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An analytical study on the nonlinear vibration of a double-walled carbon nanotube

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Abstract. In this paper, free vibrations of a clamped-clamped double-walled carbon nanotube (DWNT) under axial force is studied. By utilizing Euler–Bernoulli beam theory, each layer of DWNT is modeled as a beam. In this analysis, nonlinear form of interlayer van der Waals (vdW) forces and nonlinearities aroused from mid-plane stretching are also considered in the equations of motion. Further, direct application of multiple scales perturbation method is utilized to solve the obtained equations and to analyze free vibrations of the DWNT. Therefore, analytical expressions are found for vibrations of each layer. Linear and nonlinear natural frequencies of the system and vibration amplitude ratios of inner to outer layers are also obtained. Finally, the results are compared with the results obtained by Galerkin method.

Keywords: double-walled; carbon nanotube; nonlinear vibrations; Galerkin method; Perturbation method

1. Introduction

Multi-walled nanotubes (MWNTs) consist of multiple concentric single-walled nanotubes (SWNTs). The mechanical properties of MWNTs are superior to the mechanical properties of SWNTs. As a result, MWNTs are preferred, in some applications. On the other hand, due to excessive use of nanotubes in vibration devices, modeling their vibration behavior is essential. A number of studies in this field use Molecular Dynamics (MD) approach for modeling nanotubes (Wang *et al*. 2014) , and others model nanotubes as continuous systems (Hajnayeb *et al*. 2011, Elishakoff *et al*. 2012, Taira *et al*. 2013).

In the previous studies, usually when the aspect ratio (length to diameter ratio) of the nanotube was high, each layer was modeled as a beam (Yang *et al*. 2010, Simsek 2011, Kiani 2013). Yoon *et al*. (2002) studied free vibrations of MWNTs by utilizing linear Euler-Bernoulli beam theory. The interlayer vdW force was simplified to a linear term, and the natural frequencies were calculated. Wang and Cai (2006) conducted the same study on MWNTs under compressive and tension form of stresses. The results showed significant influence of axial stress on the natural frequencies of the nanotubes. Wang *et al*. (2006) compared the results of linear Timoshenko and Euler–Bernoulli beam theories for modeling free vibrations of MWNTs with linear interlayer

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forces. Wang and Waradan (2006) studied free vibrations of SWNTs and double-walled nanotubes utilizing nonlocal continuum mechanics. These studies presented reasonable consistency with previous experimental works. Zhang *et al*. (2005) utilized linear Euler-Bernoulli beam theory to study free vibrations of a double-walled CNT under a compressive axial load. They computed the natural frequencies of inner and outer layers as a function of axial force and the geometrical parameters of the CNT. Elishakov and Pentaras (2009) used the same model and computed the analytical forms of natural frequencies of double-walled nanotubes utilizing Galerkin's Methods. In a comprehensive study, Xu *et al*. (2008) modeled a DWNT as two elastic beams with interlayer vdW forces acting between the beams. Dissimilar boundary conditions for each tube were considered and natural frequencies and mode shapes of the system were studied.

There are many reasons to consider nonlinearity for analyzing vibrations of NEMS. Low sensitivity vibration sensors cannot accurately sense small amplitude vibrations; thereby they should operate with higher amplitudes which would result in making their behavior nonlinear. Additionally, the common forces in nano scale such as vdW forces are highly nonlinear. In mass detection, it is proved that increasing the vibrations amplitudes increases mass sensing sensitivity (Buks and Yurke 2006). Higher amplitudes make system behavior nonlinear. Postma *et al*. (2005) identified the linear dynamic range of a fixed-fixed nanotube utilizing a nonlinear beam theory with stretching effect. The results showed a narrow linear range of operation and they concluded that nonlinear models for most of the applications are necessary. Dequens *et al*. (2004) after presenting vibration instabilities of nanotubes using continuum model and molecular dynamics model, utilized Euler-Bernoulli beam theory with stretching term to study a SWNT vibration. Fu *et al*. (2006) studied free vibrations of MWNTs embedded in an elastic medium. The Euler-Bernoulli beam theory with stretching term is utilized. The obtained equations were solved utilizing the Harmonic Balance Perturbation method. The elastic medium, vdW forces and aspect ratio effects on frequency responses were discussed. Xu *et al*. (2006) utilized linear Euler-Bernoulli beam theory and considered a Taylor expansion of interlayer vdW force to the first nonlinear term in order to model MWNTs vibrations. They utilized the Harmonic Balance Perturbation method to solve the obtained equations. Besseghier *et al*. (2015) studied the influence of chirality and surrounding medium on the nonlinear vibrations of a zigzag SWNT. They used Harmonic Balance mecthod in order to derive the amplitude-frequency relations. Fu *et al*. studied the static deflection of a DWNT under uniform static loadings on the outer tube (Fu *et al*. 2006). They compared the results of linear and nonlinear beam theories. Khosrozadeh and Hajabasi (2012) studied the free vibrations of embedded DWNTs considering a nonlinear interlayer force, as well. They utilized Harmonic Balance Perturbation method to solve the obtained equations. Yan *et al*. (2011) modeled a DWNT using a double-beam model. They studied the undamped free vibrations of the system, through a perturbation method. The shortcoming of their approach is that it is effective only in no damping condition. Additionally, the interlayer vdW force was simplified to a linear function. Hajnayeb and Khadem (2012) studied the nonlinear forced vibrations and stability of a DWNT under electrostatic actuation. The electrostatic actuation comprised of AC and DC voltages. The multiple scales method and a numerical method were used and the obtained results were compared.

In this paper, free vibrations of a clamped-clamped DWNT under axial force are studied, using a nonlinear Euler-Bernoulli beam theory. The stretching term is considered in the model; however, the nonlinear inertia terms are neglected. The Taylor expansion of nonlinear interlayer vdW forces is implemented and multiple scales perturbation method is used to analytically solve the coupled nonlinear equations. It is intended to find an expression for free vibrations of each layers of the

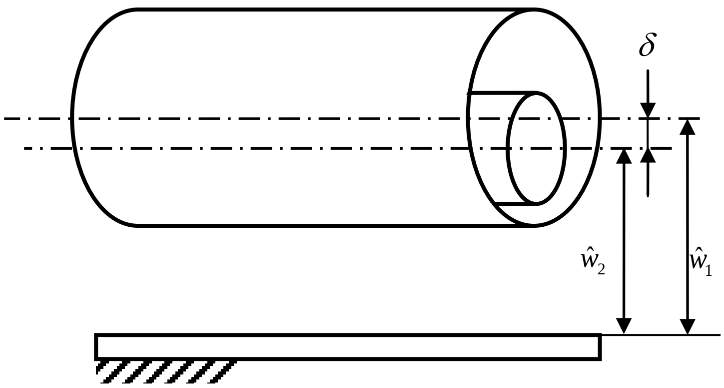

Fig. 1 The center position of the first and second layers of a DWNT and their distances

DWNT. The solutions are then compared to the solutions obtained by Galekin method. In order to verify the new results, the midpoint vibrations of the layers of arbitrary initial conditions are plotted for both methods. The influence of different parameters on the nonlinear natural frequencies of the system is also studied. These parameters include initial stress, the length of nanotube and mechanical damping. The values of obtained nonlinear natural frequencies for different values of initial stress and nanotube length are compared with previous works. Because the exact damping value is unknown in the DWNT, the changes in its nonlinear natural frequencies are studied for different values of damping coefficients in a diagram.

2. Equations of motion

As mentioned before, in nano scale, vdW forces have large values thereby have to be considered in the equations. In MWNTs, vdW forces are applied between each two neighboring layers, as well. These nonlinear forces make the vibrations of the adjacent layers of a MWNT coupled. Interlayer vdW forces, F_{int}, can be calculated in the same way as the forces between two concentric cylinders are calculated. Because the interlayer vdW force is an odd function of the interlayer distance, its Taylor expansions including the first two terms is (Xu *et al*. 2006)

$$F_{\mathrm{int}} = \hat{c}_1(\hat{w}_2 - \hat{w}_1) + \hat{c}_3(\hat{w}_2 - \hat{w})^3 \tag{1}$$

where $\hat{c}_1$ =71.11 GPa and $\hat{c}_3$ =2.57×10^4 GPa/nm^2, $\hat{w}_1$ and $\hat{w}_2$ or $\hat{w}_1(\hat{x},\hat{t})$ and $\hat{w}_2(\hat{x},\hat{t})$ are the center positions (at time $\hat{t}$ and at point $\hat{x}$) of the outer and inner layers, respectively (Fig. 1). Using Eq. (1) and applying Euler-Bernoulli beam theory with the stretching term and axial forces, the governing equations of the nonlinear free vibrations of a DWNT are written as

$$\hat{c}_3[\hat{w}_2 - \hat{w}_1]^3 + \hat{c}_1[\hat{w}_2 - \hat{w}_1] = EI_1 \frac{\partial^4 \hat{w}_1}{\partial \hat{x}^4} + \rho A_1 \frac{\partial^2 \hat{w}_1}{\partial \hat{t}^2} + \hat{\mu}_1 \frac{\partial \hat{w}_1}{\partial \hat{t}} - \left(\sigma A_1 + \frac{EA_1}{2L} \int_0^L \left(\frac{\partial \hat{w}_1}{\partial \hat{x}} \right)^2 d\hat{x} \right) \frac{\partial^2 \hat{w}_1}{\partial \hat{x}^2}$$

$$-\hat{c}_3[\hat{w}_2 - \hat{w}_1]^3 - \hat{c}_1[\hat{w}_2 - \hat{w}_1] = EI_2 \frac{\partial^4 \hat{w}_2}{\partial \hat{x}^4} + \rho A_2 \frac{\partial^2 \hat{w}_2}{\partial \hat{t}^2} + \hat{\mu}_2 \frac{\partial \hat{w}_2}{\partial \hat{t}} - \left(\sigma A_2 + \frac{EA_2}{2L} \int_0^L \left(\frac{\partial \hat{w}_2}{\partial \hat{x}} \right)^2 d\hat{x} \right) \frac{\partial^2 \hat{w}_2}{\partial \hat{x}^2} \tag{2}$$

In Eq. (2), $\hat{\mu}_1$ and $\hat{\mu}_2$ are damping coefficients of outer and inner tubes, respectively. $\hat{t}$ is time. A_1 and A_2 are the cross-sectional area of the outer and inner tubes, respectively. I_1 and I_2 are the Cross-section moments of inertia of the outer and inner tubes, respectively. ρ, E, L and σ are the density, modulus of elasticity, length and axial stress of the DWNT, respectively. Both ends of the DWNT are fixed.

Before solving the above equations, for convenience, they are rearranged to a dimensionless form. Therefore, the following dimensionless parameters are needed and defined as

$$\alpha_1 = \frac{A_1}{A_2},\ \alpha_2 = \frac{I_1}{I_2},\ \alpha_3 = \frac{\hat{\mu}_1}{\hat{\mu}_2},\ \alpha_4 = \frac{A_1 L^2}{2I_1},\ \mu_1 = \frac{\hat{\mu}_1 L^4}{EI_1 T},\ t = \frac{\hat{t}}{T},\quad x = \frac{\hat{x}}{L},$$
$$w_1 = \frac{\hat{w}_1}{L},\quad w_2 = \frac{\hat{w}_2}{L},\ c_1 = \frac{\hat{c}_1 L^4}{EI_1},\ c_3 = \frac{\hat{c}_3 L^6}{EI_1},\ N = \frac{\sigma A_1 L^2}{EI_1} \tag{3}$$

where T is defined as $T = \sqrt{\dfrac{\rho A_1 L^4}{EI_1}}$. Substituting the defined dimensionless parameters in the equations of motion, the following dimensionless equations are obtained

$$\frac{\partial^4 w_1}{\partial x^4} + \frac{\partial^2 w_1}{\partial t^2} + \mu_1 \frac{\partial w_1}{\partial t} - \left(N + \alpha_4 \int_0^1 \left(\frac{\partial w_1}{\partial x} \right)^2 dx \right) \frac{\partial^2 w_1}{\partial x^2} - c_1 [w_2 - w_1] - c_3 [w_2 - w_1]^3 = 0$$
$$\frac{\partial^4 w_2}{\partial x^4} + \frac{\alpha_2}{\alpha_1} \cdot \frac{\partial^2 w_2}{\partial t^2} + \mu_1 \cdot \frac{\alpha_2}{\alpha_3} \cdot \frac{\partial w_2}{\partial t} - \left(\frac{\alpha_2}{\alpha_1} N + \frac{\alpha_4 \alpha_2}{\alpha_1} \int_0^1 \left(\frac{\partial w_2}{\partial x} \right)^2 dx \right) \frac{\partial^2 w_2}{\partial x^2} + c_1 \alpha_2 [w_2 - w_1] + c_3 \alpha_2 [w_2 - w_1]^3 = 0. \tag{4}$$

Eliminating the nonlinear and damping terms, the linear equation of undamped free vibrations of the DWNT can be written as

$$\frac{\partial^4 w_1}{\partial x^4} + \frac{\partial^2 w_1}{\partial t^2} - N \frac{\partial^2 w_1}{\partial x^2} - c_1 [w_2 - w_1] = 0$$
$$\frac{\partial^4 w_2}{\partial x^4} + \frac{\alpha_2}{\alpha_1} \cdot \frac{\partial^2 w_2}{\partial t^2} - \frac{\alpha_2}{\alpha_1} N \frac{\partial^2 w_2}{\partial x^2} + c_1 \alpha_2 [w_2 - w_1] = 0. \tag{5}$$

To solve the above equations, w_1 and w_2 can be substituted by $w_1 = a_1 \phi(x) e^{i\omega t}$ and $w_2 = a_2 \phi(x) e^{i\omega t}$, where $\phi(x)$ is the mode shape of a clamped-clamped beam.

3. Perturbation method

In this paper, multiple scales perturbation method is applied directly to solve the problem. Because the system has cubic nonlinearities, the solution procedure is as follows; an approximate solution of system equations for small but finite amplitudes is assumed to be in the form of

$$w_1(x, T_0, T_2) = \varepsilon w_{11}(x, T_0, T_2) + \varepsilon^3 w_{13}(x, T_0, T_2) + \cdots \tag{6}$$

$$w_2(x,T_0,T_2)=\varepsilon w_{21}(x,T_0,T_2)+\varepsilon^3 w_{23}(x,T_0,T_2)+\cdots \tag{7}$$

where $T_0=t$ and $T_2=\varepsilon^2 t$. In the above equations, ε is a small dimensionless parameter. Because of the cubic nonlinearity, slow time T_1 and w_{12} terms are not included in the expansion (Nayfeh 1981). In order that the damping balances the effect of nonlinear terms, the damping coefficients must have appropriate orders, which make damping terms to be in the same perturbation equations as the nonlinear terms. So, damping coefficients, μ_i, are substituted by $\varepsilon^2\mu_i$ term. Substituting Eqs. (6) and (7) into Eq. (4) and collecting the coefficients of the same powers of ε, one can obtain

Order ε

$$\frac{\partial^4 w_{11}}{\partial x^4}+\frac{\partial^2 w_{11}}{\partial T_0^{\,2}}-N\frac{\partial^2 w_{11}}{\partial x^2}+c_1 w_{11}-c_1 w_{21}=0 \tag{8}$$

$$\frac{\partial^4 w_{21}}{\partial x^4}+\frac{\alpha_2}{\alpha_1}\frac{\partial^2 w_{21}}{\partial T_0^{\,2}}-\frac{\alpha_2}{\alpha_1}N\frac{\partial^2 w_{21}}{\partial x^2}+c_1\alpha_2 w_{21}-c_1\alpha_2 w_{11}=0 \tag{9}$$

Order ε^3

$$\begin{aligned}&\frac{\partial^4 w_{13}}{\partial x^4}+\frac{\partial^2 w_{13}}{\partial T_0^{\,2}}-N\frac{\partial^2 w_{13}}{\partial x^2}+c_1 w_{13}-c_1 w_{23}\\&=\left(\frac{\alpha_4\alpha_2}{\alpha_1}\int_0^1\left(\frac{\partial w_{11}}{\partial x}\right)^2 dx\right)\frac{\partial^2 w_{11}}{\partial x^2}+2\frac{\partial^2 w_{11}}{\partial T_2\partial T_0}+\mu_1\frac{\partial w_{11}}{\partial T_0}-c_3 w_{21}^3+3c_3 w_{21}^2 w_{11}+c_3 w_{11}^3-3c_3 w_{11}w_{11}^2\end{aligned} \tag{10}$$

$$\begin{aligned}&\frac{\partial^4 w_{23}}{\partial x^4}+\frac{\alpha_2}{\alpha_1}\frac{\partial^2 w_{23}}{\partial T_0^{\,2}}-\frac{\alpha_2}{\alpha_1}N\frac{\partial^2 w_{23}}{\partial x^2}+c_1\alpha_2 w_{23}-c_1\alpha_2 w_{13}\\&=\left(\frac{\alpha_4\alpha_2}{\alpha_1}\int_0^1\left(\frac{\partial w_{21}}{\partial x}\right)^2 dx\right)\frac{\partial^2 w_{21}}{\partial x^2}+2\frac{\partial^2 w_{21}}{\partial T_2\partial T_0}+\mu_1\frac{\alpha_2}{\alpha_3}\frac{\partial w_{21}}{\partial T_0}-c_3\alpha_2 w_{11}^3-3c_3\alpha_2 w_{21}^2 w_{11}+c_3\alpha_2 w_{21}^3+3c_3\alpha_2 w_{21}w_{11}^2\end{aligned} \tag{11}$$

Order ε equations are in fact, the linear undamped form of equations of motion, which were solved in the previous section. So, the solutions of those equations are

$$\begin{aligned}w_{11}&=\phi(x)\left(A_1(T_2)e^{i\omega_1T_0}+\bar{A}_1(T_2)e^{-i\omega_1T_0}+A_2(T_2)e^{i\omega_2T_0}+\bar{A}_2(T_2)e^{-i\omega_2T_0}\right)\\w_{21}&=\phi(x)\Delta_1\left(A_1(T_2)e^{i\omega_1T_0}+\bar{A}_1(T_2)e^{-i\omega_1T_0}\right)+\phi(x)\Delta_2\left(A_2(T_2)e^{i\omega_2T_0}+\bar{A}_2(T_2)e^{-i\omega_2T_0}\right)\end{aligned} \tag{12}$$

Substituting Eq. (12) into Eqs. (10)-(11) and collecting coefficients of $e^{i\omega T_0}$, one obtains

$$\begin{aligned}&\frac{\partial^4 w_{13}}{\partial x^4}+\frac{\partial^2 w_{13}}{\partial T_0^{\,2}}-N\frac{\partial^2 w_{11}}{\partial x^2}+c_1 w_{13}-c_1 w_{23}=G_1(x,T_2)e^{i\omega_1T_0}+H_1(x,T_2)e^{i\omega_2T_0}+C.C.+NST,\\&\frac{\partial^4 w_{23}}{\partial x^4}+\frac{\alpha_2}{\alpha_1}\frac{\partial^2 w_{23}}{\partial T_0^{\,2}}-\frac{\alpha_2}{\alpha_1}N\frac{\partial^2 w_{23}}{\partial x^2}+c_1\alpha_2 w_{23}-c_1\alpha_2 w_{13}=G_2(x,T_2)e^{i\omega_1T_0}+H_2(x,T_2)e^{i\omega_2T_0}+C.C.+NST\end{aligned} \tag{13}$$

where, *NST* stands for nonsecular terms (the terms without $e^{i\omega_1T_0}$ or $e^{i\omega_2T_0}$) and *C.C.* denotes the complex conjugate of the preceding terms. $H_1(x, T_2)$, $H_2(x, T_2)$, $G_1(x, T_2)$ and $G_2(x, T_2)$ are defined in appendix A.

According to (Nayfeh 1981), the solvability conditions would be

$$\int_0^1 \phi G_1(x,T_2)dx = 0 \tag{14}$$

$$\int_0^1 \phi G_2(x,T_2)dx = 0 \tag{15}$$

Eqs. (19)-(20) result in the following equations

$$2i\omega(Q_m A_1' + Z_m A_1) + 8S_m A_1^{\,2}\overline{A}_1 + 8\Lambda_m A_1 A_2 \overline{A}_2 = 0\,, \qquad m = 1,2 \tag{16}$$

Where, Q_1, Z_1, Λ_1, S_1, Q_2, Z_2, Λ_2 and S_2 are parameters defined in Appendix B.
If the polar form of A_1 and A_2 are substituted in Eq. (16), solving them would become more straight forward. Their polar forms are considered to be

$$A_1(T_2) = \frac{1}{2}a_1(T_2)e^{i\beta_1(T_2)}\,,\quad A_2(T_2) = \frac{1}{2}a_2(T_2)e^{i\beta_2(T_2)}\,. \tag{17}$$

Substituting the above polar forms in Eq. (16) and separating the result into real and imaginary parts, one obtains four differential equations for $a_2(T_2)$, $a_1(T_2)$, $\beta_2(T_2)$ and $\beta_1(T_2)$. By solving these four differential equations, $a_2(T_2)$, $a_1(T_2)$, $\beta_2(T_2)$ and $\beta_1(T_2)$ are obtained. Substituting the solutions into Eqs. (17)-(12), and then transforming the obtained expressions into triangular form, the total response of the system is computed as

$$\begin{aligned} w_{11} &= \phi C_1 e^{-\frac{1}{2}\frac{Z_1}{Q_1}T_2}\cos\!\left(\omega_1 T_0 - \frac{1}{8}\frac{S_1 C_1^{\,2} e^{-\frac{2Z_1}{Q_1}T_2}}{\omega_1 Z_1} - \frac{1}{8}\frac{\Lambda_1 Q_2 C_2^{\,2} e^{-\frac{2Z_2}{Q_2}T_2}}{\omega_1 Q_1 Z_2} + C_3\right) + \\ &\phi C_2 e^{-\frac{1}{2}\frac{Z_2}{Q_2}T_2}\cos\!\left(\omega_2 T_0 - \frac{1}{8}\frac{S_2 C_2^{\,2} e^{-\frac{2Z_2}{Q_2}T_2}}{\omega_2 Z_2} - \frac{1}{8}\frac{\Lambda_2 Q_1 C_1^{\,2} e^{-\frac{2Z_1}{Q_1}T_2}}{\omega_2 Q_2 Z_1} + C_4\right) \end{aligned} \tag{18}$$

$$\begin{aligned} w_{21} &= \phi \Delta_1 C_1 e^{-\frac{1}{2}\frac{Z_1}{Q_1}T_2}\cos\!\left(\omega_1 T_0 - \frac{1}{8}\frac{S_1 C_1^{\,2} e^{-\frac{2Z_1}{Q_1}T_2}}{\omega_1 Z_1} - \frac{1}{8}\frac{\Lambda_1 Q_2 C_2^{\,2} e^{-\frac{2Z_2}{Q_2}T_2}}{\omega_1 Q_1 Z_2} + C_3\right) + \\ &\phi \Delta_2 C_2 e^{-\frac{1}{2}\frac{Z_2}{Q_2}T_2}\cos\!\left(\omega_2 T_0 - \frac{1}{8}\frac{S_2 C_2^{\,2} e^{-\frac{2Z_2}{Q_2}T_2}}{\omega_2 Z_2} - \frac{1}{8}\frac{\Lambda_2 Q_1 C_1^{\,2} e^{-\frac{2Z_1}{Q_1}T_2}}{\omega_2 Q_2 Z_1} + C_4\right) \end{aligned} \tag{19}$$

The constant parameters, C_1, C_2, C_3 and C_4 are determined according to initial conditions of the system.

In these equations, the nonlinear natural frequencies are

$$\omega_1 + \frac{\beta_1(T_0)}{T_0} \text{ and } \omega_2 + \frac{\beta_2(T_0)}{T_0}\,. \tag{20}$$

Substituting $\beta_2(T_2)$ and $\beta_1(T_2)$ in Eq. (20), one can write coaxial nonlinear natural frequency (CNNF) and noncoaxial nonlinear natural frequency (NNNF) in the following form

$$CNNF = \left(-\frac{1}{8}\frac{S_1 C_1^{\,2} e^{-\frac{2Z_1}{Q_1}T_0}}{\omega Z_1} - \frac{1}{8}\frac{\Lambda_1 Q_2 C_2^{\,2} e^{-\frac{2Z_2}{Q_2}T_0}}{\omega Q_1 Z_2}\right)\frac{1}{T_0} + \omega_1$$

$$NNNF = \left(-\frac{1}{8}\frac{S_2 {C_2}^2 e^{-\frac{2Z_2}{Q_2}T_0}}{\omega Z_2} - \frac{1}{8}\frac{\Lambda_2 Q_1 {C_1}^2 e^{-\frac{2Z_1}{Q_1}T_0}}{\omega Q_2 Z_1} \right)\frac{1}{T_0} + \omega_2 \tag{21}$$

Because ε is a bookkeeping parameter, the final values of Eqs. (18)-(19) and (21) are obtained by substituting ε=1 in those equations.

4. Comparing with the Galerkin method

In order to verify the results, a comparison is made with Galerkins method (Thomsen 2003). According to this method, $w_1(x, t)$ and $w_2(x, t)$ are replaced with the product of their corresponding time functions, and a fixed-fixed beam mode shape

$$w_1(x,t) = \phi(x)P_1(t) \tag{22}$$

$$w_2(x,t) = \phi(x)P_2(t)\,. \tag{23}$$

In Eqs. (22) and (23) $\phi(x)$ is the first vibration mode shape of a clamped-clamped beam. It is necessary to mention that this approximation is acceptable for most of the initial conditions caused by the common forces (for example electrostatic forces) applied to the DWNTs in the nano scale. In these conditions, the percentage of participation of higher modes is not significantly lower than other vibration modes. Additionally, these forms of solutions also make it possible to extract the first two natural frequencies and mode shapes of the DWNT.

Substituting Eqs. (22)-(23) into Eq. (4), multiplying them by $\phi(x)$, and integrating both sides of equations on x from 0 to 1, two ordinary differential equations (ODEs) of $P_1(t)$ and $P_2(t)$ are obtained as

$$\begin{aligned}
&\left(\int_0^1 \phi\phi^{(4)}dx\right)P_1(t) + \left(\int_0^1 \phi^2 dx\right)\ddot{P}_1(t) + \mu_1\left(\int_0^1 \phi^2 dx\right)\dot{P}_1(t) - \left[N + \alpha_4 P_1^2(t)\int_0^1 \phi'^2 dx\right]\left(\int_0^1 \phi\phi'' dx\right)P_1(t) \\
&-c_1\left[P_2(t) - P_1(t)\right]\left(\int_0^1 \phi^2 dx\right) - c_3\left[P_2(t) - P_1(t)\right]^3\left(\int_0^1 \phi^4 dx\right) = 0 \\
&\left(\int_0^1 \phi\phi^{(4)}dx\right)P_2(t) + \frac{\alpha_2}{\alpha_1}\cdot\left(\int_0^1 \phi^2 dx\right)\ddot{P}_2(t) + \mu_1\cdot\frac{\alpha_2}{\alpha_3}\cdot\left(\int_0^1 \phi^2 dx\right)\dot{P}_2(t) - \left[\frac{\alpha_2}{\alpha_1}N + \frac{\alpha_4\alpha_2}{\alpha_1}P_2^2(t)\int_0^1 \phi'^2 dx\right]\left(\int_0^1 \phi\phi'' dx\right)P_2(t) \\
&+c_1\alpha_2\left[P_2(t) - P_1(t)\right]\left(\int_0^1 \phi^2 dx\right) + c_3\alpha_2\left[P_2(t) - P_1(t)\right]^3\left(\int_0^1 \phi^4 dx\right) = 0.
\end{aligned} \tag{24}$$

In Eq. (24), $\phi^{(4)} = \dfrac{d^4\phi}{dx^4}$ and $\phi=\phi(x)$ is considered to be the first vibration mode shape of a fixed-fixed beam and can be expressed as (Leissa and Qatu 2011)

$$\phi(x) = \left[\cos(4.73x) - \cosh(4.73x)\right] - 0.9825\left[\sin(4.73x) - \sinh(4.73x)\right]. \tag{25}$$

The resulting ODEs of Eq. (24) can be solved numerically using Runge-Kuta method. In the next section, the results of Galerkin method are compared with the results of multiple scales perturbation method.

5. Numerical example

In this section, free lateral vibrations of a DWNT with inner and outer diameter of d_{in}=0.7 nm and d_{out}=14 nm are analyzed (Yoon *et al*. 2002). The numerical values of the assumed parameters for simulated nanotube are given in Table 1.

$$
\begin{aligned}
&w_{11}(0.5,0) = -0.001, \quad \frac{\partial w_{11}(0.5,0)}{\partial T_0} = 0 \\
&w_{21}(0.5,0) = 0.001, \quad \frac{\partial w_{21}(0.5,0)}{\partial T_0} = 0
\end{aligned}
\tag{26}
$$

The above arbitrary non-dimensional initial conditions and the first mode shape of a clamped-clamped beam at its midpoints are used to calculate the corresponding values of C_1, C_2, C_3 and C_4.

6. Results and discussion

The vibrations of the midpoints of the DWNT with the prescribed initial conditions (Eq. (26)) are displayed in Figs. 2 and 3 during a time interval. In Figs. 2 and 3, the results of Galerkin method are compared with the results of multiple scales perturbation method for the same initial conditions. It is also obvious that vibrations of both layers are combinations of coaxial and non-

Table 1 The center position of the first and second layers of a DWNT and their distance (Xu *et al.* 2006)

Nanotube density (ρ)	1300 kg/m^3
nanotube elastic modulus (E)	1000 GPa
Nanotube length (L)	14 nm
Inner and outer tubes damping coefficients (μ_1 and μ_2)	0.0002 N.sec/m

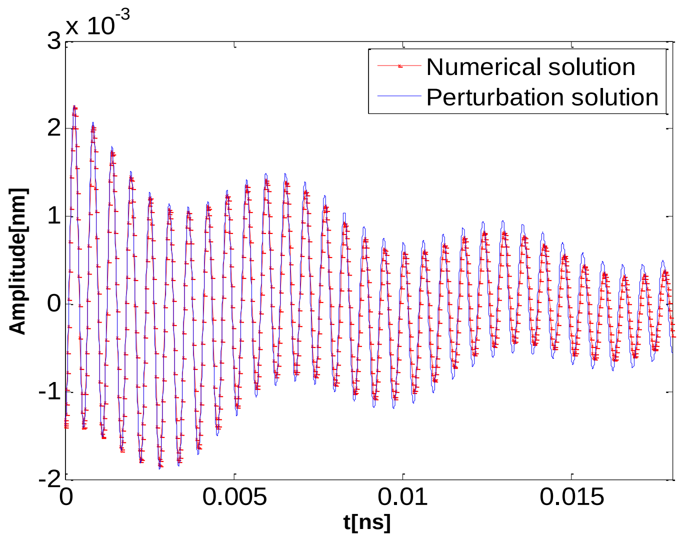

Fig. 2 The vibrations of inner layer midpoint of DWNT for the mentioned initial conditions

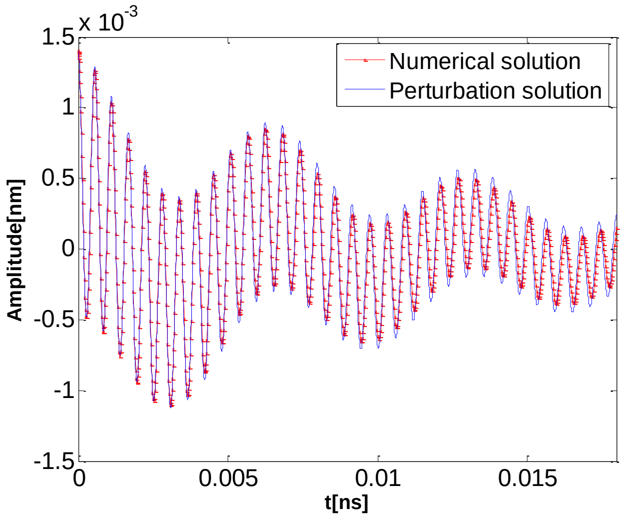

Fig. 3 The vibrations of outer layer midpoint of DWNT for the mentioned initial conditions

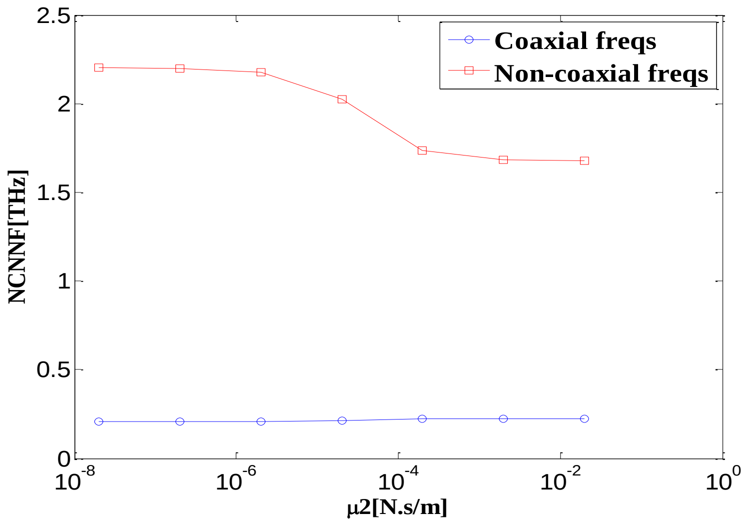

Fig. 4 Coaxial nonlinear natural frequencies for different values of damping coefficient

coaxial frequencies of the system. Also, there is an interaction between the layers vibrations which stems from the nonlinear interlayer vdW force.

Setting T_0=1 in Eq. (21), one can calculate the nonlinear natural frequencies for different values of the parameters. In order to study the influence of damping on the clamped-clamped DWNT, the damping coefficients of both layers are assumed to be equal. The axial stress in this state is set to zero. It is observed from Fig. 4 that NCNNF decreases with increasing the damping coefficients. However, comparing to NCNNF, CNNF is almost insensitive to increasing the damping coefficients. Different values of damping coefficients not only are affected by the structure of the nanotube, but also depend on the working condition of the nanotube. These values can be properly determined through experiments.

7. Conclusions

Free vibrations of double-walled carbon nanotubes (DWNTs) under axial force were studied, using both nonlinear interlayer vdW forces and nonlinear mid-plane stretching term in the model, concurrently. The obtained equations of motion were solved utilizing direct multiple scales perturbation method. It was concluded that both coaxial and non-coaxial nonlinear natural frequencies contribute in DWNT nonlinear free vibrations. For the first vibration mode of DWNT, Increasing the damping coefficient of the system, decreases non-coaxial nonlinear natural frequencies while coaxial nonlinear natural frequencies remain almost unchanged. Therefore, study of damping effects, becomes more important when the DWNT works in different vacuum levels.

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Appendix A

$$H_1(x,T_2)=12c_3\varphi^3\Delta_1A_1A_2\bar{A}_1\left(-\Delta_2+1\right)+9c_3\varphi^3A_2^{\ 2}\bar{A}_2\left(\Delta_2-\Delta_2^2\right)+\Psi\left(6\alpha_4A_1A_2\bar{A}_1+3\alpha_4A_2^{\ 2}\bar{A}_2\right)$$
$$+6c_3\varphi^3A_1A_2\bar{A}_1\left(\Delta_2\Delta_1^2+\Delta_2-\Delta_1^2-1\right)-I\varphi\mu_1A_2\omega_2-2I\varphi\left(\frac{d}{dT_2}A_2\right)\omega_2+3c_3\varphi^3A_2^{\ 2}\bar{A}_2\left(\Delta_2^3-1\right) \quad \text{(A.1)}$$

$$G_1(x,T_2)=-I\mu_1A_1\omega_1\varphi+3c_3\varphi^3A_1^2\bar{A}_1(3\Delta_1-1+\Delta_1^3-3\Delta_1^2)+6c_3\varphi^3A_1A_2\bar{A}_2\varpi+\Psi\left(6\alpha_4A_1A_2\bar{A}_2+3\alpha_4A_1^{\ 2}\bar{A}_1\right)-2I\left(\frac{d}{dT_2}A_1(T_2)\right)\omega_1\varphi \quad \text{(A.2)}$$

$$H_2(x,T_2)=\left(12c_3\alpha_3\alpha_1\Delta_1\varphi^3A_1\bar{A}_1A_2\left(1-\Delta_2\right)+I\alpha_1\Delta_2\varphi\mu_1A_2\omega_2+6c_3\alpha_3\alpha_1\varphi^3A_1A_2\bar{A}_1\left(\Delta_2\Delta_1^2+\Delta_2-\Delta_1^2-1\right)\right.$$
$$\left.+2I\alpha_3\alpha_2\Delta_2\varphi\left(\frac{d}{dT_2}A_2\right)\omega_2+3c_3\alpha_3\alpha_1\varphi^3A_2^{\ 2}\bar{A}_2\left(\Delta_2^3-1\right)+9c_3\alpha_3\alpha_1\varphi^3A_2^{\ 2}\bar{A}_2\left(\Delta_2-\Delta_2^2\right)\right)\left(\frac{-\alpha_2}{\alpha_3\alpha_1}\right) \quad \text{(A.3)}$$
$$+\frac{\alpha_4}{\alpha_1}\Psi\left(6\Delta_2\Delta_1^2A_1A_2\bar{A}_1+3\Delta_2^3A_2^{\ 2}\bar{A}_2\right)$$

$$G_2(x,T_2)=-I\mu_1A_1\omega_1\varphi\Delta_1\frac{\alpha_2}{\alpha_3}+3c_3\alpha_2\varphi^3A_1^2\bar{A}_1(3\Delta_1^2+1-3\Delta_1-\Delta_1^3)-6c_3\alpha_2\varphi^3A_1A_2\bar{A}_2\varpi$$
$$+\frac{\alpha_4}{\alpha_1}\Psi\left(6\Delta_1\Delta_2^2A_1A_2\bar{A}_2+3\Delta_1^3A_1^{\ 2}\bar{A}_1\right)-2I\left(\frac{d}{dT_2}A_1\right)\omega_1\varphi\Delta_1\frac{\alpha_2}{\alpha_1} \quad \text{(A.4)}$$

$$\Psi=\left(\int_0^1\left(\frac{d}{dx}\varphi\right)^2dx\right)\left(\frac{d^2}{dx^2}\varphi\right),\ \varpi=(\Delta_1\Delta_2^2-2\Delta_1\Delta_2-1-\Delta_2^2+\Delta_1+2\Delta_2) \quad \text{(A.5)}$$

Appendix B

$$Q_1=(1+\frac{\alpha_2}{\alpha_1}\Delta_1)\int_0^1\phi^2dx \quad \text{(B.1)}$$

$$Z_1=\frac{1}{2}\mu_1(1+\frac{\alpha_2}{\alpha_3}\Delta_1)\int_0^1\phi^2dx \quad \text{(B.2)}$$

$$\Lambda_1=-\frac{6}{8}c_3\varpi\left(-1+\alpha_2\right)\int_0^1\varphi^4dx+\frac{6}{8}\alpha_4\left(1+\frac{\alpha_2}{\alpha_1}\Delta_1\Delta_2^2\right)\left(\int_0^1\varphi'^2dx\right)\left(\int_0^1\varphi''dx\right) \quad \text{(B.3)}$$

$$S_1=\frac{3}{8}c_3\left(\Delta_1^3+3\Delta_1-1-3\Delta_1^2\right)\left(-1+\alpha_2\right)\int_0^1\phi^4dx+\frac{3}{8}\alpha_4\left(1+\frac{\alpha_2}{\alpha_1}\Delta_1^3\right)\left(\int_0^1\phi'^2dx\right)\left(\int_0^1\phi''dx\right) \quad \text{(B.4)}$$

$$Q_2=(1+\Delta_2)\int_0^1\phi^2dx \quad \text{(B.5)}$$

$$Z_2=\frac{1}{2}\mu_1\frac{(\alpha_3+\alpha_2\Delta_2)}{\alpha_3}\int_0^1\phi^2dx \quad \text{(B.6)}$$

$$\Lambda_2=\frac{6}{8}c_3\varpi\left(-1+\alpha_2\right)\int_0^1\varphi^4dx+\frac{6}{8}\alpha_4\left(1+\frac{\alpha_2}{\alpha_1}\Delta_2\Delta_1^2\right)\left(\int_0^1\varphi'^2dx\right)\left(\int_0^1\varphi''dx\right) \quad \text{(B.7)}$$

$$S_2=\frac{3}{8}c_3\left(\Delta_2^3+3\Delta_2-1-3\Delta_2^2\right)\left(-1+\alpha_2\right)\int_0^1\phi^4dx+\frac{3}{8}\alpha_4\left(1+\frac{\alpha_2}{\alpha_1}\Delta_2^3\right)\left(\int_0^1\phi'^2dx\right)\left(\int_0^1\phi''dx\right). \quad \text{(B.8)}$$