[R_0 - S_i]^b$. Generally, Eqs. (17) and (18) could be utilized to predict the expected damage accumulation value at any given point of time (usage cycles) with the given single stress level or multiple stress levels. For simplicity, a linear relationship between the damage accumulation measure and usage time can be established as follow

$$D = \frac{S_i^m n_i}{A_i} = h_i n_i, \tag{19}$$

where h_i indicates the slope of the damage accumulation trend line. While at fatigue failure life (e.g., n_i=N_i), Eq. (19) can be further written as

$$D = \frac{S_i^m N_i}{A_i} = h_i N_i. \tag{20}$$

Since the fatigue life N is following a three parameter Weibull distribution (as given in Eq. (14)), the distribution of D can be directly obtained. By substituting Eq. (20) into Eq. (14), the

CDF of D can be derived as follow

$$F_D\left(x\right)=1-\exp\left[-\left(\frac{\left(\frac{xS^m}{h_iC}\right)^{\frac{1}{b}}+S}{\beta}\right)^{\alpha}\right]. \tag{21}$$

Eq. (21) is thus the distribution of the damage accumulation at the time when the fatigue failure occurs.

3. Derived randomized approach

The determination of fatigue life has to account for the random nature of strength degradations. This covers random process of time, random process of space and random properties of the environment. All of which should be added in the deterministic model as random disturbances. With this concern, a random differential equation is usually introduced to estimate the parameter values. Then the solution of the randomized equation, the parameter estimations, can take account of the random nature of the observed data as well. By considering the randomized model in the strength degradation equation, Eq. (7) can be rewritten as

$$n=\frac{C}{S^m}\left[R_0-R\left(n\right)\right]^b\left(1+\boldsymbol{X}\left(n\right)\right), \tag{22}$$

where $\boldsymbol{X}(n)$ is a random process which represents the model errors and dependent on n (thermalelastic effect). Basically, in this randomized approach, $\boldsymbol{X}(n)$ is usually assumed to follow a log-Gaussian stochastic process with 0 mean and constant σ standard deviation (Meneghetti 2007). Therefore, by taking the logarithmic form of Eq. (22), the randomized equation can be further changed to

$$y=a_0+a_1x_1+a_2x_2+U\ , \tag{23}$$

where y=log(n), a_0=log(C), a_1=-m, a_2=b, x_1=log(S), x_2=log[R_0-R(n)], U=log[1+$\boldsymbol{X}$(n)]. Obviously, U is following a Gaussian distribution which has a mean value of 0.698 and a new variance of σ'^2. Similarly, the derived linear equation for the random variable y is also expected to follow a Gaussian distribution with a mean value of $a_0+a_1x_1+a_2x_2$+0.698 and variance of σ'^2. This gives the probability density function of the random variable y as

$$f\left(y\right)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left[-\frac{1}{2\sigma^2}\left(y-a_0-a_1x_1-a_2x_2-0.698\right)^2\right]. \tag{24}$$

Therefore, the parameter values in Eq. (22) can be determined by statistical parameter estimations based on Eq. (24). For example, the maximum likelihood estimation method can be applied. Here, the parameter values of a_0, a_1 and a_2 are yet to be determined. The values of x_1, x_2 and y are mainly obtained from the experimental tests. After the parameters in the strength degradation model are all estimated, the CDF for residual strength R(n), fatigue life N and damage accumulation value D can be constructed from Eqs. (12), (14) and (21).

However, this randomized approach is limited to cases in which probabilistic models can be specified for all the variables with sufficient confidence. Although the implementation of the stochastic variables can help to capture the randomness of the model (e.g., the use of $\boldsymbol{X}(n)$), several problems can still occur in the following situations:

• Limited number of experimental data set (available data set of x_1, x_2 and y are too small);

• Difficulties in specifying the distribution types (distribution of $\boldsymbol{X}(n)$ could be non-Gaussian);

• Imprecise uncertainties exist in the relationship between input and output variables (coefficients a_0, a_1 and a_2 may contain dubious uncertainties);

• Inadequate assumptions and distortion introduced by linearization (derived Eq. (23) can be misleading).

Therefore, alternatively, non-probabilistic methods could be applied to provide a framework for uncertainty quantification for the randomized model. Among the developed non-probabilistic concepts, fuzzy variables showed the most attractive properties in handling these imprecise information (Möller and Beer 2008, Beer *et al*. 2013). Thus, a fuzzy set based fatigue life prediction model is proposed herein. The fundamental concepts and ideas of this fuzzy based model is elucidated in the following sections.

4. Fuzzy modeling and fuzzy regression model

The fuzzy set provides a mathematical way for describing the uncertain data in which the information can only be described by a set of intervals and their associated gradual assignment (Walley 1991). The estimated interval values for a parameter a, e.g., $a \in [a_l, a_u]$, can be assessed with the aid of membership values $\mu_a(a)$. The membership value ranges from 0 to 1 and can be used to represent different meanings in real practice, for example, the similarity between different categories, preference from the decision maker and degree of uncertainty in the collected information (Zhang 2015a). In this paper, we propose to use it for representing a degree of epistemic possibility. In other words, the uncertainties of the parameters (the width of the intervals) are modeled as changes in the membership value, or so called membership function. More specifically, a fuzzy set can be described with its uncertain proposition by

$$\tilde{a} = \{(a, \mu_a(a)) | a \in \Re, 0 \le \mu_a(a) \le 1\}, \tag{25}$$

where $\tilde{a}$ is referred to as the fuzzy set on the domain of a, $\mu_a(a)$ is the membership function of the fuzzy set. In the modeling of uncertain quantities, fuzzy set can be more useful in cases where only possible bounds are known. These are due to various kinds of reasons, for example, lack of enough data, fluctuations, linguistic or vagueness (Zhang 2015b). Therefore, we propose to implement this concept into the fatigue life prediction model to offset the shortcomings as discussed in the previous section.

In the fuzzy modeling process, more commonly, the fuzzy components are assumed to be triangular fuzzy numbers (TFNs). The reason that TFN is so widely used is because it can be simply specified by three parameters. A fuzzy triangular number is determined by specifying the left spread and right spread as well as the value of mode (mean value). An illustration of the salient features of a TFN is shown in Fig. 1. Following the idea proposed by Tanaka *et al*. (1987), we propose to use the TFN model in characterizing the uncertainties in the developed fatigue model as

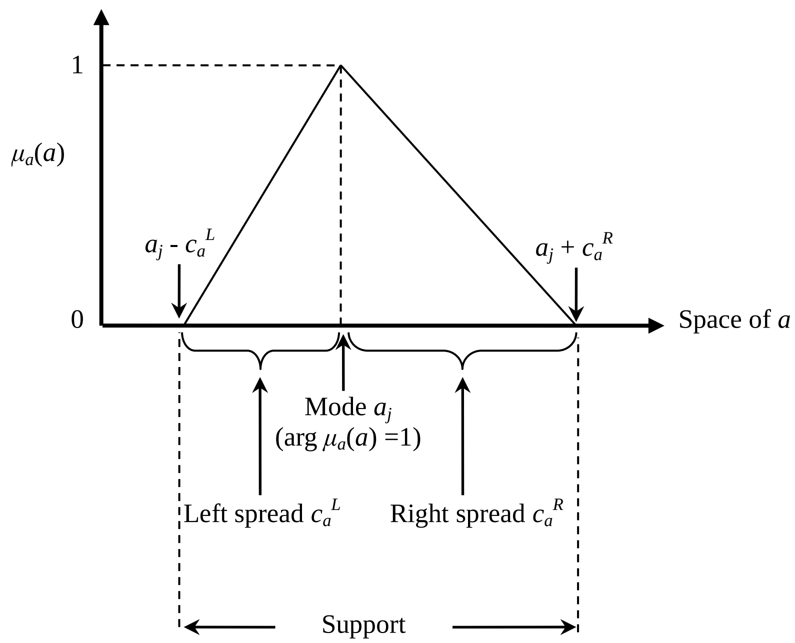

Fig. 1 Triangular fuzzy number

given in Eq. (23). In that sense, a fuzzification is applied to all the coefficients of the linear regression equation to account the uncertainties described in Section 3. This leads to a general fuzzy linear equation as

$$\tilde{y} = \tilde{a}_0 + \tilde{a}_1 x_1 + \tilde{a}_2 x_2 + U , \tag{26}$$

where $\widetilde{a_0}$, $\widetilde{a_1}$, $\widetilde{a_2}$ represent the fuzzy coefficients and $\tilde{y}$ represent the estimated fuzzy output. For the ease of simplicity, the Gaussian noise term U can be integrated into the fuzzy coefficient $\widetilde{a_0}$. Then Eq. (26) can be changed to a short form as

$$\tilde{y} = \tilde{a}_0 + \tilde{a}_1 x_1 + \tilde{a}_2 x_2 . \tag{27}$$

Therefore, the former parameter estimation method could not be applied anymore. Instead, the determination of the fuzzy model coefficients need to be conducted through a fuzzy regression analysis. Many previous works have been done to address the approaches in evaluating the values for the fuzzy regression (Peters 1994, Friedman *et al*. 1998). Most approaches tried to minimize the fuzziness of the linear model by minimizing the total spread of the fuzzy coefficients, subject to including all the observations.

To simplify matter, one of the most direct and convenient parameter estimation method, the least-squares method, is employed in this work (Friedman *et al*. 1998). This approach is primarily focusing on the minimization of distance between the output of the model and the observed data. Conceptually, the determined fuzzy coefficients should make the predicted model value deviate from the observed data as small as possible. This brings the fuzzy regression in line with the statistical regression.

By definition, from a least square perspective, the problem is to minimize the difference between a set of observed values y_i for i=1,...,n and a set of predicted fuzzy values $\tilde{y}_i$ for i=1,...,n which can be expressed as

$$\min \sum_{i=1}^{n} (\tilde{y}_i - y_i)\,. \tag{28}$$

However, the measure of the distance between a fuzzy number and a crisp value has many different definitions (Möller and Beer 2004). The most commonly applied distance measure concept is the L^2-metric $d(.,.)^2$ which is proposed by Diamond (1988). In Diamond's theory, the measure of the distance between two fuzzy numbers can be calculated based on their modes, left spreads and right spreads. A general formulation for measuring the distance between two fuzzy numbers can be given as below

$$d\left(\left\langle c_1^L, m_1, c_1^R \right\rangle, \left\langle c_2^L, m_2, c_2^R \right\rangle\right)^2 = \left(m_1 - m_2\right)^2 + \left(\left(m_1 - c_1^L\right) - \left(m_2 - c_2^L\right)\right)^2 + \left(\left(m_1 + c_1^R\right) - \left(m_2 + c_2^R\right)\right)^2 \tag{29}$$

where c^L and c^R represent the left and right spreads, m represents the mode. Therefore, by combining Eqs. (27), (28) and (29), the fuzzy coefficients in the fatigue life prediction model can be estimated by

$$\min_{\tilde{a}_0, \tilde{a}_1, \tilde{a}_2} \sum d(\tilde{a}_0 + \tilde{a}_1 x_1 + \tilde{a}_2 x_2, y)^2\,. \tag{30}$$

Since Eq. (27) is a linear and monotonic function, it would be easier to obtain the spreads and mode of $\tilde{y}$ by the following equations

$$c_{\tilde{y}}^L = c_{\tilde{a}_0}^L + c_{\tilde{a}_1}^L x_1 + c_{\tilde{a}_2}^L x_2\,, \tag{31}$$

$$c_{\tilde{y}}^R = c_{\tilde{a}_0}^R + c_{\tilde{a}_1}^R x_1 + c_{\tilde{a}_2}^R x_2\,, \tag{32}$$

$$m_{\tilde{y}} = m_{\tilde{a}_0} + m_{\tilde{a}_1 x_1} + m_{\tilde{a}_2 x_2}\,. \tag{33}$$

Then, for a set of observed values of x_1, x_2 and y, Eq. (30) can be derived as

$$\begin{aligned} &\min_{\tilde{a}_0 \tilde{a}_1 \tilde{a}_2} \sum d(\tilde{a}_0 + \tilde{a}_1 x_1 + \tilde{a}_2 x_2, y)^2 \\ &= \min_{\substack{c_{\tilde{a}_0}^L, c_{\tilde{a}_1}^L, c_{\tilde{a}_2}^L \\ c_{\tilde{a}_0}^R, c_{\tilde{a}_1}^R, c_{\tilde{a}_2}^R \\ m_{\tilde{a}_0}, m_{\tilde{a}_1}, m_{\tilde{a}_2}}} \sum_{i=1}^{n} \left(\begin{aligned} &(m_{\tilde{a}_0} + m_{\tilde{a}_1} x_1^i + m_{\tilde{a}_2} x_2^i - y_i)^2 + ((m_{\tilde{a}_0} + m_{\tilde{a}_1} x_1^i + m_{\tilde{a}_2} x_2^i - c_{\tilde{a}_0}^L - c_{\tilde{a}_1}^L x_1^i - c_{\tilde{a}_2}^L x_2^i) - y_i)^2 \\ &+ ((m_{\tilde{a}_0} + m_{\tilde{a}_1} x_1^i + m_{\tilde{a}_2} x_2^i + c_{\tilde{a}_0}^L + c_{\tilde{a}_1}^L x_1^i + c_{\tilde{a}_2}^L x_2^i) - y_i)^2 \end{aligned} \right). \end{aligned} \tag{34}$$

By solving Eq. (34), the values of $c_{\tilde{a}_0}^L, c_{\tilde{a}_1}^L, c_{\tilde{a}_2}^L, c_{\tilde{a}_0}^R, c_{\tilde{a}_1}^R, c_{\tilde{a}_2}^R, m_{\tilde{a}_0}, m_{\tilde{a}_1}, m_{\tilde{a}_2}$ are estimated and thus the fuzzy coefficients of the linear equation can be finally determined. However, the procedures in calculating the parameter values may require a robust optimization approach.

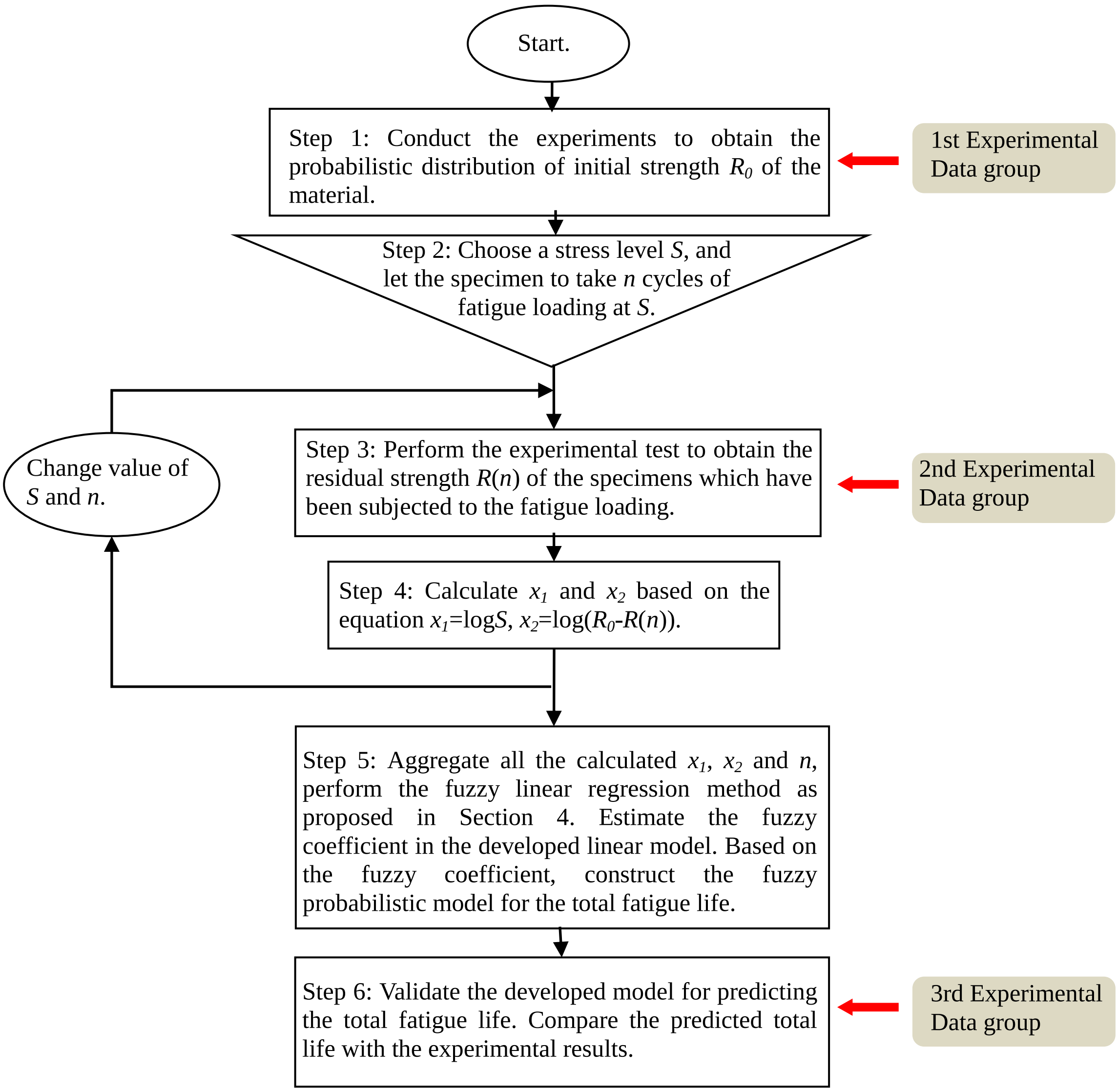

Fig. 2 Schematic diagram of the validation process

Detailed calculation steps in calculating the parameter values in Eq. (34) are provided in Appendix A. However, we should point out that the membership functions of fuzzy set should be determined empirically. The membership functions must be determined based on the continuous correction through practice and feedback. Most often, the membership functions are constructed based on statistical data or expert knowledge. Obviously, the accuracy of fuzzy fatigue life prediction depends upon the selection of membership function. For the actual analysis, whether the proposed model can accurately predict the fatigue life or not will be evaluated and compared with experimental data. These are further discussed in the later part of this paper. Nevertheless, the current work aims to demonstrate the usage of fuzzy set in the fatigue life predictions. The concepts and idea can be generalized to any types of fuzzy models.

To demonstrate the applicability of the proposed fuzzy approach, a validation study which compares the experimental results and predicted model value will be provided in the following section.

5. Model validation with experimental results

In this work, we utilize a set of experimental values measured in Zuo *et al*. (2014) for the model validation study. The tested material is 45 steels. There are three groups of measurements for the specimens. The first group of specimens were tested statically to obtain the distribution of the initial strength (R_0). The second group of specimens were subjected at various maximum stress levels (S) with certain number of cycles (n). Then the residual strengths ($R(n)$) were measured. The third group of specimens were used for validation purpose. These specimens are subjected to different stress levels to obtain the total fatigue life cycles (N). The detailed steps of the validation process are shown in Fig. 2. This is very similar with the design process in offshore engineering (Zhang *et al*. 2015).

Based on the first tested data group, the Weibull model is constructed for the initial strength R_0. Maximum likelihood method is used to estimate the statistical parameters in this work. Fig. 3 shows the empirical CDF of R_0 and the fitted Weibull model. A p-value less than 0.05 in the KS test indicates the adopted Weibull model is a valid option.

In establishing the fuzzy fatigue model, the second group of tested residual strength data are been used. To account for the measurement errors, the observed data set are assumed to have a 5% errors (e.g., the ratio of spread/mode for $\widetilde{x_1}$, $\widetilde{x_2}$, and $\tilde{y}$ equals to 0.05). Based on the proposed method and the derived parameter estimation methods (as provided in Section 4 and Appendix A), the modes and spreads of the fuzzy coefficients in Eq. (27) are computed. Figure 4 plots the results of $\widetilde{a_0}$, $\widetilde{a_1}$ and $\widetilde{a_2}$ with membership values from 0 to 1. It is now easy to see how the model parameter values vary with respect to the change of data imprecision. The ratios of the spread over mode are quite close to 5% for all the coefficients. This told us the uncertainties associated with

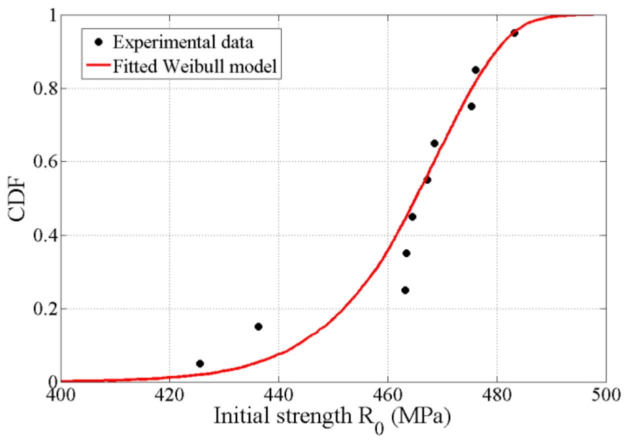

Fig. 3 Fitted Weibull model for the initial strength R_0

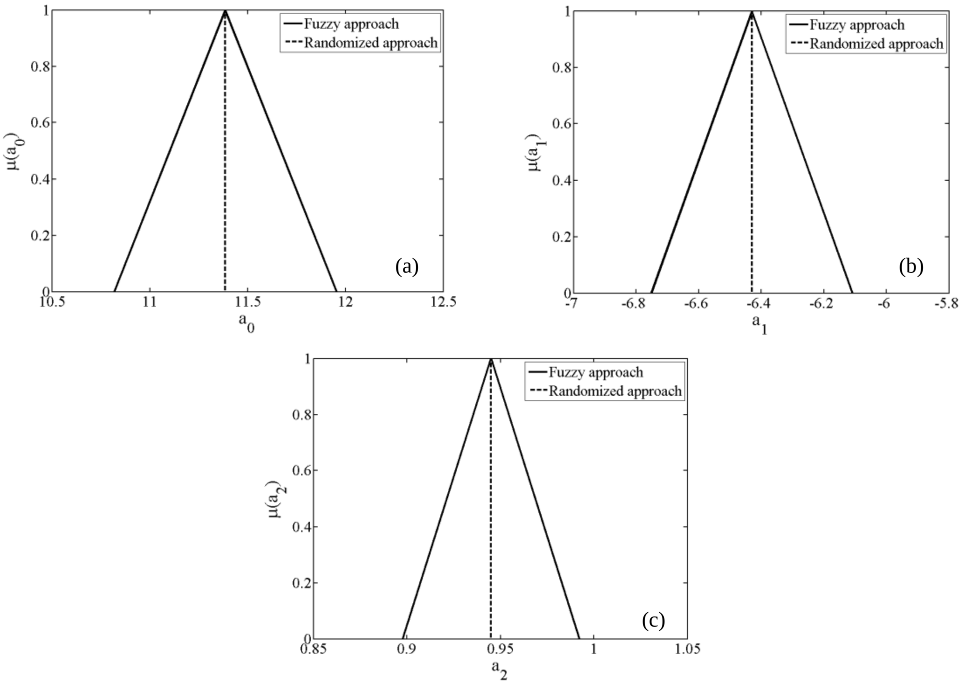

Fig. 4 Estimated model coefficients ((a) a_0; (b) a_1 and (c) a_2) in the linear fatigue life prediction model

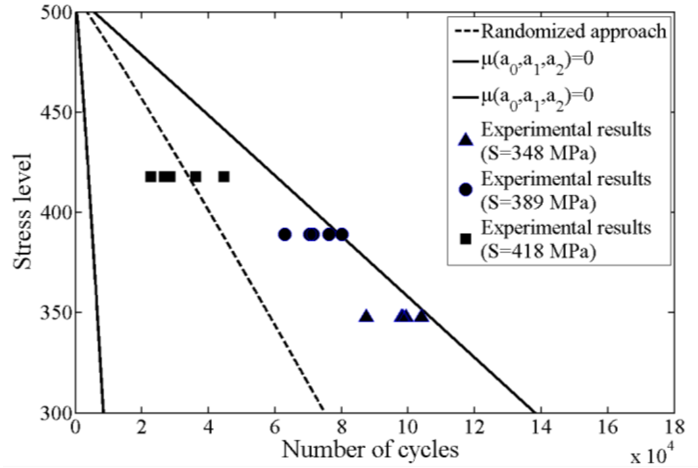

Fig. 5 Comparison of the proposed approach to the randomized approach and experimental results

the observed data are directly transferred to each of the coefficients. In order to have a better understanding of the fuzzy approach, the coefficient values are also determined by the randomized approach which was derived in Section 3. The results are also plotted in Fig. 4 (dotted lines). It can be observed from Fig. 4 that the estimated parameter value from the randomized approach is quite close to the mode values calculated from the fuzzy approach. This implies when the data imprecision (e.g., the fuzziness of y) is minimal the fuzzy approach can converge to the randomized approach. However, realizing the knowledge can never be perfectly known in real practices, the fuzzy coefficients possess more practical meanings. Normally, the case in which the membership value is quite close to 1.0 can rarely occur. In fact, for most cases, the membership value lies in the range from 0.2 to 0.7 (Zhang and Cao 2015). The interpretation of the results are more important and reasonable for the feasible range of membership values for the coefficients.

By using the estimated fuzzy coefficients, the prediction of the total fatigue life under a fixed stress level can be calculated from Eq. (13). This produces a fuzzy result for the predicted total fatigue life. To validate the proposed fuzzy approach, the results calculated from the fuzzy model are compared to three experimental data sets. These include specimens been subjected to three individual stress levels which are 348 MPa, 389 MPa and 418 MPa. The results of the total fatigue life (total loading cycles to failure) include the experimental tests and predicted values from fuzzy and randomized approaches which are computed and illustrated in Fig. 5.

As shown in the figure, the experimental values deviate quite a lot with each other (large spreads). For example, the spread of the experimental results is around $4{\cdot}10^4$ cycles for different loading intensity. Therefore, it should be realized that using only one set of these experimental data in determining the total fatigue life is quite bias and uncertain. Moreover, it should be pointed out the sample size of the measured data is quite small which may create large difficulties for the probabilistic approach (large statistical errors in the parameter estimations). This is particularly true and common in most material strength analysis since conducting fatigue test is usually quite expensive and takes a long time. However, the fuzzy model does not need to account these drawbacks. One could see that the tested fatigue life based on experiments are within the interval range of the predicted fuzzy result for membership value equal to 0. Moreover, the fuzzy membership values expresses a degree of uncertainty associated with the observed data set. For example, the membership function value can be used to represent a degree of data scarceness. The use of the fuzzy model is therefore more conservative and realistic as it contains the consideration of the dubious errors. From the plot in Fig. 5, it is clearly to see the randomized approach gives smaller values in the predictions for the fatigue life compared to the experimental tests. However, when the stress level rises (e.g., S=418 MPa), the predicted fatigue life from the randomized approach is quite close to the experimental results. The major reason is the data set been used in constructing the randomized approach (the second group of data) are based on specimens subjected to high stress levels (around 400 MPa). Therefore, the determined coefficients in the fatigue model can reflect the material behavior more realistically within the high stress level ranges. For lower range of stress levels, the prediction would become less reliable. This reveals a fact that the quality of the constructed fatigue life model based on randomized approach largely relies on the tested data set properties. Thus, the applicability of randomized approach may be constrained within certain stress level value due to the limited experiment data sample. Besides, the assumption of random process may not be appropriate (e.g., $\boldsymbol{X}(n)$ in Eq. (22) may not be log-normal). Such inappropriate assumptions in the randomized approach may introduce some subjective uncertainties through the modeling of errors. Thus, the randomized approach should be used with caution.

A further extension from the current approach is to establish the CDF of the total fatigue life N. According to Eq. (14), the CDF can be constructed based on the estimated coefficients from the linear fatigue model. However, since certain parameters are modeled as fuzzy numbers, the imprecision in the probabilistic model is realized through the fuzzy set. As such, a fuzzy set of probabilistic models can be regarded as a set of probabilistic models, namely imprecise probabilities. For instance, Fig. 6 plots out the developed imprecise probability model for N while

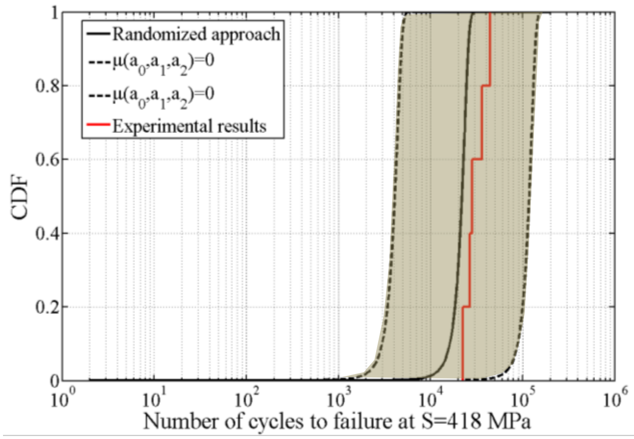

Fig. 6 Comparison of predicted model for the number of cycles to failure at stress level S=418 MPa

the model parameters C, m and b are modeled as fuzzy numbers which are determined from the above studies. For comparison purpose, the CDF based on the randomized approach is also included and compared with the experimental results. It would be easy to see that the probability of the total fatigue life N is bounded by the upper and lower probabilities in the imprecise probability model (e.g., the shaded area in Fig. 5). Compared to the randomized approach, the fuzzy based imprecise model is more robust in identifying the bounds of errors that could made in the predictions. For example, the bounds of the predicted cycles are ranging from 1000 cycles to 100000 cycles. This reflects the influence of the coefficient uncertainties (a_0, a_1 and a_2) to the finally predicted life cycles. The reason for having such a wide interval is because of the high conservatism in the modeling of uncertainty for the input parameters (e.g., the estimated coefficients in Fig. 4 are quite large).

A very good feature of the constructed imprecise probability is the identification of bounds on probabilities for the total fatigue life; the uncertainty of the predicted total fatigue life is characterized with two measure values-an upper probability and a lower probability. The width between the probability bounds reflects the indeterminacy in the fuzzy linear regression model. As discussed in Section 4, this imprecision is the concession for not using the probabilistic models. That is, the expert knowledge or data availability are often too limited, which is too difficult to use a crisp value in the model. However, this problem can be circumvented by implementing the imprecise probabilities which can provide set-valued descriptions in the probabilistic model. Although the imprecise probability introduces less information compared with a specific subjective distribution function, it reflects in the result in a form of a set of probabilities which contain the true probability. It is particular important when the calculated probabilities are been used for making critical decisions. The engineering analysis based on the imprecise probabilities can help to obtain a set of relevant results and associated decisions. Therefore, it reduces the errors in making engineering decisions due to artificial restrictions in the modeling.

More specifically, in reliability engineering, such fuzzy probabilistic analysis can be utilized to identify sensitivities of the failure probability with respect to the imprecision in the probabilistic model specification. Sensitivities of failure probability can be represented when the interval size of calculated failure probability grows strongly with a small increase in the interval size of the input parameters. If this is the case, we have to pay particular attention to the interval sizes in the input,

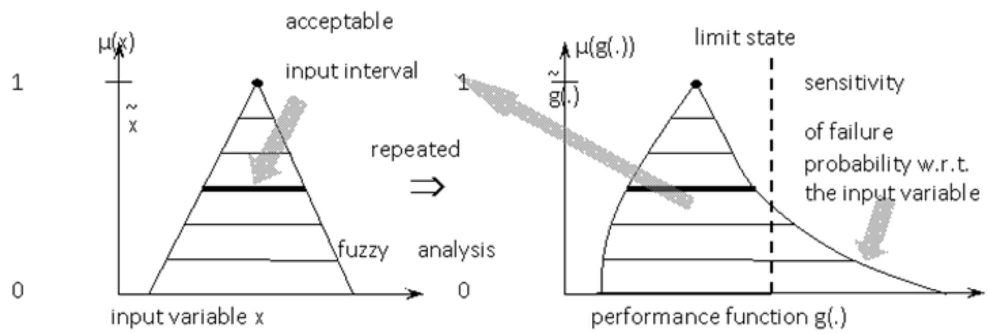

Fig. 7 Fuzzy analysis as a design process (Beer *et al*. 2013)

which may cause large intervals of failure probability. Then, a further investigation can be conducted to verify the reasoning for having such large interval sizes. And certain techniques can be performed to possibly exclude these critical cases. Such fuzzy probabilistic analysis can also provide interesting features for decision makings in the design process. The analysis can be performed with coarse specifications (e.g., large intervals for the input) for design parameters. From the results of the fuzzy analysis, acceptable intervals for the design parameters can be determined directly from the failure probability calculations without a repetition of the fuzzy analysis. Indications are provided in a quantitative manner for people to decide whether collecting additional specific information or applying certain design measures to reduce the input imprecision to an acceptable magnitude. A detailed procedure of conducting such fuzzy based design process is shown in Fig. 7.

The developed model regarding the evaluation of total fatigue life may not be generalized to all the other materials. For example, if the material contains large defects or minor cracks, the fuzzy based fatigue model may not lead to accurate predictions. Further advanced fatigue model is required to handle more complicated problems. However, the proposed method provides a general basis to assess the fatigue problem for any newly produced simple materials with respect to the considered model uncertainty. The influence of the fatigue model uncertainty to the total life prediction can be interpreted in a comprehensive way. This provides a deeper view for the engineering design works to decide whether a re-analysis is necessary (Beer *et al*. 2013).

While this work endeavor to develop a fuzzy approach for accurate modeling of parameter uncertainties for the prediction of fatigue life, many simplifications were made in the analysis. The environment conditions, including the temperature and moisture condition, for the steel elements are not considered. The current analysis only focus on the loading intensity and material parameters and the complex behavior of steel elements under different environment will not be analyzed in detail. The fatigue strength function is simplified to follow the *S-N* curve in the fuzzy analysis. With regards to the cumulative damages due to different stress levels, only linear damage accumulation law is considered. Besides, the separation of inherent, model and statistical uncertainties are not addressed in this study. Rather, the key focus is on quantification of total uncertainties in the prediction of fatigue life by using fuzzy concepts. The investigated fuzzy model in this work is only based on the collected data which is independent of the structural types. Therefore, other types of steel elements may need a further analysis. The conclusions drawn from this study should be seen in the light of these limitations. The influence of these limitations to the reliability results may need further investigations in the future.

6. Conclusions

In this paper, fuzzy theory is utilized in the regression analysis for the fatigue life prediction. The uncertainties associated with the parameters in the fatigue *S-N* curve are modeled by the triangular fuzzy numbers. The least square method is utilized to estimate the parameters in the fuzzy regression analysis. Detailed calculation steps in determining the fuzzy numbers are derived and discussed in the paper. Furthermore, the applicability of this technique is demonstrated through a validation study which is based on an comparison between the predicted values and experimental values. The randomized approach is also compared with the proposed approach in predicting both the fatigue damage and fatigue life. It is shown that the developed approach can provide a comprehensive understanding of the influence of model uncertainties in the fatigue life predictions. Compared to the traditional deterministic approaches, the fuzzy approach is also proven to be more robust and conservative in evaluating the fatigue life of materials. The fuzzy model has the flexibility to allow the consideration of different degrees of indeterminacy of the model information. This provides the opportunity to meet at optimal engineering decisions where a level of accuracy is required.

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Appendix A. Detailed parameter estimation steps for fuzzy linear equation

Finding the estimate of parameters $c^L_{\widetilde{a_0}}, c^L_{\widetilde{a_1}}, c^L_{\widetilde{a_2}}, c^R_{\widetilde{a_0}}, c^R_{\widetilde{a_1}}, c^R_{\widetilde{a_2}}, m_{\widetilde{a_0}}, m_{\widetilde{a_1}}, m_{\widetilde{a_2}}$ for a set of observed values x_1, x_2 and y in Eq. (34) looks rather daunting. The minimization of Eq. (34) first requires taking the partial derivative of the equation with respect to each of the parameter. In that sense, the following equation set provide the detailed derivations

$$\frac{\partial d(\tilde{a}_0 + \tilde{a}_1 x_1 + \tilde{a}_2 x_2, y)^2}{\partial m_{\tilde{a}_0}}$$

$$= \sum_{i=1}^{n} \left(\begin{array}{c} 2(m_{\tilde{a}_0} + m_{\tilde{a}_1} x_1^i + m_{\tilde{a}_2} x_2^i - y_i) + 2((m_{\tilde{a}_0} + m_{\tilde{a}_1} x_1^i + m_{\tilde{a}_2} x_2^i - c^L_{\tilde{a}_0} - c^L_{\tilde{a}_1} x_1^i - c^L_{\tilde{a}_2} x_2^i) - y_i) \\ + 2((m_{\tilde{a}_0} + m_{\tilde{a}_1} x_1^i + m_{\tilde{a}_2} x_2^i + c^L_{\tilde{a}_0} + c^L_{\tilde{a}_1} x_1^i + c^L_{\tilde{a}_2} x_2^i) - y_i \end{array} \right)$$

$$= 6n m_{\tilde{a}_0} + 6m_{\tilde{a}_1} \sum_{i=1}^{n} x_1^i + 6m_{\tilde{a}_2} \sum_{i=1}^{n} x_2^i - 2nc^L_{\tilde{a}_0} - 2c^L_{\tilde{a}_1} \sum_{i=1}^{n} x_1^i - 2c^L_{\tilde{a}_2} \sum_{i=1}^{n} x2_1^i + 2nc^R_{\tilde{a}_0} + 2c^R_{\tilde{a}_1} \sum_{i=1}^{n} x_1^i$$

$$+ 2c^R_{\tilde{a}_2} \sum_{i=1}^{n} x_2^i - 6\sum_{i=1}^{n} y_i \tag{A.1}$$

$$\frac{\partial d(\tilde{a}_0 + \tilde{a}_1 x_1 + \tilde{a}_2 x_2, y)^2}{\partial m\tilde{a}_1}$$

$$= \sum_{i=1}^{n} \left(\begin{array}{l} 2(m_{\tilde{a}_0} + m_{\tilde{a}_1} x_1^i + m_{\tilde{a}_2} x_2^i - y_i) x_1^i + 2((m_{\tilde{a}_0} + m_{\tilde{a}_1} x_1^i + m_{\tilde{a}_2} x_2^i - c^L_{\tilde{a}_0} - c^L_{\tilde{a}_1} x_1^i - c^L_{\tilde{a}_2} x_2^i) - y_i) x_1^i \\ + 2((m_{\tilde{a}_0} + m_{\tilde{a}_1} x_1^i + m_{\tilde{a}_2} x_2^i + c^L_{\tilde{a}_0} + c^L_{\tilde{a}_1} x_1^i + c^L_{\tilde{a}_2} x_2^i) - y_i) x_1^i \end{array} \right)$$

$$= 6m_{\tilde{a}_0} \sum_{i=1}^{n} x_1^i + 6m_{\tilde{a}_1} \sum_{i=1}^{n} x_1^i \cdot x_1^i + 6m_{\tilde{a}_2} \sum_{i=1}^{n} x_1^i \cdot x_2^i - 2c^L_{\tilde{a}_0} \sum_{i=1}^{n} x_1^i - 2c^L_{\tilde{a}_1} \sum_{i=1}^{n} x_1^i \cdot x_1^i - 2c^L_{\tilde{a}_2} \sum_{i=1}^{n} x_1^i \cdot x_2^i$$

$$+ 2c^R_{\tilde{a}_0} \sum_{i=1}^{n} x_1^i + 2c^R_{\tilde{a}_1} \sum_{i=1}^{n} x_1^i \cdot x_1^i + 2c^R_{\tilde{a}_2} \sum_{i=1}^{n} x_1^i \cdot x_2^i - 6\sum_{i=1}^{n} x_1^i \cdot y_i \tag{A.2}$$

$$\frac{\partial d(\tilde{a}_0 + \tilde{a}_1 x_1 + \tilde{a}_2 x_2, y)^2}{\partial m\tilde{a}_2}$$

$$= \sum_{i=1}^{n} \left(\begin{array}{l} 2(m_{\tilde{a}_0} + m_{\tilde{a}_1} x_1^i + m_{\tilde{a}_2} x_2^i - y_i) x_2^i + 2((m_{\tilde{a}_0} + m_{\tilde{a}_1} x_1^i + m_{\tilde{a}_2} x_2^i - c^L_{\tilde{a}_0} - c^L_{\tilde{a}_1} x_1^i - c^L_{\tilde{a}_2} x_2^i) - y_i) x_2^i \\ + 2((m_{\tilde{a}_0} + m_{\tilde{a}_1} x_1^i + m_{\tilde{a}_2} x_2^i + c^L_{\tilde{a}_0} + c^L_{\tilde{a}_1} x_1^i + c^L_{\tilde{a}_2} x_2^i) - y_i) x_2^i \end{array} \right)$$

$$= 6m_{\tilde{a}_0} \sum_{i=1}^{n} x_2^i + 6m_{\tilde{a}_1} \sum_{i=1}^{n} x_1^i \cdot x_2^i + 6m_{\tilde{a}_2} \sum_{i=1}^{n} x_2^i \cdot x_2^i - 2c^L_{\tilde{a}_0} \sum_{i=1}^{n} x_2^i - 2c^L_{\tilde{a}_1} \sum_{i=1}^{n} x_1^i \cdot x_2^i - 2c^L_{\tilde{a}_2} \sum_{i=1}^{n} x_2^i \cdot x_2^i$$

$$+ 2c^R_{\tilde{a}_0} \sum_{i=1}^{n} x_2^i + 2c^R_{\tilde{a}_1} \sum_{i=1}^{n} x_1^i \cdot x_2^i + 2c^R_{\tilde{a}_2} \sum_{i=1}^{n} x_2^i \cdot x_2^i - 6\sum_{i=1}^{n} x_2^i \cdot y_i \tag{A.3}$$

$$\frac{\partial d(\tilde{a}_0+\tilde{a}_1x_1+\tilde{a}_2x_2,y)^2}{\partial c^L_{\tilde{a}_0}}$$

$$=\sum_{i=1}^{n}\Big(-2(m_{\tilde{a}_0}+m_{\tilde{a}_1}x_1^i+m_{\tilde{a}_2}x_2^i-c^L_{\tilde{a}_0}-c^L_{\tilde{a}_1}x_1^i-c^L_{\tilde{a}_2}x_2^i-y_i)\Big)$$

$$=-2nm_{\tilde{a}_0}-2m_{\tilde{a}_1}\sum_{i=1}^{n}x_1^i-2m_{\tilde{a}_2}\sum_{i=1}^{n}x_2^i+2nc^L_{\tilde{a}_0}+2c^L_{\tilde{a}_1}\sum_{i=1}^{n}x_1^i+2c^L_{\tilde{a}_2}\sum_{i=1}^{n}x_2^i+2\sum_{i=1}^{n}y_i \quad \text{(A.4)}$$

$$\frac{\partial d(\tilde{a}_0+\tilde{a}_1x_1+\tilde{a}_2x_2,y)^2}{\partial c^L_{\tilde{a}_1}}$$

$$=\sum_{i=1}^{n}\Big(-2x_1^i(m_{\tilde{a}_0}+m_{\tilde{a}_1}x_1^i+m_{\tilde{a}_2}x_2^i-c^L_{\tilde{a}_0}-c^L_{\tilde{a}_1}x_1^i-c^L_{\tilde{a}_2}x_2^i-y_i)\Big)$$

$$=-2m_{\tilde{a}_0}\sum_{i=1}^{n}x_1^i-2m_{\tilde{a}_1}\sum_{i=1}^{n}x_1^i\cdot x_1^i-2m_{\tilde{a}_2}\sum_{i=1}^{n}x_1^i\cdot x_2^i+2c^L_{\tilde{a}_0}\sum_{i=1}^{n}x_1^i+2c^L_{\tilde{a}_1}\sum_{i=1}^{n}x_1^i+2c^L_{\tilde{a}_1}\sum_{i=1}^{n}x_1^i\cdot x_1^i$$

$$+2c^R_{\tilde{a}_2}\sum_{i=1}^{n}x_1^i\cdot x_2^i+2\sum_{i=1}^{n}x_1^i\cdot y_i \quad \text{(A.5)}$$

$$\frac{\partial d(\tilde{a}_0+\tilde{a}_1x_1+\tilde{a}_2x_2,y)^2}{\partial c^L_{\tilde{a}_2}}$$

$$=\sum_{i=1}^{n}\Big(-2x_2^i(m_{\tilde{a}_0}+m_{\tilde{a}_1}x_1^i+m_{\tilde{a}_2}x_2^i-c^L_{\tilde{a}_0}-c^L_{\tilde{a}_1}x_1^i-c^L_{\tilde{a}_2}x_2^i-y_i)\Big)$$

$$=\sum_{i=1}^{n}\Big(-2m_{\tilde{a}_0}x_2^i-2m_{\tilde{a}_1}x_1^ix_2^i-2m_{\tilde{a}_2}x_2^ix_2^i+2c^L_{\tilde{a}_0}x_2^L+2c^L_{\tilde{a}_1}x_1^Lx_2^L+2c^L_{\tilde{a}_2}x_2^Lx_2^L+2x_2^iy_i\Big)$$

$$=-2m_{\tilde{a}_0}\sum_{i=1}^{n}x_2^i-2m_{\tilde{a}_1}\sum_{i=1}^{n}x_1^ix_2^i-2m_{\tilde{a}_2}\sum_{i=1}^{n}x_2^ix_2^i+2c^L_{\tilde{a}_0}\sum_{i=1}^{n}x_2^i+2c^L_{\tilde{a}_1}\sum_{i=1}^{n}x_1^ix_2^i+2c^L_{\tilde{a}_2}\sum_{i=1}^{n}x_2^ix_2^i+2\sum_{i=1}^{n}x_2^iy_i$$

$$\text{(A.6)}$$

$$\frac{\partial d(\tilde{a}_0+\tilde{a}_1x_1+\tilde{a}_2x_2,y)^2}{\partial c^L_{\tilde{a}_0}}$$

$$=\sum_{i=1}^{n}\Big(2(m_{\tilde{a}_0}+m_{\tilde{a}_1}x_1^i+m_{\tilde{a}_2}x_2^i+c^L_{\tilde{a}_0}+c^L_{\tilde{a}_1}x_1^i+c^L_{\tilde{a}_2}x_2^i-y_i)\Big)$$

$$=-2nm_{\tilde{a}_0}+2m_{\tilde{a}_1}\sum_{i=1}^{n}x_1^i+2m_{\tilde{a}_2}\sum_{i=1}^{n}x_2^i+2nc^R_{\tilde{a}_0}+2c^R_{\tilde{a}_1}\sum_{i=1}^{n}x_1^i+2c^R_{\tilde{a}_2}\sum_{i=1}^{n}x_2^i-2\sum_{i=1}^{n}y_i \quad \text{(A.7)}$$

$$
\begin{aligned}
&\frac{\partial d(\tilde{a}_0 + \tilde{a}_1 x_1 + \tilde{a}_2 x_2, y)^2}{\partial c^L_{\tilde{a}_1}} \\
&= \sum_{i=1}^{n} \left(2x_1^i (m_{\tilde{a}_0} + m_{\tilde{a}_1} x_1^i + m_{\tilde{a}_2} x_2^i + c^R_{\tilde{a}_0} + c^R_{\tilde{a}_1} x_1^i + c^R_{\tilde{a}_2} x_2^i - y_i) \right) \\
&= 2m_{\tilde{a}_0} \sum_{i=1}^{n} x_1^i + 2m_{\tilde{a}_1} \sum_{i=1}^{n} x_1^i x_1^i + 2m_{\tilde{a}_2} \sum_{i=1}^{n} x_1^i x_2^i + 2c^R_{\tilde{a}_0} \sum_{i=1}^{n} x_1^i + 2c^R_{\tilde{a}_1} \sum_{i=1}^{n} x_1^i x_1^i + 2c^R_{\tilde{a}_2} \sum_{i=1}^{n} x_1^i x_2^i - 2\sum_{i=1}^{n} x_1^i y_i
\end{aligned} \tag{A.8}
$$

$$
\begin{aligned}
&\frac{\partial d(\tilde{a}_0 + \tilde{a}_1 x_1 + \tilde{a}_2 x_2, y)^2}{\partial c^L_{\tilde{a}_2}} \\
&= \sum_{i=1}^{n} \left(2x_2^i (m_{\tilde{a}_0} + m_{\tilde{a}_1} x_1^i + m_{\tilde{a}_2} x_2^i + c^R_{\tilde{a}_0} + c^R_{\tilde{a}_1} x_1^i + c^R_{\tilde{a}_2} x_2^i - y_i) \right) \\
&= 2m_{\tilde{a}_0} \sum_{i=1}^{n} x_2^i + 2m_{\tilde{a}_1} \sum_{i=1}^{n} x_1^i x_2^i + 2m_{\tilde{a}_2} \sum_{i=1}^{n} x_2^i x_2^i + 2c^R_{\tilde{a}_0} \sum_{i=1}^{n} x_2^i + 2c^R_{\tilde{a}_1} \sum_{i=1}^{n} x_1^i x_2^i + 2c^R_{\tilde{a}_2} \sum_{i=1}^{n} x_2^i x_2^i - 2\sum_{i=1}^{n} x_2^i y_i
\end{aligned} \tag{A.9}
$$

To complete the minimization procedure for the target function, the equation is set to zero and solved for each of the parameter. Since Eqns. (A.1)-(A.9) are all linear equations, we can implement a matrix to solve the system of linear equations. Therefore, the parameter values can be estimated by solving the following linear system

$$
\begin{bmatrix}
6n & 6\sum_{i=1}^{n} x_1^i & 6\sum_{i=1}^{n} x_2^i & -2n & -2\sum_{i=1}^{n} x_1^i & -2\sum_{i=1}^{n} x_2^i & 2n & 2\sum_{i=1}^{n} x_1^i & 2\sum_{i=1}^{n} x_2^i \\
6\sum_{i=1}^{n} x_1^i & 6\sum_{i=1}^{n} x_1^i \cdot x_1^i & 6\sum_{i=1}^{n} x_1^i \cdot x_2^i & -2\sum_{i=1}^{n} x_1^i & -2\sum_{i=1}^{n} x_1^i \cdot x_1^i & -2\sum_{i=1}^{n} x_1^i \cdot x_2^i & 2\sum_{i=1}^{n} x_1^i & 2\sum_{i=1}^{n} x_1^i \cdot x_1^i & 2\sum_{i=1}^{n} x_1^i \cdot x_2^i \\
6\sum_{i=1}^{n} x_2^i & 6\sum_{i=1}^{n} x_1^i \cdot x_2^i & 6\sum_{i=1}^{n} x_2^i \cdot x_2^i & -2\sum_{i=1}^{n} x_2^i & -2\sum_{i=1}^{n} x_1^i \cdot x_2^i & -2\sum_{i=1}^{n} x_2^i \cdot x_2^i & 2\sum_{i=1}^{n} x_2^i & 2\sum_{i=1}^{n} x_1^i \cdot x_2^i & 2\sum_{i=1}^{n} x_2^i \cdot x_2^i \\
-2n & -2\sum_{i=1}^{n} x_1^i & -2\sum_{i=1}^{n} x_2^i & 2n & 2\sum_{i=1}^{n} x_1^i & 2\sum_{i=1}^{n} x_2^i & 0 & 0 & 0 \\
-2\sum_{i=1}^{n} x_1^i & -2\sum_{i=1}^{n} x_1^i \cdot x_1^i & -2\sum_{i=1}^{n} x_1^i \cdot x_2^i & 2\sum_{i=1}^{n} x_1^i & 2\sum_{i=1}^{n} x_1^i \cdot x_1^i & 2\sum_{i=1}^{n} x_1^i \cdot x_2^i & 0 & 0 & 0 \\
-2\sum_{i=1}^{n} x_2^i & -2\sum_{i=1}^{n} x_1^i x_2^i & -2\sum_{i=1}^{n} x_2^i x_2^i & 2\sum_{i=1}^{n} x_2^i & 2\sum_{i=1}^{n} x_1^i x_2^i & 2\sum_{i=1}^{n} x_2^i x_2^i & 0 & 0 & 0 \\
2n & 2\sum_{i=1}^{n} x_1^i & 2\sum_{i=1}^{n} x_2^i & 0 & 0 & 0 & 2n & 2\sum_{i=1}^{n} x_1^i & 2\sum_{i=1}^{n} x_2^i \\
2\sum_{i=1}^{n} x_1^i & 2\sum_{i=1}^{n} x_1^i x_1^i & 2\sum_{i=1}^{n} x_1^i x_2^i & 0 & 0 & 0 & 2\sum_{i=1}^{n} x_1^i & 2\sum_{i=1}^{n} x_1^i x_1^i & 2\sum_{i=1}^{n} x_1^i x_2^i \\
2\sum_{i=1}^{n} x_2^i & 2\sum_{i=1}^{n} x_1^i x_2^i & 2\sum_{i=1}^{n} x_2^i x_2^i & 0 & 0 & 0 & 2\sum_{i=1}^{n} x_2^i & 2\sum_{i=1}^{n} x_1^i x_2^i & 2\sum_{i=1}^{n} x_2^i x_2^i
\end{bmatrix}
\begin{bmatrix}
m_{\tilde{a}_0} \\ m_{\tilde{a}_1} \\ m_{\tilde{a}_2} \\ c^L_{\tilde{a}_0} \\ c^L_{\tilde{a}_1} \\ c^L_{\tilde{a}_2} \\ c^R_{\tilde{a}_0} \\ c^R_{\tilde{a}_1} \\ c^R_{\tilde{a}_2}
\end{bmatrix}
=
\begin{bmatrix}
6\sum_{i=1}^{n} y_i \\ 6\sum_{i=1}^{n} x_1^i \cdot y_i \\ 6\sum_{i=1}^{n} x_2^i \cdot y_i \\ -2\sum_{i=1}^{n} y_i \\ -2\sum_{i=1}^{n} x_1^i \cdot y_i \\ -2\sum_{i=1}^{n} x_2^i y_i \\ 2\sum_{i=1}^{n} y_i \\ 2\sum_{i=1}^{n} x_1^i y_i \\ 2\sum_{i=1}^{n} x_2^i y_i
\end{bmatrix}. \tag{A.10}
$$