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New enhanced higher order free vibration analysis of thick truncated conical sandwich shells with flexible cores

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Abstract. This paper dealt the free vibration analysis of thick truncated conical composite sandwich shells with transversely flexible cores and simply supported boundary conditions based on a new improved and enhanced higher order sandwich shell theory. Geometries were used in the present work for the consideration of different radii curvatures of the face sheets and the core was unique. The coupled governing partial differential equations were derived by the Hamilton's principle. The in-plane circumferential and axial stresses of the core were considered in the new enhanced model. The first order shear deformation theory was used for the inner and outer composite face sheets and for the core, a polynomial description of the displacement fields was assumed based on the second Frostig's model. The effects of types of boundary conditions, conical angles, length to radius ratio, core to shell thickness ratio and core radius to shell thickness ratio on the free vibration analysis of truncated conical composite sandwich shells were also studied. Numerical results are presented and compared with the latest results found in literature. Also, the results were validated with those derived by ABAQUS FE code.

Keywords: free vibration; truncated conical sandwich shells; improved higher order theory

1. Introduction

The conical shells are often used as transition elements between cylinders of different diameters and/or end closures and sometimes as stand-alone components in various engineering applications such as tanks and pressure vessels, missiles and spacecraft, submarines, nuclear reactors, jet nozzles and such other civil, chemical, mechanical, marine and aerospace engineering structures (Sofiyev 2011). Sandwich structures due to their high strength and stiffness, low weight and durability, are widely used in many engineering applications. These structures are generally consisting of two stiff face sheets and a soft core, which are bonded together. In most cases, the core is consisting of a thick foam polymer or honeycomb material, while thin composite laminates are commonly used as the face sheets. In these structures, the core keeps the face sheets at sufficient distance and transmits the transverse normal and shear loads. Advantages of this construction method are used to obtain the plates with high bending stiffness characteristics and an

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extremely low weight. To use these structures efficiently, an excellent understanding of their mechanical behavior is needed. Extensive research has been dedicated to the free vibration behavior of circular composite laminates (For example see (Tong 1994, Shu 1996, Wu and Lee 2001)). Using first order shear deformation theory, Tong (1994) studied free vibrations of laminated composite conical shells. Shu (1996) used Love’s first approximation thin shell theory in order to free vibration analysis of composite laminated conical shells. The first-order shear deformation shell theory was used by Wu and Lee (2001) to study free vibration analysis of laminated conical shells. They assumed the stiffness coefficients were functions of the circumferential coordinate.

The work on conical truncated composite sandwich shells with flexible cores is somewhat limited. In order to investigate free vibration analysis of sandwich structures the higher-order sandwich panel theory was developed by Frostig *et al*. (1994, 2004), who considered two types of computational models for expressing the governing equations of the core. The second Frostig’s model assumed a polynomial description of the displacement fields in the core that was based on the displacement fields of the first model. This theory does not impose any restriction on the deformation distribution through the thickness of the core. The improved higher-order sandwich plate theory (IHSAPT), applying the first-order shear deformation theory for the face sheets, was introduced by Malekzadeh *et al*. (2005). Liang *et al*. (2007) used the vibration theory and transfer matrix method to study the free vibration of a thin-walled laminated conical shell. The free vibration analysis of laminated conical and cylindrical shells was done using Love's first approximation thin shell theory by Civalek (2007). Natural frequencies and forced vibration analyses of a thin, homogeneous, and isotropic conical shell were studied using Hamilton's principle and the Rayleigh-Ritz method by Li *et al.* (2009). Sofiyev *et al.* (2009) performed the free vibration and buckling analyses of truncated conical shells with non-homogeneous material properties under uniform lateral and hydrostatic pressures. Rahmani *et al.* (2009) applied a higher order sandwich panel theory to study the free vibration analysis of an open single curved composite sandwich shell with a flexible core. They used the classical shell theory and an elasticity theory for the face sheets and the core, respectively. Tornabene (2009) conducted the free vibration analysis of functionally graded conical and cylindrical shells based on the first order shear deformation theory. The buckling analysis of non-homogeneous orthotropic truncated conical shells was done under a uniform hydrostatic pressure by Joshi and Patel (2010). Also, they investigated the effects of non-homogeneity and number of layers on the critical hydrostatic pressure. Biglari and Jafari (2010) presented a complex three layer theory for the free vibration and bending analysis of open single curved sandwich structures. In their model, Donell's theory was used for the face sheets. Using differential quadrature method, free vibrations analysis of functionally graded cylindrical panel was done by Yas *et al.* (2010). The free vibration analysis of FG conical shell was performed using meshless method and first order shear deformation shell theory by Zhao and Liew (2011). Ghannad *et al.* (2012) investigated the elastic analysis of thick functionally graded truncated conical shells and used the first order shear deformation theory and the virtual work principle. Mochida *et al.* (2012) studied the free vibration response of doubly curved shallow shells using the approximate Galerkin method. Kheirikhah *et al.* (2012) applied an improved high order theory to examine the bending analysis of soft core sandwich plates. They also used the third order plate theory for face sheets and quadratic and cubic functions for transverse and in-plane displacements of the orthotropic soft core and satisfied the continuity conditions for transverse shear stresses at the interfaces and the conditions of zero transverse shear stresses on the upper and lower surfaces. Ghavanloo and Fazelzadeh (2013) investigated the free

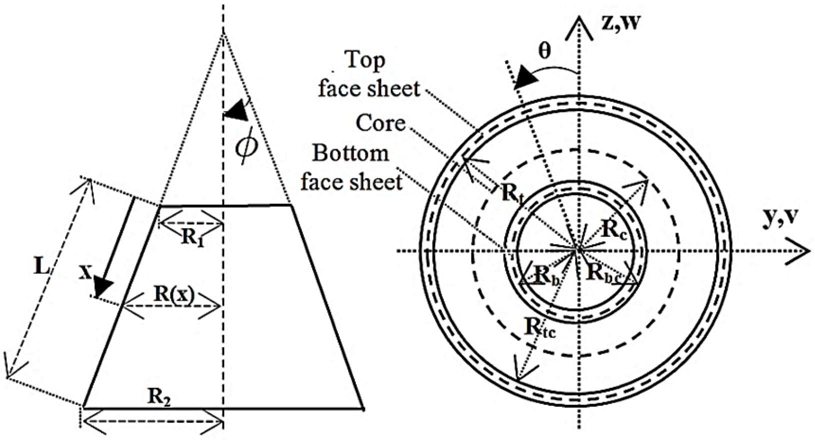


Fig. 1 Geometry parameters of the composite truncated conical sandwich shell

vibration analysis of simply supported doubly curved shallow shells. Their formulation was based on Novozhilov's linear shallow shell theory. Viola *et al.* (2013) used a 2D higher order shear deformation theory with nine parameters in order to analyze the free vibration analysis of the thick laminated doubly curved shells. Their main assumptions were based on small deflections and negligible normal stress and strain. The free vibration of variable thickness conical shell was numerically done by Viswanathan *et al.* (2013). Thickness of the layers varied linearly and exponentially in an axial direction. They considered three and five layered conical shells, which were made up of two different types of materials. Malekzadeh Fard (2014) studied the free vibration of a sandwich curved beam with a functionally graded core and used two dimensional higher order beam theory without neglecting the amount of *z/R*. Jalili *et al.* (2014) applied numerical and experimental methods for studying the buckling analysis of composite conical shells under dynamic external pressure. Also, they investigated the effect of geometrical imperfections of experimental specimens on the numerical results.

The literature survey demonstrated that most of the studies have been performed on the free vibration analysis of flat and curved composite sandwich panels using high order theory and no research is available in the field of thick sandwich truncated conical shells that use new improved high order theory. In this paper, by using a new improved higher order sandwich panel theory (Malekzadeh *et al.* 2005) and second computational Frostig's model (2004), the free vibration analysis of conical composite sandwich shells was investigated. Also, the in-plane circumferential (hoop) stresses of the core were considered. Geometries were used in the present work for the consideration of different radii curvatures of the face sheets and the core was unique. In this study, the analytical solution of the displacement field of the core in terms of the polynomials with unknown coefficients was presented according to the second computational Frostig’s model. Simply supported boundary conditions were considered in this paper. Since there are few research about the free vibration analysis of a composite truncated conical shell, to validate the results obtained in the present work, a truncated thick conical sandwich shell was modeled in ABAQUS FE code and the results obtained from analytical formulations and FE code were compared with each other. Also, for composite laminated shells, obtained results by the present method were validated by comparing them with those in the literature. Finally, the effects of various parameters including types of boundary conditions, conical angles, length to radius ratio, core to shell thickness ratio and core radius to shell thickness ratio on the free vibration response of truncated conical composite sandwich shells were studied.

2. Theoretical formulation

2.1 Basic assumptions

Consider a thick composite truncated conical sandwich shell which is composed of two composite laminated face sheets. The thickness of the top face sheet, bottom face sheet and the core is h_t, h_b and h_c, respectively, in which indices t and b refer to the top and bottom face sheets of the shell, respectively, as shown in Fig. 1. The assumption used in the present analysis was the small deformation of linearly elastic materials.

2.2 Kinematic relations

Base on the first shear deformation theory, the displacements u, v and w of the face sheets in the x, θ and z (thickness) directions with small linear displacements are expressed by following relations (Reddy 2004)

$$\begin{aligned} u_i(x,z,\theta,t) &= u_0^i(x,\theta,t) + z_i\,\psi_x^i(x,\theta,t) \\ v_i(x,z,\theta,t) &= v_0^i(x,\theta,t) + z_i\,\psi_\theta^i(x,\theta,t) \quad ; \quad (i=t,b) \\ w_i(x,z,\theta,t) &= w_0^i(x,\theta,t) \end{aligned} \tag{1}$$

where ψ_x^i and ψ_θ^i are the rotation components of the transverse normal along the x and θ-axes of mid-surface of the top and bottom face-sheets, respectively. Also u_0^i and v_0^i are displacements components in the x and θ directions, respectively and w_0^i is the vertical deflection of the top and bottom face-sheets. z_i is the vertical coordinate of the face-sheets and is measured upward from the mid-plane of each face-sheet.

The kinematic equations for the strains in the face sheets are as follows (Reddy 2003, Sofiyev *et al*. 2009, Qatu 2004)

$$\begin{aligned} &\varepsilon_{xx}^i = \varepsilon_{0xx}^i + z_i\kappa_{xx}^i,\ \varepsilon_{\theta\theta}^i = \varepsilon_{0\theta\theta}^i + z_i\kappa_{\theta\theta}^i, \varepsilon_{zz}^i = 0 \\ &\gamma_{x\theta}^i = 2\varepsilon_{x\theta}^i = \varepsilon_{0x\theta}^i + z_i\kappa_{x\theta}^i,\ \gamma_{xz}^i = 2\varepsilon_{xz}^i = \varepsilon_{0xz}^i, \gamma_{\theta z}^i = 2\varepsilon_{\theta z}^i = \varepsilon_{0\theta z}^i \quad ; \quad (i=t,b) \end{aligned} \tag{2}$$

where

$$\begin{aligned} &\varepsilon_{0xx}^i = \frac{\partial u_0^i}{\partial x}, \varepsilon_{0\theta\theta}^i = \frac{\partial v_0^i}{R_i(x)\partial\theta} + \frac{w_0^i\cos(\phi)}{R_i(x)} + \frac{u_0^i\sin(\phi)}{R_i(x)},\ \varepsilon_{0x\theta}^i = \frac{\partial v_0^i}{\partial x} + \frac{\partial u_0^i}{R_i(x)\partial\theta} - \frac{v_0^i\sin(\phi)}{R_i(x)}, \\ &\varepsilon_{0xz}^i = \frac{\partial w_0^i}{\partial x} + \psi_x^i, \varepsilon_{0\theta z}^i = \frac{\partial w_0^i}{R_i(x)\partial\theta} + \psi_\theta^i - \frac{v_0^i\cos(\phi)}{R_i(x)} \quad ; \quad (i=t,b) \\ &\kappa_{xx}^i = \frac{\partial\psi_x^i}{\partial x}, \kappa_{\theta\theta}^i = \frac{\partial\psi_\theta^i}{R_i(x)\partial\theta}, \kappa_{x\theta}^i = \frac{\partial\psi_\theta^i}{\partial x} + \frac{\partial\psi_x^i}{R_i(x)\partial\theta} - \frac{\psi_\theta^i\sin(\phi)}{R_i(x)} \end{aligned} \tag{3}$$

and

$$R_i(x) = R_{i0} + x\sin(\phi) \tag{4}$$

The displacements fields for the core are based on model II of Frostig (Frostig and Thomsen 2004) and a cubic pattern for in-plane displacements and a quadratic one for vertical ones are taken as follows

$$\begin{cases} u_c(x,\theta,z,t) = u_0^c(x,\theta,t) + z_c u_1^c(x,\theta,t) + z_c^2 u_2^c(x,\theta,t) + z_c^3 u_3^c(x,\theta,t) \\ v_c(x,\theta,z,t) = (1+\dfrac{z}{R_c(x)}) v_0^c(x,\theta,t) + z_c v_1^c(x,\theta,t) + z_c^2 v_2^c(x,\theta,t) + z_c^3 v_3^c(x,\theta,t) \\ w_c(x,\theta,z,t) = w_0^c(x,\theta,t) + z_c w_1^c(x,\theta,t) + z_c^2 w_2^c(x,\theta,t) \end{cases} \tag{5}$$

where u_k^c and v_k^c (k=0, 1, 2, 3) are the unknowns of the in-plane displacements of the core and w_k^c (k=0, 1, 2) are the unknowns of its vertical displacements. $R_c(x)$ is the radius of curvature of the core in θ-z plane that varies with x

$$R_c(x) = R_{c0} + x\sin(\phi) \tag{6}$$

The kinematic relations of the core for a conical sandwich shell based on small deformations are (Sofiyev *et al.* 2009, Qatu 2004)

$$\begin{aligned} &\varepsilon_{xx}^c = \frac{\partial u_c}{\partial x},\ \varepsilon_{\theta\theta}^c = \frac{1}{(1+z/R_c(x))}\left(\frac{\partial v_c}{a_2\partial\theta} + \frac{w_c}{R_c(x)} + \frac{u_c\partial a_2}{a_2\partial x}\right), \\ &\gamma_{x\theta}^c = 2\varepsilon_{x\theta}^c = \frac{\partial v_c}{\partial x} + \frac{1}{(1+z/R_c(x))}\left(\frac{\partial u_c}{a_2\partial\theta} - \frac{v_c\partial a_2}{a_2\partial x}\right), \gamma_{xz}^c = 2\varepsilon_{xz}^c = \frac{\partial w_c}{\partial x} + \frac{\partial u_c}{\partial z}, \\ &\gamma_{\theta z}^c = 2\varepsilon_{\theta z}^c = \frac{1}{a_2(1+z/R_c(x))}\frac{\partial w_c}{\partial\theta} - \frac{v_c}{R_c^2(x)(1+z/R_c(x))} + \frac{\partial v_c}{\partial z} \end{aligned} \tag{7}$$

where

$$a_2 = R_i(x) = R_{i0} + x\sin(\phi)\ ;\ (i = t,b,c) \tag{8}$$

The stress- strain relations for the face sheets are given as

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{\theta\theta} \\ \sigma_{xz} \\ \sigma_{\theta z} \\ \sigma_{x\theta} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & & & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & & & \bar{Q}_{26} \\ & & \bar{Q}_{44} & \bar{Q}_{45} & \\ & & \bar{Q}_{45} & \bar{Q}_{55} & \\ \bar{Q}_{16} & \bar{Q}_{26} & & & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{\theta\theta} \\ \varepsilon_{xz} \\ \varepsilon_{\theta z} \\ \varepsilon_{x\theta} \end{Bmatrix} \tag{9}$$

where the transformed reduced elasticity matrix [$\bar{Q}$] is

$$\begin{aligned} &\bar{Q}_{11} = Q_{11}\cos^4\alpha + 2(Q_{12} + 2Q_{66})\sin^2\alpha\cos^2\alpha + Q_{22}\sin^4\alpha \\ &\bar{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66})\sin^2\alpha\cos^2\alpha + Q_{12}(\sin^4\alpha + \cos^4\alpha) \\ &\bar{Q}_{22} = Q_{11}\sin^4\alpha + 2(Q_{12} + 2Q_{66})\sin^2\alpha\cos^2\alpha + Q_{22}\cos^4\alpha \\ &\bar{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66})\sin\alpha\cos^3\alpha + (Q_{12} - Q_{22} + 2Q_{66})\sin^3\alpha\cos\alpha \\ &\bar{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66})\sin^3\alpha\cos\alpha + (Q_{12} - Q_{22} + 2Q_{66})\sin\alpha\cos^3\alpha \\ &\bar{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})\sin^2\alpha\cos^2\alpha + Q_{66}(\sin^4\alpha + \cos^4\alpha) \\ &\bar{Q}_{44} = Q_{44}\cos^4\alpha + Q_{55}\sin^4\alpha \end{aligned}$$

$$\begin{aligned}\bar{Q}_{45} &= (Q_{55} - Q_{44})\cos\alpha\sin\alpha \\ \bar{Q}_{55} &= Q_{55}\cos^4\alpha + Q_{44}\sin^4\alpha\end{aligned} \tag{10}$$

where α is the angular orientation of the fibers and

$$\begin{aligned} &Q_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}}, \qquad Q_{12} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}}, \qquad Q_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}}, \\ &Q_{44} = G_{23}, \qquad Q_{55} = G_{13}, \qquad Q_{66} = G_{12}. \end{aligned} \tag{11}$$

2.3 Compatibility conditions

The compatibility conditions were presented assuming perfect bonded conditions between the core and face-sheet interfaces (Kheirikhah *et al*. 2011)

$$\begin{cases} u_c(z = z_{ci}) = u_0^i + \frac{1}{2}(-1)^k h_i \psi_x^i \\ v_c(z = z_{ci}) = v_0^i + \frac{1}{2}(-1)^k h_i \psi_\theta^i & ; i = t \rightarrow \left(k = 1\, ;\, z_{ct} = \frac{h_c}{2}\right) \\ w_c(z = z_{ci}) = w_0^i & ; i = b \rightarrow \left(k = 0\, ;\, z_{cb} = -\frac{h_c}{2}\right) \end{cases} \tag{12}$$

Using displacement fields of the core (Eq. (4)), the compatibility conditions can be written as follows

$$\begin{cases} u_2^c = \dfrac{2(u_0^t + u_0^b) - h_t\psi_x^t + h_b\psi_x^b - 4u_0^c}{h_c^2} \\ u_3^c = \dfrac{4(u_0^t - u_0^b) - 2(h_t\psi_x^t + h_b\psi_x^b) - 4h_c u_1^c}{h_c^3} \\ v_2^c = \dfrac{2(v_0^t + v_0^b) - h_t\psi_\theta^t + h_b\psi_\theta^b - 4v_0^c}{h_c^2} \\ v_3^c = \dfrac{4(v_0^t - v_0^b) - 2(h_t\psi_\theta^t + h_b\psi_\theta^b) - 4h_c v_1^c - 4h_c v_0^c / R_c(x)}{h_c^3} \\ w_1^c = \dfrac{(w_0^t - w_0^b)}{h_c} \\ w_2^c = \dfrac{2(w_0^t + w_0^b) - 4w_0^c}{h_c^2} \end{cases} \tag{13}$$

It can be seen from Eq. (13) that the number of unknowns in the core is reduced to five. These unknowns are $u_0^c, u_1^c, v_0^c, v_1^c$ and w_0^c. Therefore, in a general form, the number of unknowns for a doubly curved composite sandwich shell is fifteen as below:

$$\left\{u_0^t, v_0^t, w_0^t, y_x^t, y_q^t, u_0^b, v_0^b, w_0^b, y_x^b, y_q^b, u_0^c, u_1^c, v_0^c, v_1^c, w_0^c,\right\}$$

2.4 Governing equations

The governing equations were derived using Hamilton's principle, requiring that

$$\int_0^T \delta L dt = \int_0^T [\delta K - \delta U] dt = 0 \tag{14}$$

where δK and δU denote the variation of kinetic energy and that of strain energy, respectively. Also t is the time duration between the times t_1 and t_2, and δ denotes the variation operator.

The first variation of the kinetic energy, upon assuming the homogeneous conditions for the displacement and velocity with respect to the time coordinate, can be written as follows

$$\begin{aligned} \delta K = & \sum_{i=t,b} \left(\int_{V_i} \rho_i \, \ddot{u}_i \, \delta u_i + \ddot{v}_i \, \delta v_i + \ddot{w}_i \, \delta w_i \, dV_i \right) \\ & + \int_{V_c} \rho_c \, \ddot{u}_c \, \delta u_c + \ddot{v}_c \, \delta v_c + \ddot{w}_c \, \delta w_c \, dW_c \end{aligned} \tag{15}$$

where ($\ddot{\ }$) denotes the second derivative in time.

The first variation of the internal potential energy for a composite conical sandwich shell that includes the face sheets and the core is

$$\begin{aligned} \delta U = & \sum_{i=t,b} \left(\int_{V_i} \left(\sigma_{xx}^i \delta\varepsilon_{xx}^i + \sigma_{\theta\theta}^i \delta\varepsilon_{\theta\theta}^i + \tau_{x\theta}^i \delta\gamma_{x\theta}^i + \tau_{xz}^t \delta\gamma_{xz}^i + \tau_{\theta z}^i \delta\gamma_{\theta z}^i \right) dV_i \right) \\ & + \int_{V_c} \left(\sigma_{xx}^c \delta\varepsilon_{xx}^c + \sigma_{\theta\theta}^c \delta\varepsilon_{\theta\theta}^c + \sigma_{zz}^c \delta\varepsilon_{zz}^c + \tau_{x\theta}^c \delta\gamma_{x\theta}^c + \tau_{xz}^c \delta\gamma_{xz}^c + \tau_{\theta z}^c \delta\gamma_{\theta z}^c \right) dV_c \end{aligned} \tag{16}$$

The moments of inertia ($I_n^c \left(n = 0,\ 1, \ldots, 6\right)$) for the core layer are

$$I_n^c = \int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} \rho_c z_c^n \left(1 + \frac{z_c}{R_{c_0} + x \sin(\phi)}\right) dz_c \quad ; \quad n = 0,1,\ldots,6 \tag{17}$$

and the stress resultants per unit length can be defined as follow

$$\begin{Bmatrix} N_{xx}^t \\ N_{\theta\theta}^t \\ N_{x\theta}^t \\ N_{\theta x}^t \end{Bmatrix} = \int_{-d_t/2}^{d_t/2} \begin{Bmatrix} \sigma_{xx}^t \\ \sigma_{\theta\theta}^t \\ \sigma_{x\theta}^t \\ \sigma_{x\theta}^t \end{Bmatrix} dz_t \begin{Bmatrix} M_{xx}^t \\ M_{\theta\theta}^t \\ M_{x\theta}^t \\ M_{\theta x}^t \end{Bmatrix} = \int_{-d_t/2}^{d_t/2} z_t \begin{Bmatrix} \sigma_{xx}^t \\ \sigma_{\theta\theta}^t \\ \sigma_{x\theta}^t \\ \sigma_{x\theta}^t \end{Bmatrix} dz_t, \begin{Bmatrix} Q_{xx}^t \\ Q_{\theta z}^t \end{Bmatrix} = k_s \int_{-d_t/2}^{d_t/2} \begin{Bmatrix} Q_{xz}^t \\ Q_{\theta z}^t \end{Bmatrix} dz_t\ ,$$

$$\begin{Bmatrix} M_{xx}^b \\ M_{\theta\theta}^b \\ M_{x\theta}^b \\ M_{\theta x}^b \end{Bmatrix} = \int_{-d_b/2}^{d_b/2} z_b \begin{Bmatrix} \sigma_{xx}^b \\ \sigma_{\theta\theta}^b \\ \sigma_{x\theta}^b \\ \sigma_{x\theta}^b \end{Bmatrix} dz_b, \begin{Bmatrix} N_{xx}^b \\ N_{\theta\theta}^b \\ N_{x\theta}^b \\ N_{\theta x}^b \end{Bmatrix} = \int_{-d_b/2}^{d_b/2} z_t \begin{Bmatrix} \sigma_{xx}^b \\ \sigma_{\theta\theta}^b \\ \sigma_{x\theta}^b \\ \sigma_{x\theta}^b \end{Bmatrix} dz_b, \begin{Bmatrix} Q_{xz}^b \\ Q_{\theta z}^b \end{Bmatrix} = k_s \int_{-d_b/2}^{d_b/2} \begin{Bmatrix} Q_{xz}^b \\ Q_{\theta z}^b \end{Bmatrix} dz_b\ ,$$

$$\begin{Bmatrix} N_{\theta\theta}^c \\ N_{\theta x}^c \\ N_{\theta z}^c \end{Bmatrix} = \int_{-t_c/2}^{t_c/2} \begin{Bmatrix} \sigma_{\theta\theta}^c \\ \sigma_{x\theta}^c \\ \sigma_{\theta z}^c \end{Bmatrix} dz_c, \begin{Bmatrix} N_{xx}^c \\ N_{x\theta}^c \\ N_{xz}^c \\ N_{\theta z}^c \end{Bmatrix} = \int_{-t_c/2}^{t_c/2} z_t \begin{Bmatrix} \sigma_{xx}^c \\ \sigma_{x\theta}^c \\ \sigma_{xz}^c \\ \sigma_{\theta z}^c \end{Bmatrix} \left(1 + \frac{z_c}{R_{c_0} + x \sin(\phi)} \right) dz_c\ ,$$

$$\begin{Bmatrix} M_{1xx}^{c} \\ M_{1x\theta}^{c} \\ M_{1xz}^{*c} \\ M_{1\theta z}^{*c} \end{Bmatrix} = \int_{-t_c/2}^{t_c/2} z_c \begin{Bmatrix} \sigma_{xx}^{c} \\ \sigma_{x\theta}^{c} \\ \sigma_{xz}^{c} \\ \sigma_{\theta z}^{c} \end{Bmatrix} \left(1+\frac{z_c}{R_{c_0}+x\sin(\phi)}\right) dz_c, \begin{Bmatrix} M_{1\theta\theta}^{c} \\ M_{1\theta x}^{c} \\ M_{1\theta z}^{c} \end{Bmatrix} = \int_{-t_c/2}^{t_c/2} z_c \begin{Bmatrix} \sigma_{\theta\theta}^{c} \\ \sigma_{x\theta}^{c} \\ \sigma_{\theta z}^{c} \end{Bmatrix} dz_c\,,$$

$$\begin{Bmatrix} M_{2\theta\theta}^{c} \\ M_{2\theta x}^{c} \end{Bmatrix} = \int_{-t_c/2}^{t_c/2} z_c^2 \begin{Bmatrix} \sigma_{\theta\theta}^{c} \\ \sigma_{x\theta}^{c} \end{Bmatrix} dz_c, \begin{Bmatrix} M_{2xx}^{c} \\ M_{2x\theta}^{c} \\ M_{2xz}^{*c} \\ M_{2\theta z}^{*c} \end{Bmatrix} = \int_{-t_c/2}^{t_c/2} z_c^2 \begin{Bmatrix} \sigma_{xx}^{c} \\ \sigma_{x\theta}^{c} \\ \sigma_{xz}^{c} \\ \sigma_{\theta z}^{c} \end{Bmatrix} \left(1+\frac{z_c}{R_{c_0}+x\sin(\phi)}\right) dz_c\,, \tag{18}$$

$$\begin{Bmatrix} M_{3\theta\theta}^{c} \\ M_{3\theta x}^{c} \end{Bmatrix} = \int_{-t_c/2}^{t_c/2} z_c^3 \begin{Bmatrix} \sigma_{\theta\theta}^{c} \\ \sigma_{x\theta}^{c} \end{Bmatrix} dz_c, \begin{Bmatrix} M_{3xx}^{c} \\ M_{3x\theta}^{c} \\ M_{3xz}^{*c} \\ M_{3\theta z}^{*c} \end{Bmatrix} = \int_{-t_c/2}^{t_c/2} z_c^3 \begin{Bmatrix} \sigma_{xx}^{c} \\ \sigma_{x\theta}^{c} \\ \sigma_{xz}^{c} \\ \sigma_{\theta z}^{c} \end{Bmatrix} \left(1+\frac{z_c}{R_{c_0}+x\sin(\phi)}\right) dz_c\,,$$

$$\left\{R_z^c, M_z^c\right\} = \int_{-t_c/2}^{t_c/2} (1, z_c)\sigma_{zz}^{c} \left(1+\frac{z_c}{R_{c_0}+x\sin(\phi)}\right) dz_c\,.$$

Using the Hamilton's principle (Eqs. (14)-(16)), kinematic relations (Eqs. (1)-(8)), the equations of moments of inertia (Eq. (17)) and the stress resultants (Eq. (17), the governing equations can be obtained as

$$\begin{aligned}
&\delta u_0^t: \\
&N_{xx,x}^{t} + \frac{\sin(\phi)}{R_t(x)}(N_{xx}^{t} - N_{\theta\theta}^{t}) + \frac{1}{R_t(x)} N_{x\theta,\theta}^{t} + \frac{2}{h_c^2} M_{2xx,x}^{c} + \frac{2\sin(\phi)}{R_c(x)h_c^2} M_{2xx}^{c} + \frac{4}{h_c^3} M_{3xx,x}^{c} \\
&+\frac{4\sin(\phi)}{R_c(x)h_c^3} M_{3xx}^{c} - \frac{2\sin(\phi)}{R_c(x)h_c^2} M_{2\theta\theta}^{c} - \frac{4\sin(\phi)}{R_c(x)h_c^3} M_{3\theta\theta}^{c} + \frac{2}{R_c(x)h_c^2} M_{2\theta x,\theta}^{c} + \frac{4}{R_c(x)h_c^3} M_{3\theta x,\theta}^{c} \\
&-\frac{4}{h_c^2} M_{1xz}^{*c} - \frac{12}{h_c^3} M_{2xz}^{*c} = I_0^t \ddot{u}_0^t + I_1^t \ddot{\psi}_x^t + \chi_1^1 \ddot{u}_0^c + \chi_1^2 \ddot{u}_1^c + \chi_1^3 \ddot{u}_0^b + \chi_1^4 \ddot{u}_0^t + \chi_1^5 \ddot{\psi}_x^b + \chi_1^6 \ddot{\psi}_x^t\,.
\end{aligned} \tag{19}$$

$$\begin{aligned}
&\delta u_0^b: \\
&\frac{\partial N_{xx}^{b}}{\partial x} + \frac{\sin(\phi)}{R_b(x)}(N_{xx}^{b} - N_{\theta\theta}^{b}) + \frac{1}{R_b(x)} N_{x\theta,\theta}^{b} + \frac{2}{h_c^2} M_{2xx,x}^{c} + \frac{2\sin(\phi)}{h_c^2} M_{2xx}^{c} - \frac{4}{h_c^3} M_{3xx,x}^{c} - \\
&\frac{4\sin(\phi)}{h_c^3} M_{3xx}^{c} + \frac{2}{R_c(x)h_c^2} M_{2\theta x,\theta}^{c} - \frac{2\sin(\phi)}{R_c(x)h_c^2} M_{2\theta\theta}^{c} + \frac{4\sin(\phi)}{R_c(x)h_c^3} M_{3\theta\theta}^{c} - \frac{4}{R_c(x)h_c^3} M_{3\theta x,\theta}^{c} \\
&-\frac{4}{h_c^2} M_{1xz}^{*c} + \frac{12}{h_c^3} M_{2xz}^{*c} = I_0^b \ddot{u}_0^b + I_1^b \ddot{\psi}_x^b + \chi_2^1 \ddot{u}_0^c + \chi_2^2 \ddot{u}_1^c + \chi_2^3 \ddot{u}_0^b + \chi_2^4 \ddot{u}_0^t + \chi_2^5 \ddot{\psi}_x^b + \chi_2^6 \ddot{\psi}_x^t\,.
\end{aligned} \tag{20}$$

$$
\begin{aligned}
&\delta v_0^t: \\
&N_{x\theta,x}^t + \frac{1}{R_t(x)} N_{\theta\theta,\theta}^t + \frac{\cos(\phi)}{R_t(x)} Q_{\theta z}^t + \frac{2\sin(\phi)}{R_t(x)} N_{x\theta}^t + + \frac{2}{R_c(x) h_c^2} M_{2\theta\theta,\theta}^c + \frac{4}{R_c(x) h_c^3} M_{3\theta\theta,\theta}^c \\
&+ \frac{2}{h_c^2} M_{2x\theta,x}^c + \frac{2\sin(\phi)}{R_c(x) h_c^2} M_{2x\theta}^c + \frac{4}{h_c^3} M_{3x\theta,x}^c + \frac{4\sin(\phi)}{R_c(x) h_c^3} M_{3x\theta}^c + \frac{2\sin(\phi)}{R_c(x) h_c^2} M_{2\theta x}^c \\
&+ \frac{4\sin(\phi)}{R_c(x) h_c^3} M_{3\theta x}^c + \frac{2}{R_c(x) h_c^2} M_{2\theta z}^c + \frac{4}{R_c(x) h_c^3} M_{3\theta z}^c - \frac{4}{h_c^2} M_{1\theta z}^{*c} - \frac{12}{h_c^3} M_{2\theta z}^{*c} = I_0^t \ddot{v}_0^t + I_1^t \ddot{\psi}_\theta^t \\
&+ \chi_3^1 \ddot{v}_0^c + \chi_3^2 \ddot{v}_1^c + \chi_3^3 \ddot{v}_0^b + \chi_3^4 \ddot{v}_0^t + \chi_3^5 \ddot{\psi}_\theta^b + \chi_3^6 \ddot{\psi}_\theta^t.
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
&\delta v_0^b: \\
&N_{x\theta,x}^b + \frac{1}{R_b(x)} N_{\theta\theta,\theta}^b + \frac{\cos(\phi)}{R_b(x)} Q_{\theta z}^b + \frac{2\sin(\phi)}{R_b(x)} N_{x\theta}^b + \frac{2}{R_c h_c^2} M_{2\theta\theta,\theta}^c - \frac{4}{R_c h_c^3} M_{3\theta\theta,\theta}^c + \frac{2}{h_c^2} M_{2x\theta,x}^c \\
&+ \frac{2\sin(\phi)}{R_c(x) h_c^2} M_{2x\theta}^c - \frac{4}{h_c^3} M_{3x\theta,x}^c - \frac{4\sin(\phi)}{R_c(x) h_c^3} M_{3x\theta}^c + \frac{2\sin(\phi)}{R_c(x) h_c^2} M_{2\theta x}^c - \frac{4\sin(\phi)}{R_c(x) h_c^3} M_{3\theta x}^c \\
&+ \frac{2}{R_c(x) h_c^2} M_{2\theta z}^c - \frac{4}{R_c(x) h_c^3} M_{3\theta z}^c - \frac{4}{h_c^2} M_{1\theta z}^{*c} + \frac{12}{h_c^3} M_{2\theta z}^{*c} = I_0^b \ddot{v}_0^b + I_1^b \ddot{\psi}_\theta^b + \chi_4^1 \ddot{v}_0^c + \chi_4^2 \ddot{v}_1^c \\
&+ \chi_4^3 \ddot{v}_0^b + \chi_4^4 \ddot{v}_0^t + \chi_4^5 \ddot{\psi}_\theta^b + \chi_4^6 \ddot{\psi}_\theta^t.
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
&\delta w_0^t: \\
&Q_{xz,x}^t + \frac{1}{R_t(x)} Q_{\theta z,\theta}^t + \sin(\phi) Q_{xz,x}^t - \frac{\cos(\phi)}{R_t(x)} N_{\theta\theta}^t - \frac{1}{h_c} R_z^c - \frac{4}{h_c^2} M_z^c - \frac{1}{R_c(x) h_c} M_{1\theta\theta}^c \\
&- \frac{2}{R_c(x) h_c^2} M_{2\theta\theta}^c + \frac{1}{h_c} M_{1xz,x}^c + \frac{\sin(\phi)}{R_c(x) h_c} M_{1xz}^c + \frac{2}{h_c^2} M_{2xz,x}^c + \frac{2\sin(\phi)}{R_c(x) h_c^2} M_{2xz}^c \\
&+ \frac{1}{R_c h_c} M_{1\theta z,\theta}^c + \frac{2}{R_c h_c^2} M_{2\theta z,\theta}^c = I_0^t \ddot{w}_0^t + \chi_5^1 \ddot{w}_0^c + \chi_5^2 \ddot{w}_0^t + \chi_5^3 \ddot{w}_0^b.
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
&\delta w_0^b: \\
&Q_{xz,x}^b + \frac{1}{R_b(x)} Q_{\theta z,\theta}^b + \sin(\phi) Q_{xz,x}^b - \frac{\cos(\phi)}{R_b(x)} N_{\theta\theta}^b + \frac{1}{h_c} R_z^c - \frac{4}{h_c^2} M_z^c + \frac{1}{R_c h_c} M_{1\theta\theta}^c \\
&- \frac{2}{R_c h_c^2} M_{2\theta\theta}^c - \frac{1}{h_c} M_{1xz,x}^c - \frac{\sin(\phi)}{R_c(x) h_c} M_{1xz}^c + \frac{2}{h_c^2} M_{2xz,x}^c + \frac{2\sin(\phi)}{R_c(x) h_c^2} M_{2xz}^c - \frac{1}{h_c} M_{1\theta z,\theta}^c \\
&+ \frac{2}{h_c^2} M_{2\theta z,\theta}^c = I_0^b \ddot{w}_0^b + \chi_6^1 \ddot{w}_0^c + \chi_6^2 \ddot{w}_0^b + \chi_6^3 \left(-\frac{I_2^c}{h_c^2} + \frac{4 I_4^c}{h_c^4} \right) \ddot{w}_0^t.
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
&\delta \psi_x^t: \\
&M_{xx,x}^t + \frac{1}{R_t(x)} M_{x\theta,\theta}^t + \frac{\sin(\phi)}{R_t(x)} \left(M_{xx}^t - M_{\theta\theta}^t \right) - Q_{xz}^t - \frac{h_t}{h_c^2} M_{2xx,x}^c - \frac{h_t \sin(\phi)}{R_c(x) h_c^2} M_{2xx}^c \\
&- \frac{2h_t}{h_c^3} M_{3xx,x}^c - \frac{2h_t \sin(\phi)}{R_c(x) h_c^3} M_{3xx}^c - \frac{h_t}{R_c h_c^2} M_{2\theta x,\theta}^c - \frac{2h_t}{R_c h_c^3} M_{3\theta x,\theta}^c + \frac{2h_t}{h_c^2} M_{1xz}^{*c} + \frac{6h_t}{h_c^3} M_{2xz}^{*c} \\
&+ \frac{h_t \sin(\phi)}{R_c(x) h_c^2} M_{2\theta\theta}^c + \frac{h_t \sin(\phi)}{R_c(x) h_c^3} M_{3\theta\theta}^c = I_1^t \ddot{u}_0^t + I_2^t \ddot{\psi}_x^t + \chi_7^1 \ddot{u}_0^c + \chi_7^2 \ddot{u}_1^c + \chi_7^3 \ddot{u}_0^b + \chi_7^4 \ddot{u}_0^t \\
&+ \chi_7^5 \ddot{\psi}_x^b + \chi_7^6 \ddot{\psi}_x^t.
\end{aligned}
\tag{25}
$$

$$\begin{aligned}
&\delta\psi_x^b:\\
&M^b_{xx,x}+\frac{1}{R_b(x)}\partial M^b_{x\theta,\theta}+\frac{\sin(\phi)}{R_b(x)}\left(M^b_{xx}-M^b_{\theta\theta}\right)-Q^b_{xz}+\frac{h_b}{h_c^2}M^c_{2xx,x}+\frac{h_t\sin(\phi)}{R_c(x)h_c^2}M^c_{2xx}\\
&-\frac{2h_b}{h_c^3}M^c_{3xx,x}-\frac{2h_t\sin(\phi)}{R_c(x)h_c^3}M^c_{3xx}-\frac{h_b}{R_ch_c^2}M^c_{2\theta x,\theta}-\frac{2h_b}{R_ch_c^3}M^c_{3\theta x,\theta}-\frac{2h_b}{h_c^2}M^{*c}_{1xz}+\frac{6h_b}{h_c^3}M^{*c}_{2xz}\\
&-\frac{h_t\sin(\phi)}{R_c(x)h_c^2}M^c_{2\theta\theta}+\frac{h_t\sin(\phi)}{R_c(x)h_c^3}M^c_{3\theta\theta}=I_1^b\ddot{u}_0^b+I_2^b\ddot{\psi}_x^b+\chi_8^1\ddot{u}_0^c+\chi_8^2\ddot{u}_1^c+\chi_8^3\ddot{u}_0^b+\chi_8^4\ddot{u}_0^t+\chi_8^5\ddot{\psi}_x^b\\
&+\chi_8^6\ddot{\psi}_x^t.
\end{aligned}\tag{26}$$

$$\begin{aligned}
&\delta\psi_\theta^t:\\
&M^t_{x\theta,x}+\frac{1}{R_t(x)}M^t_{\theta\theta,\theta}+\frac{2\sin(\phi)}{R_t(x)}M^t_{x\theta}-Q^t_{\theta z}-\frac{h_t}{R_c(x)h_c^2}M^c_{2\theta\theta,\theta}-\frac{2h_t}{R_c(x)h_c^3}M^c_{3\theta\theta,\theta}\\
&-\frac{h_t}{h_c^2}M^c_{2x\theta,x}-\frac{h_t\sin(\phi)}{R_c(x)h_c^2}M^c_{2x\theta}-\frac{2h_t}{h_c^3}M^c_{3x\theta,x}-\frac{2h_t\sin(\phi)}{R_c(x)h_c^3}M^c_{3x\theta}+\frac{2h_t}{h_c^2}M^{*c}_{1\theta z}+\frac{6h_t}{h_c^3}M^{*c}_{2\theta z}\\
&-\frac{h_t\sin(\phi)}{R_c(x)h_c^2}M^c_{2\theta x}-\frac{2h_t\sin(\phi)}{R_c(x)h_c^3}M^c_{3\theta x}=I_1^t\ddot{v}_0^t+I_2^t\ddot{\psi}_\theta^t+\chi_9^1\ddot{v}_0^c+\chi_9^2\ddot{v}_1^c+\chi_9^3\ddot{v}_0^b+\chi_9^4\ddot{v}_0^t+\chi_9^5\ddot{\psi}_\theta^b\\
&+\chi_9^6\ddot{\psi}_\theta^t.
\end{aligned}\tag{27}$$

$$\begin{aligned}
&\delta\psi_\theta^b:\\
&M^b_{x\theta,x}+\frac{1}{R_b(x)}M^b_{\theta\theta,\theta}+\frac{2\sin(\phi)}{R_b(x)}M^b_{x\theta}-Q^b_{\theta z}+\frac{h_b}{R_c(x)h_c^2}M^c_{2\theta\theta,\theta}-\frac{2h_b}{R_c(x)h_c^3}M^c_{3\theta\theta,\theta}\\
&+\frac{h_b}{h_c^2}M^c_{2x\theta,x}+\frac{h_t\sin(\phi)}{R_c(x)h_c^2}M^c_{2x\theta}-\frac{2h_b}{h_c^3}M^c_{3x\theta,x}+\frac{2h_t\sin(\phi)}{R_c(x)h_c^3}M^c_{3x\theta}-\frac{h_b}{R_c(x)h_c^2}M^c_{2\theta z}\\
&-\frac{2h_b}{R_c(x)h_c^2}M^c_{3\theta z}-\frac{2h_b}{h_c^2}M^{*c}_{1\theta z}+\frac{6h_b}{h_c^3}M^{*c}_{2\theta z}+\frac{h_t\sin(\phi)}{R_c(x)h_c^2}M^c_{2\theta x}-\frac{2h_t\sin(\phi)}{R_c(x)h_c^3}M^c_{3\theta x}=I_1^b\ddot{v}_0^b\\
&+I_2^b\ddot{\psi}_\theta^b+\chi_{10}^1\ddot{v}_0^c+\chi_{10}^2\ddot{v}_1^c+\chi_{10}^3\ddot{v}_0^b+\chi_{10}^4\ddot{v}_0^t+\chi_{10}^5\ddot{\psi}_\theta^b+\chi_{10}^6\ddot{\psi}_\theta^t.
\end{aligned}\tag{28}$$

$$\begin{aligned}
&\delta u_0^c:\\
&N^c_{xx,x}+\frac{\sin(\phi)}{R_c(x)}N^c_{xx,x}+\frac{1}{R_c}N^c_{\theta x,\theta}-\frac{4}{h_c^2}M^c_{2xx,x}-\frac{4\sin(\phi)}{R_c(x)h_c^2}M^c_{2xx}-\frac{4}{R_ch_c^2}M^c_{2\theta x,\theta}+\frac{8}{h_c^2}M^{*c}_{1xz}\\
&-\frac{\sin(\phi)}{R_c(x)}N^c_{\theta\theta}+\frac{4\sin(\phi)}{h_c^2R_c(x)}M^c_{2\theta\theta}=\chi_{11}^1\ddot{u}_0^c+\chi_{11}^2\ddot{u}_1^c+\chi_{11}^3\ddot{u}_0^b+\chi_{11}^4\ddot{u}_0^t+\chi_{11}^5\ddot{\psi}_x^b+\chi_{11}^6\ddot{\psi}_x^t.
\end{aligned}\tag{29}$$

$$\begin{aligned}
&\delta u_1^c:\\
&M^c_{1xx,x}+\frac{\sin(\phi)}{R_c(x)}M^c_{1xx}-N^c_{xz}-\frac{4}{h_c^2}M^c_{3xx,x}-\frac{4\sin(\phi)}{R_c(x)h_c^2}M^c_{3xx}+\frac{1}{R_c(x)}M^c_{1\theta x,\theta}-\frac{4}{R_c(x)h_c^2}M^c_{3\theta x,\theta}\\
&+\frac{12}{h_c^2}M^{*c}_{2xz}-\frac{\sin(\phi)}{R_c(x)}M^c_{1\theta\theta}+\frac{4\sin(\phi)}{h_c^2R_c(x)}M^c_{3\theta\theta}=\chi_{12}^1\ddot{u}_0^c+\chi_{12}^2\ddot{u}_1^c+\chi_{12}^3\ddot{u}_0^b+\chi_{12}^4\ddot{u}_0^t+\chi_{12}^5\ddot{\psi}_x^b+\chi_{12}^6\ddot{\psi}_x^t.
\end{aligned}\tag{30}$$

$$\begin{aligned}
&\delta v_0^c:\\
&\frac{1}{R_c(x)}N^c_{\theta\theta,\theta}+N^c_{x\theta,x}+\frac{\sin(\phi)}{R_c(x)}N^c_{x\theta}-\frac{4}{R_c(x)h_c^2}M^c_{2\theta\theta,\theta}-\frac{4}{R_c^2(x)h_c^2}M^c_{3\theta\theta,\theta}+\frac{1}{R_c(x)}M^c_{1x\theta,x}\\
&-\frac{4}{h_c^2}M^c_{2x\theta,x}-\frac{4\sin(\phi)}{R_c(x)h_c^2}M^c_{2x\theta}-\frac{4}{R_c(x)h_c^2}M^c_{3x\theta,x}+\frac{\sin(\phi)}{R_c(x)}N^c_{\theta x}+\frac{\sin(\phi)}{R_c^2(x)}M^c_{1\theta x}+\\
&\frac{1}{R_c(x)}N^c_{\theta z}-\frac{4\sin(\phi)}{R_c(x)h_c^2}M^c_{2\theta x}-\frac{4\sin(\phi)}{R_c^2(x)h_c^2}M^c_{3\theta x}\ \frac{4}{R_c(x)h_c^2}M^c_{2\theta z}-\frac{4}{R_c^2(x)h_c^2}M^c_{3\theta z}+\\
&\frac{8}{h_c^2}M^{*c}_{1\theta z}+\frac{12}{R_c(x)h_c^2}M^{*c}_{2\theta z}-\frac{1}{R_c(x)}N^{*c}_{\theta z}=\chi_{13}^1\ddot{v}_0^c+\chi_{13}^2\ddot{v}_1^c+\chi_{13}^3\ddot{v}_0^b+\chi_{13}^4\ddot{v}_0^t+\chi_{13}^5\ddot{\psi}_\theta^b+\chi_{13}^6\ddot{\psi}_x^t.
\end{aligned}\tag{31}$$

$$\begin{aligned}&\delta v_1^c:\\&\frac{1}{R_c(x)}M_{1\theta\theta,\theta}^c-N_{\theta z}^{*c}-\frac{4}{R_c(x)h_c^2}M_{3\theta\theta,\theta}^c+M_{1x\theta,x}^c+\frac{\sin(\phi)}{R_c(x)}M_{1x\theta}^c-\frac{4}{h_c^2}M_{3x\theta,x}^c\\&-\frac{4\sin(\phi)}{h_c^2R_c(x)}M_{3x\theta}^c+\frac{\sin(\phi)}{R_c(x)}M_{1\theta x}^c-\frac{4\sin(\phi)}{h_c^2R_c(x)}M_{3\theta x}^c+\frac{1}{R_c(x)}M_{1\theta z}^c-\frac{4}{R_c(x)h_c^2}M_{3\theta z}^c+\frac{12}{h_c^2}M_{2\theta z}^{*c}=\\&\chi_{14}^1\ddot{v}_0^c+\chi_{14}^2\ddot{v}_1^c+\chi_{14}^3\ddot{v}_0^b+\chi_{14}^4\ddot{v}_0^t+\chi_{14}^5\ddot{\psi}_\theta^b+\chi_{14}^6\ddot{\psi}_\theta^t.\end{aligned}\tag{32}$$

$$\begin{aligned}&\delta w_o^c:\\&N_{xz,x}^c+\frac{\sin(\phi)}{R_c(x)}N_{xz}^c+\frac{1}{R_c(x)}N_{\theta z,\theta}^c+\frac{8}{h_c^2}M_z^c-\frac{1}{R_c(x)}N_{\theta\theta}^c+\frac{4}{R_c(x)h_c^2}M_{2\theta\theta}^c-\frac{4}{h_c^2}M_{2xz,x}^c\\&-\frac{4}{h_c^2}\frac{\sin(\phi)}{R_c(x)}M_{2xz}^c-\frac{4}{R_c(x)h_c^2}M_{2\theta z,\theta}^c=\chi_{15}^1\ddot{w}_0^c+\chi_{15}^2\ddot{w}_0^t+\chi_{15}^3\ddot{w}_0^b.\end{aligned}\tag{33}$$

where χ_j^i $(i=1,2,...,6\&j=1,2,...,15)$ are given in Appendix A.

3. Analytical solution

Also, using the Hamilton's principle (Eqs. (14)-(16)) and kinematic relations (Eqs. (1)-(8)), the boundary conditions equations can be obtained. The simply supported geometrical and physical boundary conditions for a truncated conical shell at the edges x=0, a of the top, bottom face-sheets and the core are

$$\begin{aligned}&N_{xx}^i=0,v_0^i=0,M_{xx}^i=0,\ \psi_\theta^i=0,w_0^i=0,(i=t,b)\\&N_{xx}^c=0\ ,M_{1xx}^c=0\ ,M_{2xx}^c=0\ ,M_{3xx}^c=0\ ,v_0^c=0\\&v_1^c=0,v_2^c=0,\ v_3^c=0,w_0^c=0,w_1^c=0,w_2^c=0.\end{aligned}\tag{34}$$

The displacement fields based on double Fourier series for a composite conical sandwich panel with simply supported boundary condition at the top and bottom face-sheets are assumed to be in the following form (j=t,b)

$$\begin{bmatrix}u_0^j(x,\theta,t)\\v_0^j(x,\theta,t)\\w_0^j(x,\theta,t)\\\psi_x^j(x,\theta,t)\\\psi_\theta^j(x,\theta,t)\\u_k^c(x,\theta,t)\\v_k^c(x,\theta,t)\\w_l^c(x,\theta,t)\end{bmatrix}=\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\begin{bmatrix}U_{0mn}^j(t)\cos(\alpha_mx)\cos(n\theta)\\V_{0mn}^j(t)\sin(\alpha_mx)\sin(n\theta)\\W_{0mn}^j(t)\sin(\alpha_mx)\cos(n\theta)\\\Psi_{xmn}^j(t)\cos(\alpha_mx)\cos(n\theta)\\\Psi_{\theta mn}^j(t)\sin(\alpha_mx)\sin(n\theta)\\U_{kmn}^c(t)\cos(\alpha_mx)\cos(n\theta)\\V_{kmn}^c(t)\sin(\alpha_mx)\sin(n\theta)\\W_{lmn}^c(t)\sin(\alpha_mx)\cos(n\theta)\end{bmatrix},\ \text{(k=0,1,2,3 , l=0,1,2)}\tag{35}$$

where $U_{0mn}^j(t)$, $V_{0mn}^j(t)$, $W_{0mn}^j(t)$, $\Psi_{xmn}^j(t)$, $\Psi_{\theta mn}^j(t)$, $U_{kmn}^c(t)$, $V_{kmn}^c(t)$ and $W_{lmn}^c(t)$ are time-dependent Fourier coefficients and m and n are the half wave numbers along x and θ directions, respectively. Also $\alpha_m=m\pi x/l$. The above double Fourier series functions assumed to satisfy the geometrical simply supported boundary conditions on all edges for a conical composite sandwich shell (See also, Qatu 2004, Reddy 2003).

By substituting the stress resultants, compatibility conditions, and displacement field (Eq. (35)) in the governing equations (Eqs. (19)-(33)) and by applying the Galerkin method, the governing

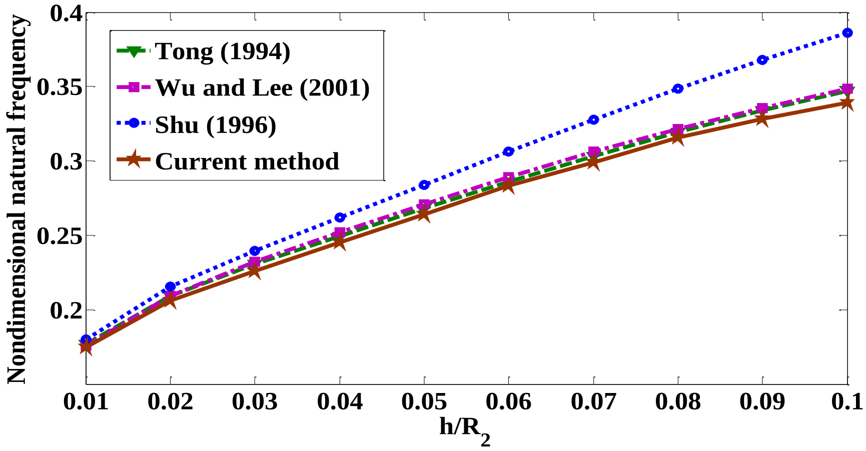


Fig. 2 Variations of the nondimensional natural frequency with respect to the ratio of h/R_2

equations were reduced to the following system of the coupled ordinary differential equations

$$[M]\{\ddot{c}\}+[K]\{c\}=\{0\} \tag{36}$$

where:

$$\{c\}=\{U^t_{0mn}(t),U^b_{0mn}(t),V^t_{0mn}(t),V^b_{0mn}(t),W^t_{0mn}(t),W^b_{0mn}(t),\psi^t_{xmn}(t),\psi^b_{xmn}(t),\psi^t_{\theta mn}(t),$$
$$\psi^b_{\theta mn}(t),U^c_{0mn}(t),V^c_{0mn}(t),U^c_{1mn}(t),V^c_{1mn}(t),W^c_{0mn}(t)\}^T$$

then the eigenvalue equation is obtained as follows

$$[K-\lambda_{mn}M]\{c\}=\{0\} \tag{37}$$

where $\lambda_{mn}=\omega^2_{mn}$, $\{c\}$ is displacement vector for all the values of m and n, $[K]$ is the $(15\times m\times(n+1))\times(15\times m\times(n+1))$ stiffness matrix and $[M]$ is the $(15\times m\times(n+1))\times(15\times m\times(n+1))$ square mass matrix. Some of the mass and stiffness matrix elements for simply supported conical composite sandwich shells are given in Appendix B.

4. Results and discussion

In this section, some examples are considered and the obtained results are validated and discussed. To validate the present results and demonstrate their capability in predicting the free vibration analysis of a composite truncated conical sandwich panel, some examples are presented. To validate obtained results, a truncated conical sandwich panel was modeled in ABAQUS FE code and results obtained from the analytical formulations and FE code were compared together. Also, for composite laminated shells, obtained results by the present method were validated by comparing them with those in the literature. The agreement between the results was very good.

Example 1: Free vibration analysis of conical laminated composite shell with simply supported boundary conditions (S. S. B.C.s)

In this example, in order to validate the present formulation, free vibration of a truncated conical composite laminated shell with two layered anti-symmetric cross-ply laminate and simply supported boundary conditions was investigated. Nondimensional natural frequencies were obtained from $\bar{\omega}=\omega R_2(\rho h/A_{11})^{1/2}$. The conical angle is α=30° and the material properties of the individual layers are considered as: E_x/E_θ=15, $G_{x\theta}/E_\theta$=0.5, $\upsilon_{x\theta}$=0.25, $\upsilon_{xz}=\upsilon_{\theta z}$=0.3, $G_{x\theta}=E_\theta/2(1+\upsilon_{xz})$ and $G_{\theta z}=E_\theta/2(1+\upsilon_{\theta z})$. In Fig. 2, the dimensionless natural frequencies obtained from the presented improved higher order theory were compared with those obtained from first order shear deformation theory (FSDT) (Tong 1994, Wu and Lee 2001) and Love's first approximation thin shell theory (Shu 1996) for $0.01<h/R_2<0.1$. The results obtained by the present method converged after 169 expressions (m=n=13). It can be seen that the agreement between the results obtained from current method and FSDT was very good but there was a little difference between the current results and those obtained by Love's first approximation thin shell theory. It is clear, that the current high order improved theory reduces to FSDT with decreasing of the core thickness up to zero. Note that, with increasing the h/R_2 ratio, the differences among results would increase.

Example 2: Free vibration of a conical composite sandwich shell with S.S. B.C.s

In this example, the free vibration of a truncated conical sandwich shell with a foam core and composite face sheets (Table 1) with simply supported (S.S. B.C.s) boundary conditions was investigated. Non-dimensional natural frequencies were obtained from $\bar{\omega}=\omega L^2(\rho/E_2)_t^{1/2}/h$, core to panel thickness ratio was h_c/h=0.88, radius of the mid-plane of core to panel thickness ratio was R_{c1}/h=10 and length to radius of the mid-plane of core ratio was L/R_{c1}=1. In Table 2 obtained results were presented. In this table, the results were compared with presented FE results by ABAQUS code. The results obtained by the present method converged after 169 expressions (m=n=13). In this paper, to investigate the free vibration problem, in addition to the presented method, a finite element procedure was considered. Therefore, for free vibration analysis, ABAQUS software (version 6.8-1) was used. The modal frequency analysis in ABQAUS is performed in ABAQUS/Standard software, which uses a central difference rule to integrate the equations of motion explicitly. In this study, the face sheets and the foam core were meshed using SC8R and C3D8R elements, respectively. There was quite good agreement between the results and there was a little difference between them. Also, this table shows that the lowest non-dimensional natural frequency for S.S. B.C.s occurred at mode number (m, n)= (1, 3).

Table 1 Material properties of a conical composite sandwich panel

Foam core	Composite face sheets
E_1=E_2=E_3=0.10363 GPa	E_1=24.51 GPa, E_2=7.77 GPa
G_{12}=G_{13}=G_{23}=0.05 GPa	G_{12}=G_{13}=3.34 GPa, G_{23}=1.34 GPa
v=0.32, ρ=130 kg/m^3	v_{12}=0.3, ρ=1800 kg/m^3

Table 2 Nondimensional natural frequencies of a conical composite sandwich panel

Mode No. (m,n)	$\bar{\omega}=\omega L^2(\rho/E_2)_t^{1/2}/h,\ \phi=30$		
	Present model	ABAQUS	Discrepancy (%)
(1,3)	211.82	202.33	4.6
(1,2)	220.82	204.66	7.8
(1,4)	224.85	211.14	6.5

Table 3 Mechanical and geometrical properties of a truncated conical composite sandwich panel

Foam core	$E_1=E_2=E_3=0.10363$ GPa, $G_{12}=G_{13}=G_{23}=0.05$ GPa, $v=0.36$, $\rho=130$ kg/m^3
Composite face sheets	$E_1=131$ GPa, $E_2=10.34$ GPa, $G_{12}=G_{13}=6.895$ GPa, $G_{23}=0.05$ GPa, $v=0.22$, $\rho=1627$ kg/m^3
Geometry	$h=0.03$, $h_c/h=0.8$, $R_{c1}=10$ h, $L=R_{c1}$, [0 90 0/core/0 90 0]

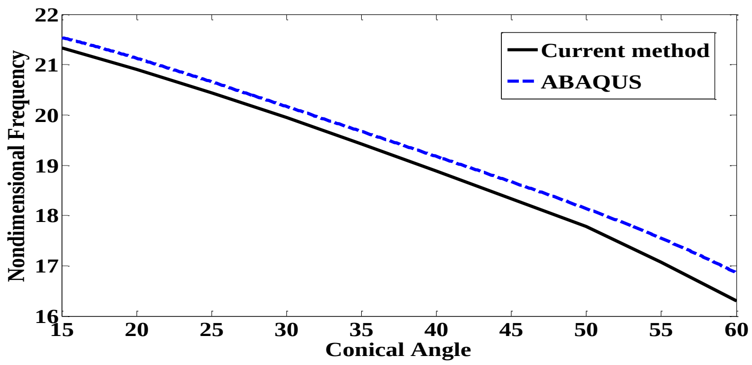
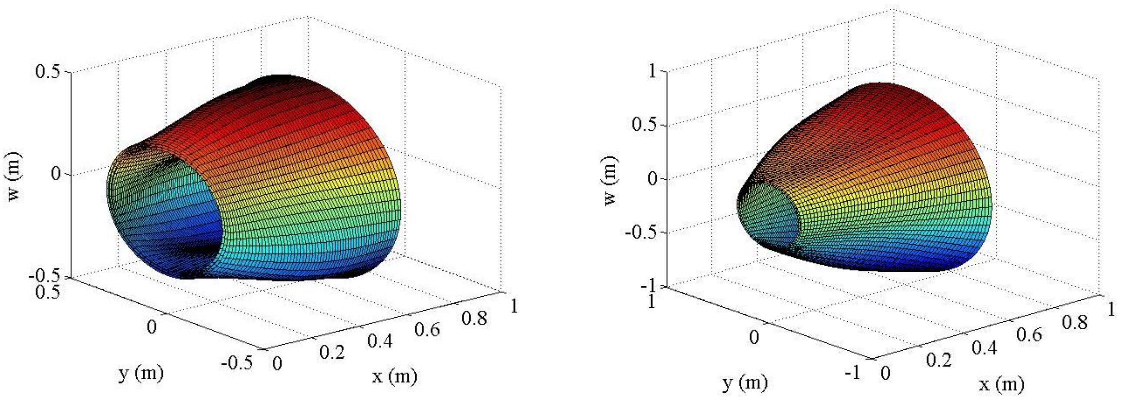


Fig. 3 Variation of the non-dimensional fundamental frequency with respect to the conical angles for the truncated conical composite sandwich panel with S.S. B.C.s obtained by ABAQUS FE code and presented formulations



(a) φ=15 (b) φ=60

Fig. 4 3D view of mode shapes of the conical sandwich shell for φ=15 and φ=60 for S.S. B.C.s

Example 3 Effect of the conical angle on the free vibration of a composite truncated conical sandwich shell

In this example the effect of the conical angle on the free vibration of a composite truncated conical sandwich shell with S.S. B.C.s were investigated. The mechanical and geometrical properties of the composite truncated conical sandwich panel were given in Table 3.

Fig. 3 shows the variation of the non-dimensional fundamental frequency with the conical angles. In this figure, the results obtained from the presented improved higher order theory were compared with presented FE results by ABAQUS code. As can be seen in Fig. 3, by increasing the conical angle, the non-dimensional fundamental frequency decreased uniformly because, by

Table 4 Mechanical and geometrical properties of a conical composite sandwich shell

Foam core	E_1=E_2=E_3=0.10363 GPa, G_{12}=G_{13}=G_{23}=0.05 GPa, v=0.36, ρ=130 kg/m^3
Composite face sheets	E_1=131 GPa, E_2=10.34 GPa, G_{12}=G_{13}=6.895 GPa, G_{23}=0.05 GPa, v=0.22, ρ=1627 kg/m^3
Geometry	h=0.03, h_c/h=0.8, R_{c1}=10 h, L=R_{c1}, [0 90 0/core/0 90 0]

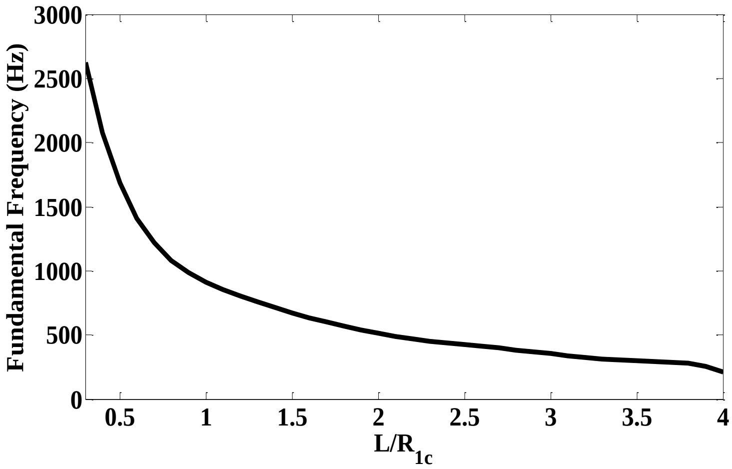


Fig. 5 Variation of the fundamental frequency (Hz) with respect to the length to radius ratio for the truncated conical composite sandwich shell with S.S. B.C.s

increasing the conical angle, a decrease occurred in axial stiffness. Also this figure shows that the agreement between the results obtained from the current method and presented FE results by ABAQUS code was very good and acceptable. The 3D view of mode shape of the truncated conical sandwich panel for two different values of the conical angle is given in Fig. 4.

Example 4 Effect of the length to radius ratio on the free vibration of a composite truncated conical sandwich shell

In this example the effect of the length to radius ratio (L/R_{c1}) on the free vibration of a composite truncated conical sandwich shell with S.S. B.C.s was investigated. The mechanical and geometrical properties of the composite truncated conical sandwich shell were given in Table 4.

The variation of the fundamental frequency (Hz) with the length to radius ratio is presented in Fig. 5. As can be observed in this figure, by increasing the length to radius ratio, fundamental frequency decreased rapidly because, by decreasing the length to radius ratio, the sandwich shell became stiffer.

Example 5 Effect of the core to panel thickness ratio on the free vibration of a composite truncated conical sandwich shell

In this example the effect of the core to panel thickness ratio (h_c/h) on the free vibration of a composite truncated conical sandwich shell with S.S. B.C.s was investigated. The mechanical and geometrical properties of the composite truncated conical sandwich shell given in Table 4 were also used in this example. The variation of the fundamental frequency (Hz) with the core to panel thickness ratio is presented in Fig. 6, in which increasing the core to panel thickness ratio resulted

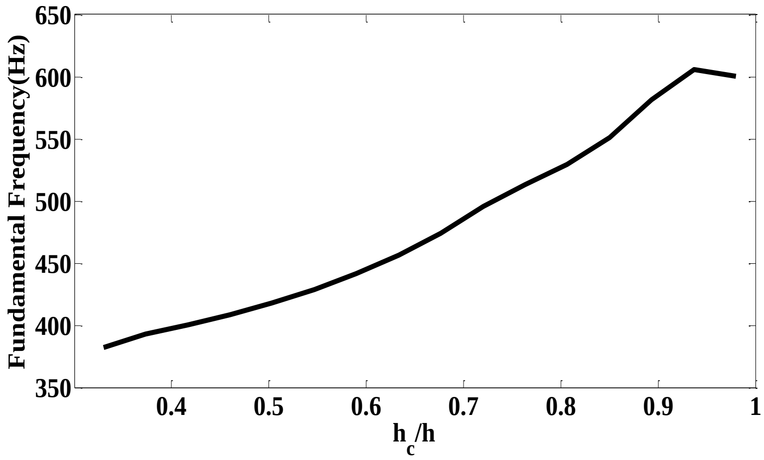


Fig. 6 Variation of the fundamental frequency (Hz) with respect to the core to panel thickness ratio for the sandwich shell with S.S. B.C.s

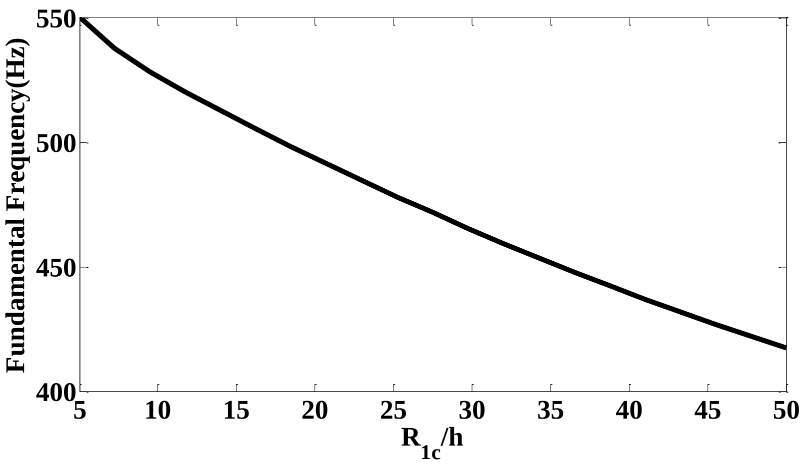


Fig. 7 Variation of the fundamental frequency (Hz) with respect to the core radius to panel thickness ratio for the truncated conical panel with S.S. B.C.s

in the fundamental frequency increase because, by increasing the core to panel thickness ratio, the flexibility of the structure decreased.

Example 6 Effect of the core radius to panel thickness ratio on the free vibration of a composite truncated conical sandwich shell

In this example the effect of the core radius to panel thickness ratio (R_{1c}/h) on the free vibration of a composite truncated conical sandwich shell with S.S. B.C.s were investigated. The mechanical and geometrical properties of the composite truncated conical sandwich shell given in Table 4 were also used in this example. The variation of the fundamental frequency (Hz) with the core radius to panel thickness ratio is presented in Fig. 7. This figure shows that, by increasing the core radius to panel thickness ratio, the fundamental frequency decreased. It means that for larger core radius to panel thickness ratio, the structure would be more flexible.

5. Conclusions

Using a new improved and enhanced higher order sandwich plate theory (IHSAPT) based on a three-layer model, the coupled partial differential governing equations on the composite truncated conical sandwich shell were derived based on the Hamilton's principle. The above analysis is quite general and valid for any type of core, any type of boundary conditions, as well as for the cases where the conditions at the top (outer) face sheet are different from those at the bottom (inner) one along the same edge. The thickness of the top (outer) face sheet may be different from that of the bottom (inner) face sheet. Transverse shear and rotary inertia effects of face sheets have been taken into consideration. To validate obtained results, a truncated conical sandwich shell was modeled in ABAQUS FE code and results obtained from analytical formulations and FE code were compared together. Also, the present method is validated by comparing the present results with those in the literature. It is clear, that the current high order improved theory reduces to FSDT with decreasing of the core thickness up to zero. Therefore, using this comprehensive high order theory, various free vibration problems like free vibration of conical laminated shells can be analyzed easily. By increasing the conical angle for both boundary conditions, the non-dimensional fundamental frequency decreased uniformly because, by increasing the conical angle, a decrease occurred in axial stiffness of the conical panel. By increasing the length to radius ratio, fundamental frequency decreased rapidly because, by decreasing the length to radius ratio, the sandwich shell became stiffer. By increasing the core to shell thickness ratio, resulted in the fundamental frequency increase because, by increasing the core to panel thickness ratio, the flexibility of the structure decreased. The results show that, by increasing the core radius to shell thickness ratio, the fundamental frequency decreased. It means that for larger core radius to shell thickness ratio, the structure would be more flexible. Using standard optimization programs like the commercial Genetic algorithm software, one can optimize the design parameters. The present approach can be linked with the standard optimization programs and it can be used in the iteration process of the structural optimization.

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CC

Nomenclature

E_1, E_2	Young's modules
$G_{x\theta}$,G_{xz},$G_{\theta z}$	Shear modules
H	Thickness of conical sandwich shells
h_t, h_c, h_b	Thickness of the top face sheet, core and bottom face sheet, respectively.
$I_n^i \ (i=t,b,c)$	The moments of inertia of the top and bottom face sheets and the core
K_s	Shear correction factor
M_z^c	Out-of-plane bending moment resultant of the core per unit length of the cross-section of the conical panel
$M_{x\theta}^i, M_{\theta x}^i, M_{xx}^i, M_{\theta\theta}^i$	Bending and twisting moments resultants per unit length of the cross-section of the conical panel (i=t,b)
$M_{nxx}^c, M_{nx\theta}^c, M_{n\theta\theta}^c, M_{n\theta x}^c, M_{nxz}^c, M_{n\theta z}^c, M_{nxz}^{*c}, M_{n\theta z}^{*c}$	Bending and twisting moments resultants per unit length of the cross-section of the conical panel (n=1,2,3)
$N_{x\theta}^i, N_{\theta x}^i, N_{xx}^i, N_{\theta\theta}^i$	In-plane force resultants per unit length of the cross-section of the conical panel (i=t,b)
$N_{qq}^c, N_{qx}^c, N_{qz}^c, N_{xx}^c, N_{xq}^c, N_{xz}^c, N_{qz}^{*c}$	In-plane force resultants per unit length of the cross-section of the conical panel
Q_{ij}	Laminate stiffness referred to the principal material coordinates
$\bar{Q}_{ij}$	Transformed stiffness
Q_{xz}^i, Q_{qz}^i	Transverse shear force resultants (i=t,b)
R_{xi}, R_{xb}, R_{xc}	Principal radius of middle surface of the top and bottom face sheets and the core
R_z^c	Normal force resultant of the core per unit length of the cross-section of the conical panel
u_k^c, v_k^c, w_k^c	Unknowns of the in-plane and out of plane displacements of the core (k=0,1,2,3)
u_c, v_c, w_c	Displacement components of the core
u_0^i, v_0^i, w_0^i	Displacement components of the face-sheets, (i = t, b)
z_t , z_b ,z_c	Normal coordinates in the mid-plane of the top and bottom face-sheets and the core

Greek letters

ρ_t, ρ_b, ρ_c	Material densities of the face-sheets and the core
δ	Variational operator
n_{12}, n_{21}	Poisson's ratio
σ_{ii}^j	Normal stress in the face sheets, (i=x,y), j=(t,b)
σ_{ii}^c	Normal stress in the core, (i=x,y,z)
$\tau_{x\theta}^j, \tau_{xz}^j, \tau_{\theta z}^j$	Shear stress in the face sheets, j=(t,b)
$\tau_{x\theta}^c$, τ_{xz}^c , $\tau_{\theta z}^c$	Shear stresses in the core

$\varepsilon_{0xx}^{i}, \varepsilon_{0x\theta}^{i}, \varepsilon_{0\theta\theta}^{i}, \varepsilon_{0xz}^{i}, \varepsilon_{0xz}^{i}$	The mid-plane strain components, (i=t,b)
$\varepsilon_{zz}^{c}, \varepsilon_{xx}^{c}, \varepsilon_{\theta\theta}^{c}$	Normal strains components of the core
$\gamma_{xz}^{c}, \gamma_{\theta z}^{c}, \gamma_{x\theta}^{c}$	Shear strains components of the core
ϕ_1, ϕ_2	Rotation of the normal section of mid-surface along x, θ
ϕ_n	Rotations about the transverse normal to the face sheets

Appendix A.

$$\chi_1^1=\frac{2}{h_c^2}I_2^c+\frac{4}{h_c^3}I_3^c-\frac{8I_4^c}{h_c^4}-\frac{16I_5^c}{h_c^5},\chi_1^2=\frac{2I_3^c}{h_c^2}+\frac{4I_4^c}{h_c^3}-\frac{8I_5^c}{h_c^4}-\frac{16I_6^c}{h_c^5},\chi_1^3=\frac{4I_4^c}{h_c^4}-\frac{16I_6^c}{h_c^6},$$
$$\chi_1^4=\frac{4I_4^c}{h_c^4}+\frac{16I_5^c}{h_c^5}+\frac{16I_6^c}{h_c^6},\chi_1^5=\frac{2h_bI_4^c}{h_c^4}-\frac{8h_bI_6^c}{h_c^6},\chi_1^6=-\frac{2h_tI_4^c}{h_c^4}-\frac{8h_tI_5^c}{h_c^5}-\frac{8h_tI_6^c}{h_c^6},$$

$$\chi_3^1=\frac{2}{h_c^2}\left(I_2^c+\frac{I_3^c}{R_c(x)}\right)+\frac{4}{h_c^3}\left(I_3^c+\frac{I_4^c}{R_c(x)}\right)-\frac{8I_4^c}{h_c^4}-\frac{8I_5^c}{h_c^4R_c(x)}-\frac{16I_5^c}{h_c^5}-\frac{16I_6^c}{R_c(x)h_c^5},$$
$$\chi_3^2=\frac{2I_3^c}{h_c^2}+\frac{4I_4^c}{h_c^3}-\frac{8I_5^c}{h_c^4}-\frac{16I_6^c}{h_c^5},\chi_3^3=\frac{4I_4^c}{h_c^4}-\frac{16I_6^c}{h_c^6},\chi_3^4=\frac{4I_4^c}{h_c^4}+\frac{16I_5^c}{h_c^5}\frac{16I_6^c}{h_c^6},\chi_3^5=\frac{2h_bI_4^c}{h_c^4}-\frac{8h_bI_6^c}{h_c^6},$$
$$\chi_3^6=-\frac{2h_tI_4^c}{h_c^4}-\frac{8h_tI_5^c}{h_c^5}-\frac{8h_tI_6^c}{h_c^6},$$

$$\chi_4^1=\frac{2}{h_c^2}\left(I_2^c+\frac{I_3^c}{R_c(x)}\right)-\frac{4}{h_c^3}\left(I_3^c+\frac{I_4^c}{R_c(x)}\right)-\frac{8I_4^c}{h_c^4}-\frac{8I_5^c}{h_c^4R_c(x)}+\frac{16I_5^c}{h_c^5}+\frac{16I_6^c}{R_c(x)h_c^5},$$
$$\chi_4^2=\frac{2I_3^c}{h_c^2}-\frac{4I_4^c}{h_c^3}-\frac{8I_5^c}{h_c^4}+\frac{16I_6^c}{h_c^5},\chi_4^3=\frac{4I_4^c}{h_c^4}-\frac{16I_5^c}{h_c^5}+\frac{16I_6^c}{h_c^6},\chi_4^4=\frac{4I_4^c}{h_c^4}-\frac{16I_6^c}{h_c^6},$$
$$\chi_4^5=\frac{2h_bI_4^c}{h_c^4}-\frac{8h_bI_5^c}{h_c^5}+\frac{8h_bI_6^c}{h_c^6},\chi_4^6=-\frac{2h_tI_4^c}{h_c^4}+\frac{8h_tI_6^c}{h_c^6}.$$

$$\chi_5^1=\frac{I_1^c}{h_c}+\frac{2I_2^c}{h_c^2}-\frac{4I_3^c}{h_c^3}-\frac{8I_4^c}{h_c^4},\chi_5^2=\frac{I_2^c}{h_c^2}+\frac{4I_3^c}{h_c^3}+\frac{4I_4^c}{h_c^4},\chi_5^3=-\frac{I_2^c}{h_c^2}+\frac{4I_4^c}{h_c^4},$$

$$\chi_6^1=-\frac{I_1^c}{h_c}+\frac{2I_2^c}{h_c^2}+\frac{4I_3^c}{h_c^3}-\frac{8I_4^c}{h_c^4},\chi_6^2=\frac{I_2^c}{h_c^2}-\frac{4I_3^c}{h_c^3}+\frac{4I_4^c}{h_c^4},\chi_6^3=-\frac{I_2^c}{h_c^2}+\frac{4I_4^c}{h_c^4},$$

$$\chi_7^1=-\frac{h_tI_2^c}{h_c^2}-\frac{2h_tI_3^c}{h_c^3}+\frac{4h_tI_4^c}{h_c^4}+\frac{8h_tI_5^c}{h_c^5},\chi_7^2=-\frac{h_tI_3^c}{h_c^2}-\frac{2h_tI_4^c}{h_c^3}+\frac{4h_tI_5^c}{h_c^4}+\frac{8h_tI_6^c}{h_c^5},$$
$$\chi_7^3=-\frac{2h_bI_4^c}{h_c^4}+\frac{8h_bI_6^c}{h_c^6},\chi_7^4=-\frac{2h_tI_4^c}{h_c^4}+\frac{8h_tI_5^c}{h_c^5}-\frac{8h_tI_6^c}{h_c^6},\chi_7^5=-\frac{h_th_bI_4^c}{h_c^4}+\frac{4h_th_bI_6^c}{h_c^6},$$
$$\chi_7^6=\frac{h_t^2I_4^c}{h_c^4}+\frac{4h_t^2I_5^c}{h_c^5}+\frac{4h_t^2I_6^c}{h_c^6},$$

$$\chi_8^1=\frac{h_bI_2^c}{h_c^2}-\frac{2h_bI_3^c}{h_c^3}-\frac{4h_bI_4^c}{h_c^4}+\frac{8h_bI_5^c}{h_c^5},\chi_8^2=\frac{h_bI_3^c}{h_c^2}-\frac{2h_bI_4^c}{h_c^3}-\frac{4h_bI_5^c}{h_c^4}+\frac{8h_bI_6^c}{h_c^5},$$
$$\chi_8^3=\frac{2h_bI_4^c}{h_c^4}-\frac{8h_bI_5^c}{h_c^5}+\frac{8h_bI_6^c}{h_c^6},\chi_8^4=\frac{2h_tI_4^c}{h_c^4}-\frac{8h_tI_6^c}{h_c^6},\chi_8^5=\frac{h_b^2I_4^c}{h_c^4}-\frac{4h_b^2I_5^c}{h_c^5}+\frac{4h_b^2I_6^c}{h_c^6},$$
$$\chi_8^6=-\frac{h_th_bI_4^c}{h_c^4}+\frac{4h_th_bI_6^c}{h_c^6},$$

$$\chi_9^1=-\frac{h_t}{h_c^2}\left(I_2^c+\frac{I_3^c}{R_c(x)}\right)-\frac{2h_t}{h_c^3}\left(I_3^c+\frac{I_4^c}{R_c}\right)+\frac{4h_tI_4^c}{h_c^4}+\frac{4h_tI_5^c}{R_c(x)h_c^4}+\frac{8h_tI_5^c}{h_c^5}+\frac{8h_tI_6^c}{R_c(x)h_c^5},$$
$$\chi_9^2=-\frac{h_tI_3^c}{h_c^2}-\frac{2h_tI_4^c}{h_c^3}+\frac{4h_tI_5^c}{h_c^4}+\frac{8h_tI_6^c}{h_c^5},\chi_9^3=-\frac{2h_tI_4^c}{h_c^4}+\frac{8h_tI_6^c}{h_c^6},\chi_9^4=-\frac{2h_tI_4^c}{h_c^4}+\frac{8h_tI_5^c}{h_c^5}-\frac{8h_tI_6^c}{h_c^6},$$
$$\chi_9^5=-\frac{h_th_bI_4^c}{h_c^4}+\frac{4h_th_bI_6^c}{h_c^6},\chi_9^6=\frac{h_t^2I_4^c}{h_c^4}+\frac{4h_t^2I_5^c}{h_c^5}+\frac{4h_t^2I_6^c}{h_c^6},$$

$$\chi_{10}^{1}=\frac{h_b}{h_c^2}\left(I_2^c+\frac{I_3^c}{R_c(x)}\right)-\frac{2h_b}{h_c^3}\left(I_3^c+\frac{I_4^c}{R_c(x)}\right)-\frac{4h_bI_4^c}{h_c^4}-\frac{4h_bI_5^c}{R_c(x)h_c^4}+\frac{8h_bI_5^c}{h_c^5}+\frac{8h_bI_6^c}{R_c(x)h_c^5},$$
$$\chi_{10}^{2}=\frac{h_bI_3^c}{h_c^2}-\frac{2h_bI_4^c}{h_c^3}-\frac{4h_bI_5^c}{h_c^4}+\frac{8h_bI_6^c}{h_c^5},\chi_{10}^{3}=\frac{2h_bI_4^c}{h_c^4}-\frac{8h_bI_5^c}{h_c^5}+\frac{8h_bI_6^c}{h_c^6},\chi_{10}^{4}=\frac{2h_tI_4^c}{h_c^4}-\frac{8h_tI_6^c}{h_c^6},$$
$$\chi_{10}^{5}=\frac{h_b^2I_4^c}{h_c^4}-\frac{4h_b^2I_5^c}{h_c^5}+\frac{4h_b^2I_6^c}{h_c^6},\chi_{10}^{6}=-\frac{h_th_bI_4^c}{h_c^4}+\frac{4h_th_bI_6^c}{h_c^6},$$

$$\chi_{11}^{1}=I_0^c-\frac{8I_2^c}{h_c^2}+\frac{16I_4^c}{h_c^4},\chi_{11}^{2}=I_1^c-\frac{8I_3^c}{h_c^2}+\frac{16I_5^c}{h_c^4},\chi_{11}^{3}=\frac{2I_2^c}{h_c^2}-\frac{4I_3^c}{h_c^3}-\frac{8I_4^c}{h_c^4}+\frac{16I_5^c}{h_c^5},$$
$$\chi_{11}^{4}=\frac{2I_2^c}{h_c^2}+\frac{4I_3^c}{h_c^3}-\frac{8I_4^c}{h_c^4}-\frac{16I_5^c}{h_c^5},\chi_{11}^{5}=\frac{h_bI_2^c}{h_c^2}-\frac{2h_bI_3^c}{h_c^3}-\frac{4h_bI_4^c}{h_c^4}+\frac{8h_bI_5^c}{h_c^5},$$
$$\chi_{11}^{6}=-\frac{h_tI_2^c}{h_c^2}-\frac{2h_tI_3^c}{h_c^3}+\frac{4h_tI_4^c}{h_c^4}+\frac{8h_tI_5^c}{h_c^5},$$

$$\chi_{12}^{1}=I_1^c-\frac{8I_3^c}{h_c^2}+\frac{16I_5^c}{h_c^4},\chi_{12}^{2}=I_2^c-\frac{8I_4^c}{h_c^2}+\frac{16I_6^c}{h_c^4},\chi_{12}^{3}=\frac{2I_3^c}{h_c^2}-\frac{4I_4^c}{h_c^3}-\frac{8I_5^c}{h_c^4}+\frac{16I_6^c}{h_c^5},$$
$$\chi_{12}^{4}=\frac{2I_3^c}{h_c^2}+\frac{4I_4^c}{h_c^3}-\frac{8I_5^c}{h_c^4}-\frac{16I_6^c}{h_c^5},\chi_{12}^{5}=\frac{h_bI_3^c}{h_c^2}-\frac{2h_bI_4^c}{h_c^3}-\frac{4h_bI_5^c}{h_c^4}+\frac{8h_bI_6^c}{h_c^5},$$
$$\chi_{12}^{6}=-\frac{h_tI_3^c}{h_c^2}-\frac{2h_tI_4^c}{h_c^3}+\frac{4h_tI_5^c}{h_c^4}+\frac{8h_tI_6^c}{h_c^5},$$

$$\chi_{13}^{1}=I_0^c+\frac{I_2^c}{R_c^2(x)}+\frac{2I_1^c}{R_c(x)}-\frac{8}{h_c^2}\left(I_2^c+\frac{I_3^c}{R_c(x)}\right)-\frac{8}{h_c^2R_c(x)}\left(I_3^c+\frac{I_4^c}{R_c(x)}\right)+\frac{16I_4^c}{h_c^4}+\frac{32I_5^c}{R_c(x)h_c^4}$$
$$+\frac{16I_6^c}{R_c^2(x)h_c^4},\chi_{13}^{2}=I_1^c+\frac{I_2^c}{R_c(x)}-\frac{8I_3^c}{h_c^2}-\frac{8I_4^c}{h_c^2R_c(x)}+\frac{16I_5^c}{h_c^4}+\frac{16I_6^c}{R_c(x)h_c^4},$$
$$\chi_{13}^{3}=\frac{2}{h_c^2}\left(I_2^c+\frac{I_3^c}{R_c}\right)-\frac{4}{h_c^3}\left(I_3^c+\frac{I_4^c}{R_c(x)}\right)-\frac{8I_4^c}{h_c^4}+\frac{16I_5^c}{h_c^5}-\frac{8I_5^c}{R_c(x)h_c^4}+\frac{16I_6^c}{R_c(x)h_c^5},$$
$$\chi_{13}^{4}=\frac{2}{h_c^2}\left(I_2^c+\frac{I_3^c}{R_c(x)}\right)+\frac{4}{h_c^3}\left(I_3^c+\frac{I_4^c}{R_c(x)}\right)-\frac{8I_4^c}{h_c^4}-\frac{16I_5^c}{h_c^5}-\frac{8I_5^c}{R_c(x)h_c^4}-\frac{16I_6^c}{R_c(x)h_c^5},$$
$$\chi_{13}^{5}=\frac{h_b}{h_c^2}\left(I_2^c+\frac{I_3^c}{R_c(x)}\right)-\frac{2h_b}{h_c^3}\left(I_3^c+\frac{I_4^c}{R_c(x)}\right)-\frac{4h_bI_4^c}{h_c^4}+\frac{8h_bI_5^c}{h_c^5}-\frac{4h_ch_bI_5^c}{R_c(x)h_c^5},$$
$$\chi_{13}^{6}=-\frac{h_t}{h_c^2}\left(I_2^c+\frac{I_3^c}{R_c(x)}\right)-\frac{2h_t}{h_c^3}\left(I_3^c+\frac{I_4^c}{R_c(x)}\right)+\frac{4h_tI_4^c}{h_c^4}+\frac{8h_tI_5^c}{h_c^5}+\frac{4h_ch_tI_5^c}{R_c(x)h_c^5}+\frac{8h_ch_tI_6^c}{R_c(x)h_c^6},$$

$$\chi_{14}^{1}=I_1^c+\frac{I_2^c}{R_c(x)}-\frac{8I_3^c}{h_c^2}-\frac{8I_4^c}{h_c^2R_c(x)}+\frac{16I_5^c}{h_c^4}+\frac{16I_6^c}{R_c(x)h_c^4},\chi_{14}^{2}=I_2^c-\frac{8I_4^c}{h_c^2}+\frac{16I_6^c}{h_c^4},$$
$$\chi_{14}^{3}=\frac{2I_3^c}{h_c^2}-\frac{4I_4^c}{h_c^3}-\frac{8I_5^c}{h_c^4}+\frac{16I_6^c}{h_c^5},\chi_{14}^{4}=\frac{2I_3^c}{h_c^2}+\frac{4I_4^c}{h_c^3}-\frac{8I_5^c}{h_c^4}-\frac{16I_6^c}{h_c^5},$$
$$\chi_{14}^{5}=\frac{h_bI_3^c}{h_c^2}-\frac{2h_bI_4^c}{h_c^3}-\frac{4h_bI_5^c}{h_c^4}+\frac{8h_bI_6^c}{h_c^5},\chi_{14}^{6}=-\frac{h_tI_3^c}{h_c^2}-\frac{2h_tI_4^c}{h_c^3}+\frac{4h_tI_5^c}{h_c^4}+\frac{8h_tI_6^c}{h_c^5},$$

$$\chi_{15}^{1}=I_0^c-\frac{8I_2^c}{h_c^2}+\frac{16I_4^c}{h_c^4},\chi_{15}^{2}=\frac{I_1^c}{h_c}+\frac{2I_2^c}{h_c^2}-\frac{4I_3^c}{h_c^3}-\frac{8I_4^c}{h_c^4},\chi_{15}^{3}=-\frac{I_1^c}{h_c}+\frac{2I_2^c}{h_c^2}+\frac{4I_3^c}{h_c^3}-\frac{8I_4^c}{h_c^4},$$

Appendix B.

Some stiffness matrix coefficients for SS B.Cs.

$$K(3,15) = -\frac{2q}{h_c^2}\left(-e_2^{c\theta\theta} + 4\frac{e_4^{c\theta\theta}}{h_c^2}\right)\pi T_4 - \frac{4}{h_c^3}\left(-e_3^{c\theta\theta} + 4\frac{e_5^{c\theta\theta}}{h_c^2}\right)\pi T_4 - \frac{2q}{h_c^2}\left[-H_3^{c\theta z} + 4\frac{H_5^{cxz}}{h_c^2}\right]\pi T_4 -$$
$$\frac{4q}{h_c^3}\left[-H_4^{c\theta z} + 4\frac{H_6^{cxz}}{h_c^2}\right]\pi T_4 + \frac{4q}{h_c^2}\left[-g_1^{c\theta z} + 4\frac{g_3^{cxz}}{h_c^2}\right]\pi T_3 + \frac{12q}{h_c^3}\left[-g_2^{c\theta z} + 4\frac{g_4^{cxz}}{h_c^2}\right]\pi T_3,$$

$$K(7,14) = \frac{h_t q\sin(\phi)}{h_c^2}\left(e_3^{c\theta\theta} - 4\frac{e_5^{c\theta\theta}}{h_c^2}\right)T_6 + \frac{2h_t q\sin(\phi)}{h_c^3}\left(e_4^{c\theta\theta} - 4\frac{e_6^{c\theta\theta}}{h_c^2}\right)\pi T_6$$
$$-\frac{qh_t}{h_c^2}\left[\alpha_m\left(g_3^{c\theta x} - 4\frac{g_5^{c\theta x}}{h_c^2}\right)\pi T_2 - \sin(\phi)\left(-H_3^{c\theta x} + 4\frac{H_5^{c\theta x}}{h_c^2}\right)\pi T_6\right]$$
$$-\frac{2qh_t}{h_c^3}\left[\alpha_m\left(g_4^{c\theta x} - 4\frac{g_6^{c\theta x}}{h_c^2}\right)\pi T_2 - \sin(\phi)\left(-H_4^{c\theta x} + 4\frac{H_6^{c\theta x}}{h_c^2}\right)\pi T_6\right],$$

$$K(8,9) = -\frac{h_t^2 q\sin(\phi)}{h_c^2}\left(\frac{e_4^{c\theta\theta}}{h_c^2} - 2\frac{e_5^{c\theta\theta}}{h_c^3}\right)\pi T_6 + \frac{2h_t^2 q\sin(\phi)}{h_c^3}\left(\frac{e_6^{c\theta\theta}}{h_c^2} - 2\frac{e_7^{c\theta\theta}}{h_c^3}\right)\pi T_6$$
$$-\frac{qh_t^2}{h_c^2}\left[\alpha_m\left(-\frac{g_4^{c\theta x}}{h_c^2} - 2\frac{g_5^{c\theta x}}{h_c^3}\right)\pi T_1 + \sin(\phi)\left(\frac{H_4^{c\theta x}}{h_c^2} + 2\frac{H_5^{c\theta x}}{h_c^3}\right)\pi T_6\right]$$
$$-\frac{2qh_t^2}{h_c^3}\left[\alpha_m\left(-\frac{g_5^{c\theta x}}{h_c^2} - 2\frac{g_6^{c\theta x}}{h_c^3}\right)\pi T_1 + \sin(\phi)\left(\frac{H_5^{c\theta x}}{h_c^2} + 2\frac{H_6^{c\theta x}}{h_c^3}\right)\pi T_6\right],$$

$$K(14,5) = -\frac{q}{h_c}\left(e_2^{c\theta\theta} - 4\frac{e_4^{c\theta\theta}}{h_c^2}\right)\pi T_4 - \frac{2}{h_c^2}\left(e_3^{c\theta\theta} - 4\frac{e_5^{c\theta\theta}}{h_c^2}\right)\pi T_4 +$$
$$\frac{q}{h_c}\left[\left(-H_2^{c\theta z} + 4\frac{H_4^{c\theta z}}{h_c^2}\right)\pi T_4 + \alpha_m^2\left(g_1^{c\theta z} - 12\frac{g_3^{c\theta z}}{h_c^2}\right)\pi T_3\right] +$$
$$\frac{2q}{h_c^2}\left[\left(-H_3^{c\theta z} + 4\frac{H_5^{c\theta z}}{h_c^2}\right)\pi T_4 + \alpha_m^2\left(g_2^{c\theta z} - 12\frac{g_4^{c\theta z}}{h_c^2}\right)\pi T_3\right],$$

Some mass matrix coefficients for SS B.C.s:

$$M(1,1) = \frac{\pi L}{2}\left[I_0^t + \left(\frac{4I_4^c}{h_c^4} + \frac{16I_5^c}{h_c^5} + \frac{16I_6^c}{h_c^6}\right)\right], M(2,12) = \frac{\pi L}{2}\left(\frac{2I_3^c}{h_c^2} - \frac{4I_4^c}{h_c^3} - \frac{8I_5^c}{h_c^4} + \frac{16I_6^c}{h_c^5}\right),$$
$$M(3,8) = \frac{\pi L}{2}\left(\frac{4I_4^c}{h_c^4} - \frac{16I_6^c}{h_c^6}\right), \qquad M(4,14) = \frac{\pi L}{2}\left(\frac{2I_3^c}{h_c^2} - \frac{4I_4^c}{h_c^3} - \frac{8I_5^c}{h_c^4} + \frac{16I_6^c}{h_c^5}\right),$$
$$M(5,15) = \frac{\pi L}{2}\left(\frac{I_1^c}{h_c} + \frac{2I_2^c}{h_c^2} - \frac{4I_3^c}{h_c^3} - \frac{8I_4^c}{h_c^4}\right), \quad M(6,6) = \frac{\pi L}{2}\left(\frac{I_2^c}{h_c^2} - \frac{4I_3^c}{h_c^3} + \frac{4I_4^c}{h_c^4}\right).$$

where:

$$T_1=\frac{2}{L}\int_0^L\frac{\cos(\alpha_m x)\cos(\alpha_p x)}{R_{c1}(x)}dx\,, T_2=\frac{2}{L}\int_0^L\frac{\cos(\alpha_m x)\cos(\alpha_p x)}{R_{c1}^2(x)}dx\,, T_3=\frac{2}{L}\int_0^L\frac{\sin(\alpha_m x)\sin(\alpha_p x)}{R_{c1}(x)}dx\,,$$

$$T_4=\frac{2}{L}\int_0^L\frac{\sin(\alpha_m x)\sin(\alpha_p x)}{R_{c1}^2(x)}dx\,, T_5=\frac{2}{L}\int_0^L\frac{\sin(\alpha_m x)\cos(\alpha_p x)}{R_{c1}(x)}dx\,, T_6=\frac{2}{L}\int_0^L\frac{\sin(\alpha_m x)\cos(\alpha_p x)}{R_{c1}^2(x)}dx\,,$$

$$T_7=\frac{2}{L}\int_0^L\frac{\cos(\alpha_m x)\sin(\alpha_p x)}{R_{c1}(x)}dx\,, T_8=\frac{2}{L}\int_0^L\frac{\cos(\alpha_m x)\sin(\alpha_p x)}{R_{c1}^2(x)}dx\,, T_9=\frac{2}{L}\int_0^L\frac{\sin(\alpha_m x)\cos(\alpha_p x)}{R_{c1}^3(x)}dx\,,$$

$$T_{10}=\frac{2}{L}\int_0^L\frac{\sin(\alpha_m x)\sin(\alpha_p x)}{R_{c1}^3(x)}dx\,, T_{11}=\frac{2}{L}\int_0^L\frac{\cos(\alpha_m x)\sin(\alpha_p x)}{R_{c1}^3(x)}dx\,, T_{12}=\frac{2}{L}\int_0^L\frac{\sin(\alpha_m x)\sin(\alpha_p x)}{R_{c1}^4(x)}dx\,,$$

$$T_{13}=\frac{2}{L}\int_0^L\frac{\cos(\alpha_m x)\sin(\alpha_p x)}{R_{c1}^4(x)}dx\,, T_{14}=\frac{2}{L}\int_0^L\frac{\cos(\alpha_m x)\cos(\alpha_p x)}{R_{c1}^3(x)}dx\,, T_{15}=\frac{2}{L}\int_0^L\frac{\cos(\alpha_m x)\cos(\alpha_p x)}{R_{c1}^4(x)}dx\,,$$

$$T_{16}=\frac{2}{L}\int_0^L\frac{\sin(\alpha_m x)\cos(\alpha_p x)}{R_{c1}^4(x)}dx\,, T_{3i}=\frac{2}{L}\int_0^L\frac{\sin(\alpha_m x)\sin(\alpha_p x)}{R_{i1}(x)}dx\,, T_{4i}=\frac{2}{L}\int_0^L\frac{\sin(\alpha_m x)\sin(\alpha_p x)}{R_{i1}^2(x)}dx\,,$$

$$T_{7i}=\frac{2}{L}\int_0^L\frac{\cos(\alpha_m x)\sin(\alpha_p x)}{R_{i1}(x)}dx\,, j_{\mathrm{m}i}=\int_0^L\cos(\alpha_m x)\sin(\alpha_p x)dx\,, \qquad i=t,b$$

$$\int_0^L\cos(\alpha_m x)\cos(\alpha_p x)dx=\int_0^L\sin(\alpha_m x)\sin(\alpha_p x)dx=\frac{L}{2},$$

$$\int_0^L\cos(n\theta)\cos(q\theta)dx=\int_0^L\sin(n\theta)\sin(q\theta)dx=\pi,$$