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Efficient methods for integrating weight function: a comparative analysis

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Abstract. This paper introduces Romberg-Richardson's method as one of the numerical integration tools for computation of stress intensity factor in a pre-cracked specimen subjected to a complex stress field across the crack faces. Also, the computation of stress intensity factor for various stress fields using existing three methods: average stress over interval method, piecewise linear stress method, piecewise quadratic method are modified by using Richardson extrapolation method. The direct integration method is used as reference for constant and linear stress distribution across the crack faces while Gauss-Chebyshev method is used as reference for nonlinear distribution of stress across the crack faces in order to obtain the stress intensity factor. It is found that modified methods (average stress over intervals-Richardson method, piecewise linear stress-Richardson method, piecewise quadratic-Richardson method) yield more accurate results after a few numbers of iterations than those obtained using these methods in their original form. Romberg-Richardson's method is proven to be more efficient and accurate than Gauss-Chebyshev method for complex stress field.

Keywords: pre-cracked specimen; stress intensity factor; weight function; mode-I loading; numerical integration

1. Introduction

To determine the stress intensity factor (SIF) for a pre-cracked specimen, the standard solutions available in the handbook (Tada *et al*. 2000) are derived for simple geometrical conditions and stress distributions while the actual condition may not be the same. Because of the fact that the real environment is influenced by the actual crack configurations subjected to complex stress field, more effective tools for calculating stress intensity factor should be explored. The major contribution in this field was made by Rice (1968a, b, 1972, 1974) whose work in the area of elasticity and plasticity opened the door for computing stress intensity factor, strain energy rate and *J*-integral.

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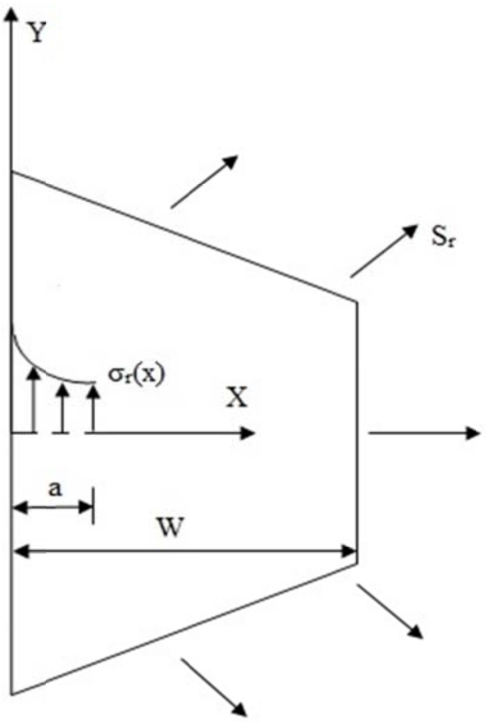
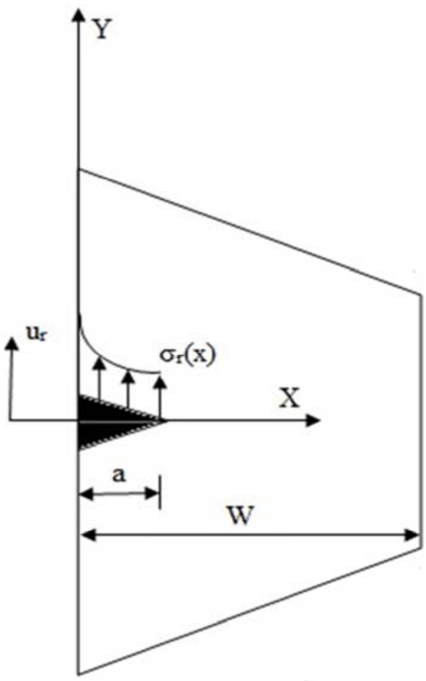
Further advancement in this field took place by the work of Rice and Rosengren (1968) which helped in predicting the relationship between crack length and size of plastic zone according to which the crack length was detected to be much larger than the size of plastic zone. Also, material deformation considering plastic behavior was also analyzed by Drucker and Rice (1970). A milestone in this area was the work of Bueckner (1970) for computation of SIF by introducing the method of weight function. This proved to be a very helpful and flexible method of calculating stress intensity factors because it requires a weight function for the cracked specimen for any loading system applied to the specimen. Also, this method can be applied to a variety of crack configurations, especially the cracks subjected to non-uniform stress fields. Because of uniqueness of the weight function method, the method of deriving the weight function proposed by Petroski and Achenbach (1978) started gaining a lot of attention. However, it was observed that Petroski-Achenbach's (1978) method was limited to constant and linear stress distribution of stress along the crack faces. Gorner *et al*. (1985) showed that Petroski-Achenbach's crack opening displacement (COD) function could be used to limited cases where the reference stress intensity factor was known which was further improved by Niu and Glinka (1987). Extending this method to structures under different boundary conditions, computation of SIF using superposition technique and weight function method was presented by Aaghaakouchak *et al*. (1990). Further advancement in this field was made by Glinka and Shen (1991) who showed that same common form of weight function can be used for a variety of crack configuration and thus weight function is having universal characteristics irrespective of the crack configuration. Later, the investigation on the use of the weight function method increased drastically as many authors used it for different cases. Niu and Glinka (1990) used the weight function method to compute the SIF for cracks in flat plates and plates having corners which showed that this COD function can be applied to semi-elliptical surface cracks but justifying the weight functions for deepest point on crack. Since it is important to know the weight function parameters in order to define the complete form of weight function, Shen and Glinka (1991) showed the method of two reference intensity factors and weight function characteristics to get weight function parameters. Shen *et al*. (1991) used two reference stress intensity factors and general form of weight function to compute SIF for surface of semi-elliptical crack in an infinitely wide plate. Zheng *et al*. (1995) used the weight function method to determine SIF for an internal semielliptical crack in a thick cylinder considering fixed ratio of inner radius to wall thickness, but the weight function parameter was calculated using two reference stress intensity factors given by Shiratori and Miyoshi (1992). The weight function approach with the indirect boundary integral method was also used by Lee and Hong (1996). Ferahi and Meguid (1998) used a new approach for getting SIF by discretized initial weight function and getting SIF through finite element calibration for crack emanating from a semi-circular edge notch. Finite element analysis was used by Pastrama and Castro (1998) on the displacement of crack faces to show that SIF for any loading can be obtained from known solutions for one loading system. Ng and Lau (1999) used another form of weight function to determine the stress intensity factor for through cracked specimens. Fett and Bahr (1999) came up with new approach of Boundary Collocation Method for computing SIF. In the study, it was shown that stress intensity factors and weight functions depend on Poisson's ratio for pure displacement condition but independent for mixed boundary conditions at end of plate for mode-I case. Fett (2001) used boundary collocation method to compute SIF with weight function and *T*-stress with Green's function for internally cracked specimen under different boundary conditions. Finite element analysis was carried out by Jones and Rothwell (2001) to obtain SIF solutions for internal cracks in cylinder component. Li *et al*. (2001) employed a new technique of Laplace inversion to obtain dynamic SIF of a finite crack in an unbounded functionally graded material under the influence of an antiplane shear loading. Finite element method was used by Rubio-Gonzalez and Mason (2001) to compute the SIF for any cracked configuration under any loading provided that complete solution for one loading is known. A new approach, i.e., modeling approach of crack was presented by Kim *et al*. (2003) who used crack-bridging model to model partially patched crack plate with weight function for computing stress intensity factor. Mattoni and Zok (2003) presented a method which involved the use of crack mouth opening displacement, COD and available SIF solution for uniform tension and pure bending of single edge notched specimen. Fett *et al*. (2004) also used the weight function for the kinked semi-infinite cracked specimens in their study. Lira-Vergara and Rubio-Gonzalez (2005) employed Laplace and Fourier transforms to solve equations of motions which are used to get dynamic stress intensity factor for orthotropic material. Shahani and Nabavi (2006) took the problem of steady state thermo-elasticity in an internally axial cracked semi-elliptical thick-walled cylinder by directing thermal and mechanical boundary conditions. Jankowiak *et al*. (2009) presented a new method of point load weight function for calculating SIF and also analyzed crack growth of planar crack under mode-I loading. Closed-form thermal stress intensity factors for an internal circumferential crack in cylinders with a variety of ratios of external to internal radii were derived by Nabavi and Ghajar (2010) using the weight function method. Das *et al*. (2011) considered the problem of an edge crack under normal point loading to the surface of a two orthotropic strip stacked together having finite thickness in plane strain condition using Hilbert transform technique to compute SIF and expressing displacements and stress in terms of harmonic functions. Ghajar and Saeid Googarchin (2012) employed general point load weight function for plates with finite thickness of semi-elliptical crack to calculate stress intensity factors for any point along the crack front for two dimensional cases. Since the weight function method has been applied to a wide range of problems, it is necessary to develop some technique so that stress intensity factors can be computed with efficiency for cracks in complex stress fields using this approach.

Due to singularity problem at the integral boundary, the SIF can be obtained using specialized numerical integration i.e., Gauss-Chebyshev method (Yang *et al*. 2005) for any kind of stress field available across the crack faces. Further, closed form solution for the SIF can be obtained using universal weight function for linear and constantly varying stress field, whereas the same is not possible for non-linear stress field acting across the crack faces. Anderson and Glinka (2006) came up with a powerful technique to integrate the weight functions for any type stress fields. They proposed three methods, i.e., average stress over interval, piecewise linear and piecewise quadratic method to calculate the stress intensity factor of nonlinear stress fields. Average stress over interval is much easier to implement, but less accurate than the other two while piecewise linear method is modest among three and provides more accurate results than average stress over intervals method. The piecewise quadratic method provides most accurate result out of three methods, though it has a disadvantage i.e., difficulty in its implementation. In light of above facts, this paper is aimed at reducing the errors in the above existing three methods (average stress over interval, piecewise linear and piecewise quadratic method) by modifying them with the help of Richardson's extrapolation method (1911). Thus, even the least accurate method of those three (average stress over the interval) can be used with good accuracy for calculating the SIF. Also, Romberg-Richardson's numerical integration method is introduced in the present investigation for determining the SIF. The results are systematically compared. The direct integration method is used as reference for constant and linear stress distribution across the crack faces while Gauss-Chebyshev method is used as reference for nonlinear distribution of stress across the crack faces.



(a) Applied nominal reference stress system S_r and corresponding stress field $\sigma_r(x)$ in prospective crack plane



(b) Local stress field $\sigma_r(x)$and corresponding crack opening displacement function $u_r(x,a)$

Fig. 1 Stress systems S_r, $\sigma_r(x)$ and displacements $u_r(x,a)$ required for deriving weight function

2. Stress intensity factors using the weight function method

The principle of superposition holds true for weight function technique which eases the effort of calculating stress intensity factors. According to this principle, it can be shown from Fig. 1 that the stress intensity factor for a cracked specimen under the influence of any external load S is the same as the stress intensity factor in a geometrically identical body subjected to local stress field $\sigma(x)$ applied to the crack faces. The local stress field $\sigma(x)$, induced in the prospective crack plane, can be easily predicted due to action of the external load S for an uncracked specimen.

2.1 Background

According to the weight function method, if weight function for a particular cracked body is known, then stress intensity factor for any loading system applied to the body can be calculated by simply integrating, the product of the weight function $m(x,a)$ and the stress field $\sigma(x)$ in the prospective crack plane (Fig. 1), over the entire length of crack a. Mathematically, the SIF can be given as

$$K = \int_0^a \sigma(x) m(x,a) dx \tag{1}$$

Buekner (1970) and Rice (1972) showed that weight function can be predicted in terms of crack opening displacement function u_r and reference stress intensity factor K_r (Fig.1) which can be expressed as

$$m(x,a)=\frac{H}{K_r}\frac{\partial u_r}{\partial a} \tag{2}$$

where, H is Generalized modulus of elasticity whose value is E for plain stress and $E/(1-\upsilon^2)$ for plain strain and υ is Poisson's ratio. Petroski and Achenbach (1978) provided an approximate crack opening displacement function which can be given as

$$u_r(x,a)=\frac{\sigma_0}{H\sqrt{2}}\left\{\frac{4K_r\sqrt{a(1-\zeta)}}{\sigma_0\sqrt{\pi}}+Ga(1-\zeta)^{3/2}\right\} \tag{3}$$

If the reference stress intensity factor K_r is known, the only unknown function G in Eq.(3) can be derived from self consistency of Eq. (1) by putting, $K=K_r$, $\sigma(x)=\sigma_r(x)$, and $\sigma_r(x)=\sigma_0\,p(x)$ and substituting Eq. (2) for $m(x,a)$. In which, ζ is the ratio of distance at which load is applied to the crack length i.e., x/a and $p(x)$ is the non-dimensionalized stress distribution function along the crack length. This results into the expression of weight function in the form of Eq. (4) (Niu and Glinka 1987,1990).

$$m(x,a)=\frac{2}{\sqrt{2\pi(a-x)}}\left[1+M_1\left(1-\frac{x}{a}\right)+M_2\left(1-\frac{x}{a}\right)^2\right] \tag{4}$$

In case of multiple loading, when there are multiple forces applied at a distance x from the crack-tip to the crack face, the solution of the stress intensity factor can be simplified by using the weight function method. However, in the present study, the cases of multiple loading are not considered.

2.2 Universal weight function for one-dimensional cracks

Sha and Yang (1986) introduced a new form of weight function as expressed in Eq. (5), because of inaccuracy in results predicted while using Eq. (4). Further, Glinka and Shen (1991) found that one universal weight function expression can be used to approximate weight functions for a variety of geometrical configurations of cracked bodies with one dimensional cracks of Mode I type loading.

$$m(x,a)=\frac{2}{\sqrt{2\pi(a-x)}}\left[1+M_1\left(1-\frac{x}{a}\right)^{1/2}+M_2\left(1-\frac{x}{a}\right)+M_3\left(1-\frac{x}{a}\right)^{3/2}+.....+M_n\left(1-\frac{x}{a}\right)^{n/2}\right] \tag{5}$$

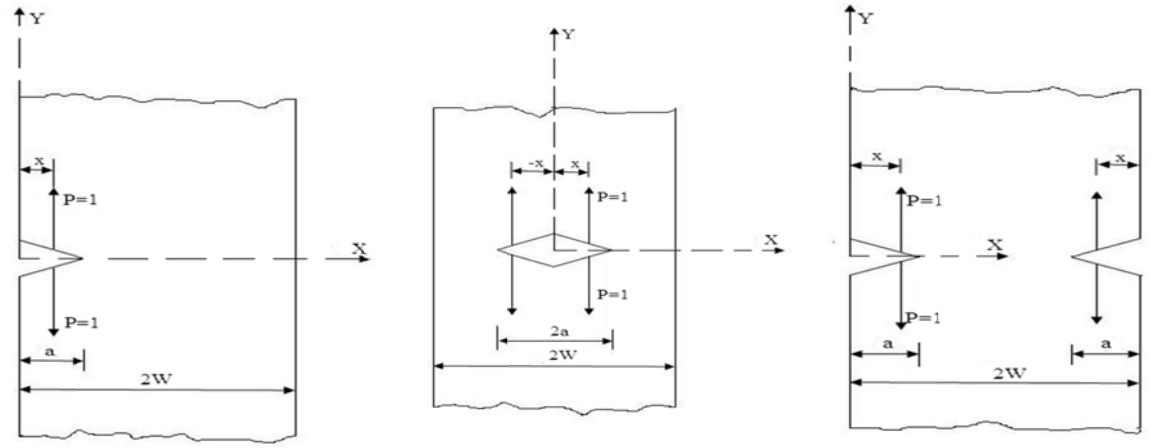
Considering the four terms in Eq. (5), the universal weight function is expressed as

$$m(x,a)=\frac{2}{\sqrt{2\pi(a-x)}}\left[1+M_1\left(1-\frac{x}{a}\right)^{1/2}+M_2\left(1-\frac{x}{a}\right)+M_3\left(1-\frac{x}{a}\right)^{3/2}\right] \tag{6}$$

In order to determine the universal weight function $m(x,\,a)$ of Eq. (6) for a particular cracked body, it is necessary to determine the three parameters M_1, M_2, and M_3. Because the form of universal weight function is the same for all cracks, the same integration procedure can be

Table 1 Weight function parameters for various cracked geometries

Weight function parameter (a/W=0.2)	Edge crack with finite width plate	Finite width plate with central crack	Double edge crack with finite width plate
M_1	−0.0832121	0.1122954	0.0624526
M_2	1.53307	−0.0693821	0.287762
M_3	0.263238	0.434237	0.460859



(a) Edge crack in finite width plate

(b) Central through crack in finite width plate

(c) (c) Double edge crack in finite width plate

Fig. 2 Loading and dimension of various specimen geometries

followed for calculating the stress intensity factors using Eq. (1).

Three specimen geometries i.e., edge crack with finite width plate, double edge crack with finite width plate, and finite width plate with central crack as shown in Fig. 2 are considered in the present study. Similar to the work of Glinka and Shen (1991), the weight function parameters for various specimens under consideration are computed by using least square fitting method for the given ratio of crack length to width of plate (*a/W*) as 0.2. These parameters are tabulated in Table1.

In order to determine SIF, Anderson and Glinka (2006) proposed three methods (Method-I: average stress over interval, Method-II: piecewise linear method, Method-III: piecewise quadratic method), for integrating the weight function by considering a power law of stress field $\sigma(x)=\sigma_h\zeta^h$ in Eq. (1). These methods are presented in subsequent para in which h shows number of intervals, ζ shows the ratio of distance at which load is applied to the crack length i.e., *x/a*.

Method- I (Average Stress Over Interval) Anderson and Glinka (2006) - In this method the stress is taken as the average over the interval ζ_{i-1} to ζ_i. Since the stress is considered as constant over each segment, h=0, indefinite integrals of weight function is given by

$$I_0 = \frac{\sigma_0}{3}\sqrt{\frac{a}{2\pi}}\left[6M_1(\zeta-1)+4M_2(\zeta-1)\sqrt{1-\zeta}+3M_3\left(2\zeta-\zeta^2\right)-12\sqrt{1-\zeta}\right] \quad (7)$$

Since stress is average over the interval i-1 to i, the average stress can be given by

$$\sigma_{0(i)} = \frac{\sigma_{i-1} + \sigma_i}{2} \tag{8}$$

The stress intensity factor can now be found by adding every segment

$$K_I = \sum_{i=1}^{2^{n+1}} \left\{ I_0(\zeta_i, \sigma_{0(i)}) - I_0(\zeta_{i-1}, \sigma_{0(i)}) \right\} \tag{9}$$

where, n shows number of iterations and vary from 0 to 7.

Method- II (Piecewise linear Method) Anderson and Glinka (2006) - The method discussed above can be more refined by considering the variation of stress distribution in linear way such that its normal stress can be given by

$$\sigma(\zeta) = \sigma_0 + \sigma_1 \zeta \tag{10}$$

where, σ_0 and σ_1 are constants which can be computed for segment i-1 to i by

$$\sigma_{0(i)} = \sigma_i - \frac{\zeta_i(\sigma_i - \sigma_{i-1})}{(\zeta_i - \zeta_{i-1})} \tag{11}$$

$$\sigma_{1(i)} = \frac{(\sigma_i - \sigma_{i-1})}{(\zeta_i - \zeta_{i-1})} \tag{12}$$

In this case the indefinite integral of weight function is given for n=1 as

$$I_1 = \frac{\sigma_1}{15}\sqrt{\frac{a}{2\pi}} \begin{bmatrix} 15 M_1 \left(\zeta^2 - 1\right) + 4 M_2 (3\zeta^2 - \zeta - 2)\sqrt{1-\zeta} \\ + 5 M_3 \zeta^2 \left(3 - 2\zeta\right) - 20(\zeta + 2)\sqrt{1-\zeta} \end{bmatrix} \tag{13}$$

Now, stress intensity factor can be given as

$$K_I = \sum_{i=1}^{2^{n+1}} \left\{ \left[I_0(\zeta_i, \sigma_{0(i)}) - I_0(\zeta_{i-1}, \sigma_{0(i)}) \right] + \left[I_0(\zeta_i, \sigma_{0(i)}) - I_0(\zeta_{i-1}, \sigma_{0(i)}) \right] \right\} \tag{14}$$

Method-III (Piecewise Quadratic Method) Anderson and Glinka (2006) - If quadratic polynomial in place of linear stress distribution is used to specify the stress variation, the stress function is expressed as

$$\sigma(\zeta) = \sigma_0 + \sigma_1 \zeta + \sigma_2 \zeta^2 \tag{15}$$

where σ_0, σ_1, and σ_2 are constants which can be computed for segment i-2 to i by

$$\sigma_{0(j)} = - \frac{\sigma_{i-2}(\zeta_{i-1}^2 \zeta_i - \zeta_{i-1}\zeta_i^2) + \sigma_{i-1}(\zeta_i^2 \zeta_{i-2} - \zeta_i \zeta_{i-2}^2) + \sigma_i(\zeta_{i-2}^2 \zeta_{i-1} - \zeta_{i-2}\zeta_{i-1}^2)}{(\zeta_i - \zeta_{i-1})(\zeta_{i-1}\zeta_i - \zeta_{i-2}\zeta_i - \zeta_{i-2}\zeta_{i-1} + \zeta_{i-2}^2)} \tag{16}$$

$$\sigma_{1(j)} = - \frac{\sigma_{i-2}(\zeta_i^2 - \zeta_{i-1}^2) + \sigma_{i-1}(\zeta_{i-2}^2 - \zeta_i^2) + \sigma_i(\zeta_{i-1}^2 - \zeta_{i-2}^2)}{(\zeta_i - \zeta_{i-1})(\zeta_{i-1}\zeta_i - \zeta_{i-2}\zeta_i - \zeta_{i-2}\zeta_{i-1} + \zeta_{i-2}^2)} \tag{17}$$

$$\sigma_{2(j)} = - \frac{\sigma_{i-2}(\zeta_{i-1} - \zeta_i) + \sigma_{i-1}(\zeta_i - \zeta_{i-2}) + \sigma_i(\zeta_{i-2} - \zeta_{i-1})}{(\zeta_i - \zeta_{i-1})(\zeta_{i-1}\zeta_i - \zeta_{i-2}\zeta_i - \zeta_{i-2}\zeta_{i-1} + \zeta_{i-2}^2)} \tag{18}$$

where $j=i/2$ and weight function integral for $n=2$ will be given by

$$I_2 = \frac{\sigma_2}{210}\sqrt{\frac{a}{2\pi}}\begin{bmatrix} 140\, M_1\zeta^3 + 8M_2(15\zeta^3 - 3\zeta^2 - 4\zeta - 8)\sqrt{1-\zeta} + \\ 7M_3\zeta^3(20 - 15\zeta) - 8(21\zeta^2 + 28\zeta + 56)\sqrt{1-\zeta} \end{bmatrix} \tag{19}$$

Hence, stress intensity factor will be

$$K_I = \sum_{j=1}^{J}\sum_{k=0}^{2}\left\{I_k(\zeta_i, \sigma_{k(j)}) - I_k(\zeta_{i-2}, \sigma_{k(j)})\right\} \tag{20}$$

where, $J=2^{n+1}/2$. Since, we are only taking even values so the method for odd intervals is not shown here. The method of integration of Eq. (1) using Gauss-Chebyshev and the introduction of a new method using Romberg-Richardson method is presented as the Method-IV and Method-V respectively.

Method- IV (Gauss-Chebyshev's Method)- Let us say there is a function $f(x)$ required to be integrated from a to b then using this method the integral can be obtained as

$$\int_a^b f(x)dx = u\sum_{i=1}^{10^n} w_i f(x_i)\sqrt{1-z_i^2} \tag{21}$$

where, w_i is weight function having a value of π/h, nodes: $x_i = (u_* z_i) + v$, $z_i = \cos\frac{(2i-1)\pi}{2h}$ in which $u = \frac{b-a}{2}, v = \frac{b+a}{2}$

Method-V (Romberg's Integration and Richardosn's Extrapolation)- Romberg-Richardson method is introduced in the present work for determining the stress intensity factor. This method takes into consideration the effect of singularity and can be divided into two types.

(a) Midpoint rule

$$R(n,0) = h_n\sum_{j=1}^{2^n} f(a + (j-0.5)h_n) \tag{22}$$

where $h_n = \frac{b-a}{2^n}$

(b) Nonlinear substitution in x

$$\int_a^b f(x)dx = \int_{-1}^{1} u_* l_* f(x)_* dg \tag{23}$$

where, $x = (u_* z) + v$, $z = \frac{g(3-g^2)}{2}$, $l = \frac{3(1-g^2)}{2}$, $u = \frac{b-a}{2}$, $v = \frac{b+a}{2}$

In present work, the nonlinear substitution method is used because of its higher accuracy over midpoint rule. Further, certain modification is also introduced in existing Method-I, Method-II, Method-III in their original form of integration using Richardson's extrapolation method. By and large, the above discussed methods (Method-I, Method-II, Method-III and Method-V) except Gauss-Chebyshev method (Method-IV) are coupled with Richardson's extrapolation method which reduces the error during the segments. The Richardson's extrapolation method is expressed as

$$R(n,k+1)=\frac{4^k R(n+1,k)-R(n,k)}{4^k-1} \tag{24}$$

where, n is the number of iterations and k is the length of element which will vary from 1 to n.

3. Results and discussion

The above mentioned methods (direct integration method, Gauss-Chebyshev method, Romberg-Richardson method, average stress interval- Richardson method, piecewise linear-Richardson method, piecewise quadratic- Richardson method) are applied to determine the stress intensity factor for three types of specimen geometries under consideration i.e., edge crack with finite width plate, central crack with finite width plate, double edge crack with the finite width plate. The value of *a/W* ratio is kept constant as 0.2 for all the specimen geometries. The results of non-dimensionalized stress intensity factors for different crack configurations and stress distributions across crack faces obtained using various methods are presented in Tables 2-4.

Table 2 Non-dimensional SIF obtained from Romberg-Richardson Method and Gauss-Chebyshev Method

Crack configuration	Non-dimensional stress intensity factor $K_I/\sigma_0\sqrt{a}$											
	Constant Loading $\sigma(x)=\sigma_0$				Linearly Varying Loading $\sigma(x)=\sigma_0(1-\zeta)$				Nonlinear Loading $\sigma(x)=\sigma_0(1+cos(\pi\zeta))$			
	Romberg-Richardson Method		Gauss-Chebyshev Method		Romberg-Richardson Method		Gauss-Chebyshev Method		Romberg-Richardson Method		Gauss-Chebyshev Method	
Number of iterations	n=0	n=7	n=0	n=7	n=0	n=7	n=0	n=7	n=0	n=7	n=0	n=7
Edge crack	3.047	2.449	3.191	2.449	1.523	1.058	1.595	1.058	3.047	2.092	3.191	2.092
Central crack	2.028	1.821	2.123	1.821	1.014	0.670	1.061	0.670	2.028	1.286	2.123	1.286
Double edge crack	2.286	1.982	2.394	1.982	1.143	0.771	1.197	0.771	2.286	1.497	2.394	1.497

Table 3 Non-dimensional SIF obtained using various modified methods for constant and linearly varying loading

Crack configuration	Non-dimensional stress intensity factor $K_I/\sigma_0\sqrt{a}$						
	Constant Loading $\sigma(x)=\sigma_0$			Linearly Varying Loading $\sigma(x)=\sigma_0(1-\zeta)$			
	Average stress interval-Richardson method	Piecewise linear-Richardson method	Piecewise quadratic-Richardson method	Average stress interval-Richardson method		Piecewise linear-Richardson method	Piecewise quadratic-Richardson method
Number of iterations	n=0 to 7	n=0 to 7	n=0 to 7	n=0	n=7	n=0 to 7	n=0 to 7
Edge crack	2.449	2.449	2.449	1.132	1.058	1.058	1.058
Central crack	1.821	1.821	1.821	0.764	0.670	0.670	0.670
Double cdge crack	1.982	1.982	1.982	0.860	0.771	0.771	0.771

Table 4 Non-dimensional SIF obtained using various modified methods for nonlinear loading

Crack configuration	Non-dimensional stress intensity factor $K_I/\sigma_0\sqrt{a}$					
	Nonlinear Loading $\sigma(x)=\sigma_0(1+cos(\pi\zeta))$					
	Average stress interval-Richardson method		Piecewise linear-Richardson method		Piecewise quadratic-Richardson method	
Number of iterations	n=0	n=7	n=0	n=7	n=0	n=7
Edge crack	2.265	2.092	2.116	2.092	2.116	2.092
Central crack	1.529	1.286	1.340	1.286	1.340	1.286
Double edge crack	1.720	1.497	1.542	1.497	1.542	1.497

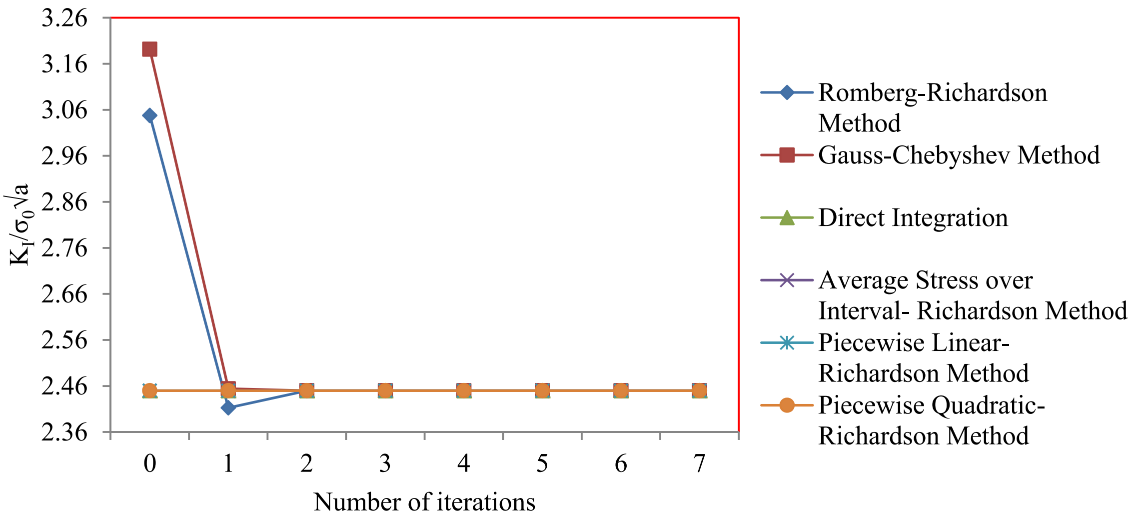


Fig. 3 Plot of non-dimensionalized parameter vs number of iterations for edge crack specimen subjected to constant loading

As such the following cases are studied.

Case-I (Constant Loading i.e., $\sigma(x)=\sigma_0$): The stress intensity factors determined for standard specimen geometries i.e., edge crack with finite width plate, central crack with finite width plate, double edge crack with the finite width plate subjected to constant stress distribution of $\sigma(x)=\sigma_0$ across the crack faces (Tables 2-3) are presented in Figs. 3-5 respectively. In these figures, the non-dimensioned parameter $K_I/\sigma_0\sqrt{a}$ versus number of iterations are plotted. The values of K_I obtained using direct integration (closed form solution) are considered as the reference for the comparison purpose. The legends shown in the figures are self explanatory.

From Fig. 3, it can be seen that at the start of iteration i.e., n=0, Gauss-Chebyshev method and Romberg-Richardson method yield the results with an error 30.28% and 24.41% respectively. Similarly, Fig. 4 shows that at the start of iteration, Gauss-Chebyshev method and Romberg-Richardson method yield the results with an error 16.585% and 11.33% respectively. Further, it can be observed from Fig. 5 that at n=0, Gauss-Chebyshev method and Romberg-Richardson method yield the results with an error 20.78% and 15.34% respectively. After 7 iterations i.e., n=7,

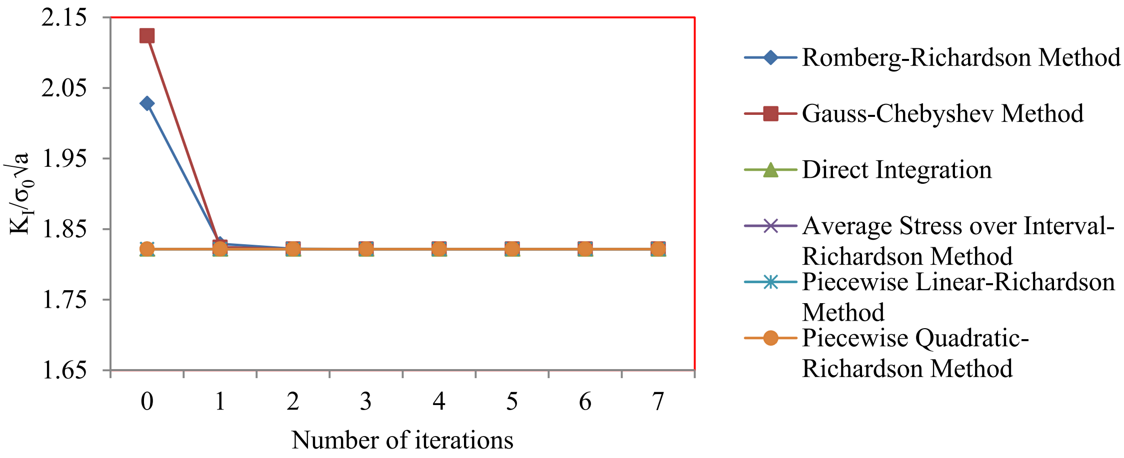


Fig. 4 Plot of non-dimensionalized parameter vs number of iterations for central crack specimen subjected to constant loading

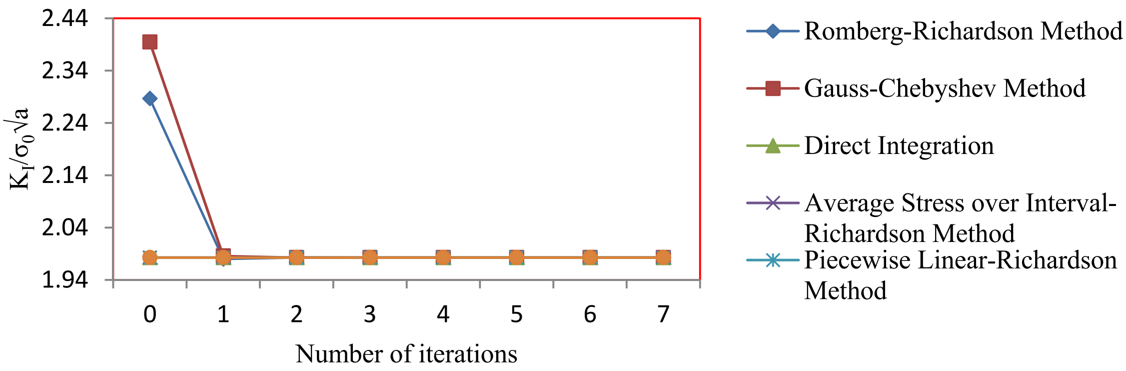


Fig. 5 Plot of non-dimensionalized parameter vs number of iterations for double edge crack specimen subjected to constant loading

both the methods (Gauss-Chebyshev method, Romberg-Richardson method) predict the result without any error for all the three specimen geometries. The figures also show that the other three methods (average stress interval- Richardson method, piecewise linear- Richardson method, piecewise quadratic-Richardson method) give the constant value of SIF throughout the iterations with insignificant error.

Case-II (Linearly Varying Loading i.e., $\sigma(x)=\sigma_0(1-\zeta)$): The stress intensity factors determined for standard specimen geometries i.e., edge crack with finite width plate, central crack with finite width plate, double edge crack with finite width plate subjected linearly varying stress distribution of $\sigma(x)=\sigma_0(1-\zeta)$ (Tables 2-3) are plotted in non-dimensional form through Figs. 6-8 respectively. In this case, the values of K_I obtained using direct integration method (closed form solution) are also considered as the reference.

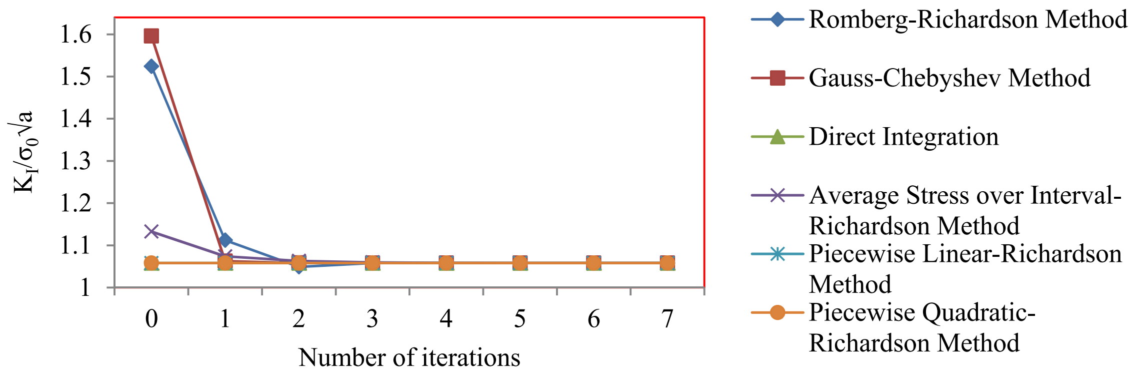


Fig. 6 Plot of non-dimensionalized parameter vs number of iterations for edge crack specimen subjected to linearly varying loading

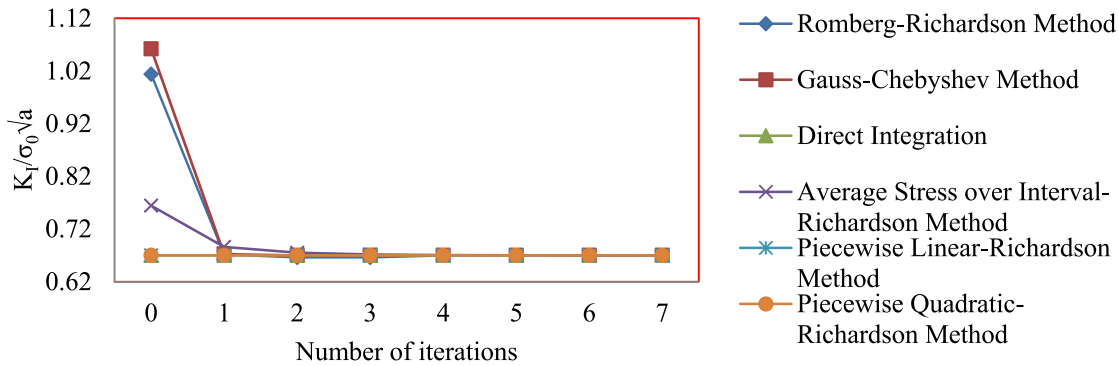


Fig. 7 Plot of non-dimensionalized parameter vs number of iterations for central crack specimen subjected to linearly varying loading

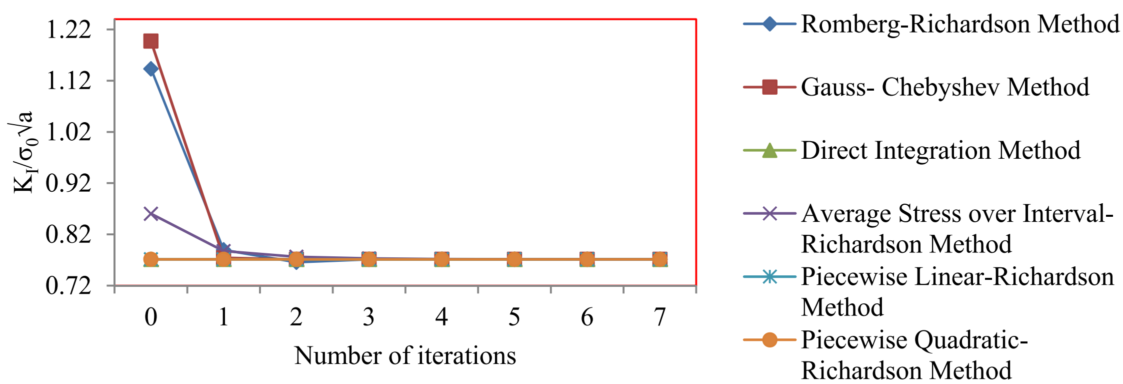


Fig. 8 Plot of non-dimensionalized parameter vs number of iterations for double edge crack specimen subjected to linearly varying loading

It can be seen from Fig. 6 that at the start of iteration, Gauss-Chebyshev method, Romberg-Richardson method and average stress interval- Richardson method yield the results with an error 50.84%, 44.04% and 7.041% respectively. Similarly, Fig. 7 shows that at the start of iteration, Gauss-Chebyshev method, Romberg-Richardson method and average stress interval-Richardson method yield the results with an error 58.478%, 51.335% and 14.101% respectively. Further, it can be observed from Fig. 8 that at n=0, Gauss-Chebyshev method, Romberg-Richardson method and average stress interval- Richardson method yield the results with an error 55.238%, 48.241% and 11.533% respectively. Figs. 6-8 show that after 7 iterations, all the five methods (Gauss-Chebyshev method, Romberg-Richardson method, average stress interval- Richardson method, piecewise linear- Richardson method, piecewise quadratic- Richardson method) predict the result without any error for the three specimen geometries under consideration.

Case-III-(Nonlinear Loading i.e., $\sigma(x)=\sigma_0(1+cos(\pi\zeta))$)

The stress intensity factors determined for standard specimen geometries i.e., edge crack with finite width plate, central crack with finite width plate, double edge crack with finite width plate subjected nonlinear stress distribution of $\sigma(x)=\sigma_0(1+cos(\pi\zeta))$ (Tables 2 and 4) are plotted in non-dimensional form through Figs. 9-11 respectively. In this case, the values of K_I obtained using Gauss-Chebyshev method after 7 iterations are considered as the reference.

It can be seen from Fig. 9 that at the start of iteration, Romberg-Richardson method, average stress interval- Richardson method, piecewise linear- Richardson method and piecewise quadratic-Richardson method yield the results with an error of 45.67%, 8.26%, 1.139% and 1.139% respectively. Similarly, Fig. 10 shows that at the start of iteration, Romberg-Richardson method, average stress interval- Richardson method, piecewise linear- Richardson method and piecewise quadratic- Richardson method yield the results with an error of 57.732%, 18.884%, 4.192% and 4.192% respectively. It can also be observed from Fig. 11 that at n=0, Romberg-Richardson

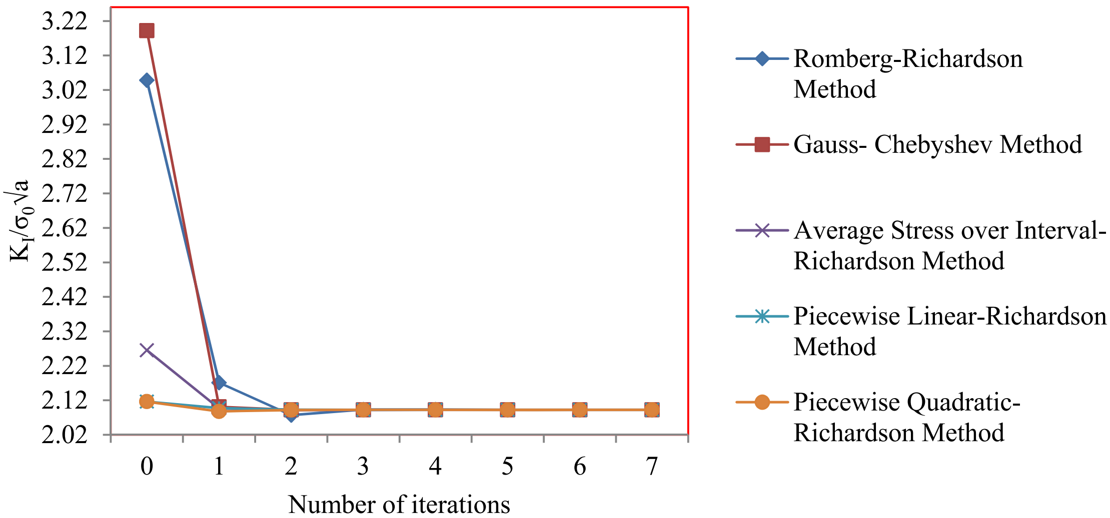


Fig. 9 Plot of non-dimensionalized parameter vs number of iterations for edge crack specimen subjected to nonlinear loading

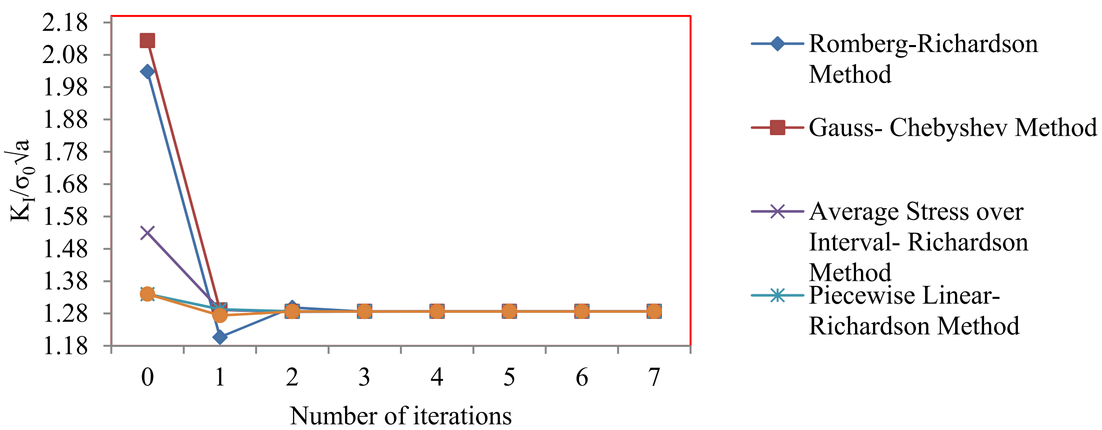


Fig. 10 Plot of non-dimensionalized parameter vs number of iterations for central crack specimen subjected to nonlinear loading

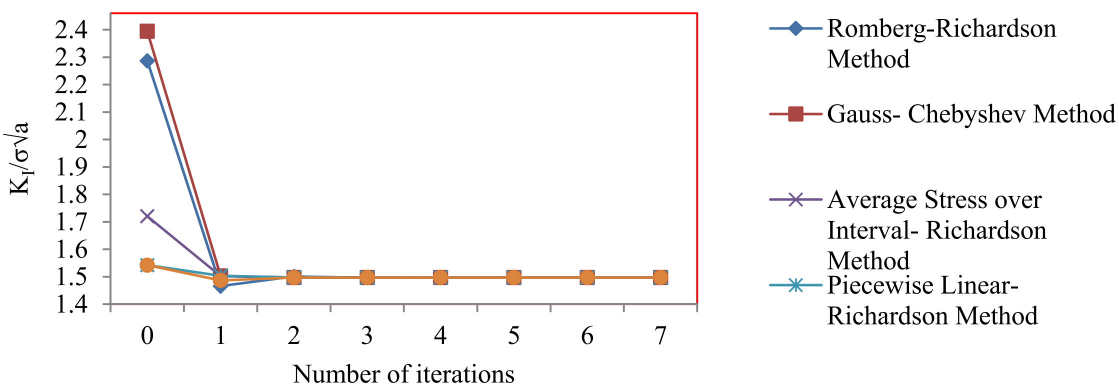


Fig. 11 Plot of non-dimensionalized parameter vs number of iterations for double edge crack specimen subjected to nonlinear loading

method, average stress interval- Richardson method, piecewise linear- Richardson method and piecewise quadratic - Richardson method yield the results with an error of 52.732%, 14.912%, 3.029% and 3.029% respectively. Further, Figs. 6-8 show that after 7 iterations, all the four methods (Romberg-Richardson method, average stress interval- Richardson method, piecewise linear- Richardson method, piecewise quadratic-Richardson method) predict the result with insignificant error for the three specimen geometries.

4. Conclusions

From the present study, the following conclusions can be drawn.

• For obtaining the closed form solution, the method of average stress over intervals is not the accurate method when compared to piecewise linear and piecewise quadratic method. But, modifying the existing methods (average stress over intervals, piecewise linear, and piecewise quadratic method) by using the Richardson's extrapolation method provides more accurate results than the methods in their original form. The Richardson's extrapolation method improves accuracy of all three methods and thus, average stress over interval method which is the weakest method among all the three methods can also be used with good accuracy for computing stress intensity factor even for non-linear distribution of stress along the crack faces.

• The Gauss-Chebyshev method provides an accurate result but its iteration time and requirement for number of iterations are higher. Hence, the Romberg's method with Richardson extrapolation is introduced in the present work which is an accurate method and it does not need those many numbers of iterations. Hence, it can be a powerful tool for determining stress intensity factor using numerical integration which reduces the effort of doing more number of iterations.

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