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Nonlocal integral elasticity analysis of beam bending by using finite element method

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Abstract. In this study, a 2-D finite element formulation in the frame of nonlocal integral elasticity is presented. Subsequently, the bending problem of a nanobeam under different types of loadings and boundary conditions is solved based on classical beam theory and also 3-D elasticity theory using nonlocal finite elements (NL-FEM). The obtained results are compared with the analytical and numerical results of nonlocal differential elasticity. It is concluded that the classical beam theory and the nonlocal differential elasticity can separately lead to significant errors for the problem under consideration as distinct from 3-D elasticity and nonlocal integral elasticity respectively.

Keywords: nonlocal integral elasticity; nonlocal differential elasticity; finite element; beam; bending

1. Introduction

Nowadays, nano-sized structures (Wang *et al.* 2006, Wang *et al.* 2008, Ghannadpour and Mohammadi 2006, Ghannadpour *et al.* 2014, Wang and Wang 2007) are widely used in the industrial and commercial productions due to the outstanding physical and chemical characteristics and also economic advantages in terms of mass production capability of these structures. For instance, nanobeams, nanoplates, and nanoshells have common application in nano-electromechanical systems (NEMS) and micro-electromechanical systems (MEMS) devices. In recent years, continuum and semi-continuum models have drawn extra attention in studying of nanostructures since the experimental studies and atomic-scale modeling of large scale nanostructures require considerable time and computation efforts.

One of the widely accepted micro-continuum theories is nonlocal elasticity, which appeared in the late sixties. For example, Kröner (1967) investigated elastic materials with regards to the long range cohesive forces and Krumhansl (1968) applied continuum approach based on atomic lattice theory. Nonlocal differential elasticity and nonlocal integral elasticity (Hu *et al.* 2008) are two

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general forms of nonlocal elasticity theory. In fact, nonlocal integral elasticity is reduced to nonlocal differential elasticity in special condition for particular class of materials. Nevertheless, the latter has attracted more attention due to its simplicity.

Not only in nanoscale problems but in cases where the nature of physical phenomena is happening at atomic or microstructure level, the local theory fails to accurately predict the results (Eringen 2002). In the latter cases, it is worth referring to the weakness of local theories in dealing with elastic continuum in the presence of geometrical singularities (Eringen and Kim 1974, Polizzotto 2002).

Applications of nonlocal elastic continuum methods in elastic behavior of nanostructures have been widely studied. Peddieson *et al.* (2003), who investigated the bending behavior of an Euler-Bernoulli beam with differential version of nonlocal elasticity theory, published the first work on the flexural properties of nanobeams.

Wang and Liew (2007) investigated the influence of scale effect on static deformation of micro- and nano-rods or tubes based on nonlocal Euler-Bernoulli beam theory and Timoshenko beam theory. They showed that the scale effect would not manifest itself for microstructures with length of the order of micro-meters, however, will be noticeable for nanostructures in their static responses.

Berrabah *et al.* (2013) proposed a unified nonlocal shear deformation theory to study bending, buckling and free vibration of nanobeams. They assumed that the in-plane and transverse displacements consist of bending and shear components in which the components do not contribute toward each other. They presented analytical solutions for the deflection, buckling load, and natural frequency for a simply supported nanobeam and compared the result with those predicted by the nonlocal Timoshenko beam theory.

Polizzotto (2001) examined three variational principles, nonlocal counterparts of classical ones, i.e. the total potential energy, the complementary energy, and the mixed Hu-Washizu principles. In his work, a proper framework for applying numerical methods such as FEM and symmetric boundary element method in nonlocal elasticity has been suggested.

Adali (2008) derived variational principles and also natural and geometric boundary conditions for multi-walled carbon nanotubes considering small scale effects using the semi-inverse method via the nonlocal differential elasticity.

The problems of the nonlocal elastic mechanic can be solved via analytical and numerical methods. The analytical solutions are often complex even for one-dimensional problems and hard to solve for two- and three-dimensional ones with general boundary conditions (Pisano *et al.* 2009). Therefore, the only effective method for dealing with nonlocal elasticity problems is applying numerical solutions such as FEM and Ritz method. The only available study on the nonlocal elastic problems using nonlocal differential elasticity theory and FEM has been published by phadikar and Pradhan (2010). They reported the formulation of the Galerkin finite element for nonlocal differential elastic Euler-Bernoulli Beam and Kirchhoff plate.

Finite element method based on the nonlocal integral elasticity was formulated by the Polizzotto (2001) for the first time. He called it nonlocal finite element method (NL-FEM). Pisano *et al.* (2009) used NL-FEM to analyze a nanoplate under tension and compared the results with local elasticity theory ones.

In this study, a 2-D finite element formulation in the frame of nonlocal integral elasticity is presented. Subsequently, the bending problem of a nanobeam under different types of loadings and boundary conditions is solved based on classical beam theory and also 3-D elasticity theory using NL-FEM. The obtained results are compared with the analytical and numerical results of nonlocal differential elasticity.

2. Nonlocal finite element method

In general, a response object is nonlocal if it depends not only on its own independent object but also on independent objects of other points. For a linear homogeneous isotropic continuum as shown in Eq. (1), nonlocality reduces to only stress-strain relation (Eringen and Kim 1974, Eringen *et al.* 1977). In fact the stress at a reference point is assumed to be a functional of the strain field or local stress field at each point in the body with a weighted average.

$$\sigma(\mathbf{x}) = \int_V \alpha(\mathbf{x}, \mathbf{x}')\sigma(\mathbf{x}')\mathrm{dV}(\mathbf{x}') \qquad \forall \mathbf{x} \in \mathrm{V} \tag{1}$$

where $\sigma(\mathbf{x})$ is second-order stress tensor, $\sigma(\mathbf{x}')$ is second order local stress tensor and $\alpha(\mathbf{x}, \mathbf{x}')$ is a positive scalar function which is called attenuation function, weighted function, or nonlocal modulus showing the level of effect of $\mathbf{x}'$ strain on $\mathbf{x}$ strain that depends on $|\mathbf{x}-\mathbf{x}'|$ and a/l where a is an internal characteristic length (e.g., lattice parameter, granular distance) and l is an external characteristic length (e.g., crack length, wave length). The attenuation function may be determined by experiments or atomic lattice dynamics for a special material and various weighting functions have been suggested for it such as triangular, spike, bilateral exponential, Cauchy distribution, error function, bell shape function and conical shape function. However, as Bažant *et al.* (1984) showed the fourier transform of the weighting function must be positive for all real ω, i.e.

$$\alpha^*(\omega) = \int_{-\infty}^{\infty} e^{-i\omega s}\alpha(s)ds > 0 \tag{2}$$

Some of the previous functions such as triangular weighting function, used in some literature (Eringen and Balta 1978, Eringen 1978 and Eringen and Kim 1977) don't satisfy Eq. (2) so they cannot be used. For correction of these attenuation functions, they were combined with a spike in the form of Delta Dirac function.

Eringen showed that integropartial differential equations of linear theory of nonlocal elasticity could reduce to singular partial differential equations if nonlocal modulus be Green's function of a linear differential operator. For example for the following two-dimensional moduli

$$\alpha(|\mathbf{x}|, \tau) = (2\pi l^2\tau^2)K_0(\sqrt{\mathbf{x}.\mathbf{x}} / l\tau) \tag{3}$$

Where $\tau = e_0 a/l$ (e_0 is a constant appropriate to each material) and K_0 is the modified Bessel function, integral Eq. (1) could be converted into the partial differential Eq. (4) (Eringen 1983). It is obvious that the solution of a latter partial differential equation is far easier than solving an integropartial differential equation. It is noted that the modulus of Eq. (3) is a two-dimensional modulus and it is more appropriate to use Eq. (3) in two-dimensional form. But, due to the beam theory assumptions, most of researchers concerning beam bending analysis have used Eq. (4) in one-dimensional form.

$$(1 - \tau^2 l^2 \nabla^2)\mathbf{t} = \boldsymbol{\sigma} \tag{4}$$

Where $\mathbf{t}$ and $\boldsymbol{\sigma}$ are the nonlocal stress tensor and local stress tensor respectively.

In this paper, the results from the nonlocal integral finite element method have been compared to the results from nonlocal differential theory, so the attenuation function corresponds to that expressed in Eq. (3) and as this function meets the requirement of the Eq. (2), it is not necessary to combine it with Dirac delta function.

Polizzotto has extracted the nonlocal form of variational principles and has provided a suitable framework to expand local finite element method to nonlocal finite element method. Relations associated with nonlocal finite element according to papers of Polizzotto (2001) and Pisano *et al.* (2009) are briefly summarized in the following.

The nonlocal total potential energy functional can be written as

$$\begin{aligned} \Pi\left[\mathbf{u}(\mathbf{x})\right] &= \frac{1}{2}\int_V \int_V \alpha(\mathbf{x},\mathbf{x}')\nabla^s \mathbf{u}(\mathbf{x}) : \mathbf{D} : \nabla^s \mathbf{u}(\mathbf{x}')\mathrm{dV}'\mathrm{dV} \\ &\quad - \int_V \overline{\mathbf{b}}(\mathbf{x}).\mathbf{u}(\mathbf{x})\mathrm{dV} - \int_{S_t} \overline{\mathbf{t}}(\mathbf{x}).\mathbf{u}(\mathbf{x})\mathrm{dS} \end{aligned} \tag{5}$$

Where $\overline{\mathbf{b}}(\mathbf{x})$ is the volume force in V and $\overline{\mathbf{t}}(\mathbf{x})$ is the surface force on S_t. $\overline{\mathbf{u}}(\mathbf{x})$ is the external displacement applied on S_u.

Now divide the domain into n elements and approximate displacement field $\mathbf{u}(\mathbf{x})$ and relevant strain field, considering appropriate shape functions as below.

$$\mathbf{u}(\mathbf{x}) = \mathbf{N}_n(\mathbf{x})\mathbf{d}_\mathrm{n}\,; \quad \varepsilon(\mathbf{x}) = \mathbf{B}_n(\mathbf{x})\mathbf{d}_\mathrm{n}\,; \quad \forall \mathbf{x} \in \mathrm{V}_n \tag{6}$$

Where $\mathbf{N}_n(\mathbf{x})$ are shape functions. In the case of simple tension problem, continuity class C^0 is enough, however, in order to analyze bending problem by using classical beam theory, continuity class for the elements should be minimum C^1. $\mathbf{d}_n$ is node displacement vector of the n-th element. By substituting Eq. (6) into Eq. (5), energy functional is written as follows

$$\begin{aligned} \Pi &= \frac{1}{2}\sum_{n=1}^{N_e}\sum_{m=1}^{N_e} \mathbf{d}_\mathrm{n}^\mathrm{T}\left(\int_{V_n}\int_{V_m} \alpha(\mathbf{x},\mathbf{x}')\mathbf{B}_n^T(\mathbf{x})\mathbf{D}\mathbf{B}_m(\mathbf{x}')\mathrm{dV}'\mathrm{dV}\right)\mathbf{d}_m \\ &\quad - \sum_{n=1}^{N_e} \mathbf{d}_\mathrm{n}^\mathrm{T}\left(\int_{V_n} \mathbf{N}_\mathrm{n}^\mathrm{T}(\mathbf{x})\overline{\mathbf{b}}(\mathbf{x})\mathrm{dV} + \int_{S_{t(n)}} \mathbf{N}_\mathrm{n}^\mathrm{T}(\mathbf{x})\overline{\mathbf{t}}(\mathbf{x})\mathrm{dS}\right) \end{aligned} \tag{7}$$

Where $S_{t(n)}=S_t \cap \partial V_n$. By considering Eq. (7), nonlocal element stiffness matrices and element force vector can be given in the shapes

$$\begin{aligned} \mathbf{k}_{nm}^{nonloc} &= \int_{V_n}\int_{V_m} \alpha(\mathbf{x},\mathbf{x}')\mathbf{B}_n^T(\mathbf{x})\mathbf{D}\mathbf{B}_m(\mathbf{x}')\mathrm{dV}'\mathrm{dV} \\ \mathbf{f}_n &= \int_{V_n} \mathbf{N}_\mathrm{n}^\mathrm{T}(\mathbf{x})\overline{\mathbf{b}}(\mathbf{x})\mathrm{dV} + \int_{S_{t(n)}} \mathbf{N}_\mathrm{n}^\mathrm{T}(\mathbf{x})\overline{\mathbf{t}}(\mathbf{x})\mathrm{dS} \end{aligned} \tag{8}$$

By inspecting Eq. (8) it is seen that nonlocal element force vector is similar to local one, but local element stiffness matrix which is obtained for n-th element, has been converted into $\mathbf{k}_{nm}^{nonloc}$ and as a result there will be N_e^2 element stiffness matrices instead of N_e which is the number of matrices in the local case. With increasing distance, the value of $\alpha(\mathbf{x}, \mathbf{x}')$ is rapidly reduced and so it is effective only within a limited distance that is called influence zone (Eringen 2002) or influence distance (Pisano *et al.* 2009). For a given influence distance and assigned geometrical boundaries, the final number of element stiffness matrices will be less than N_e^2.

In finite element method, integrations are usually performed by numerical methods. In the case of mathematically simple shape functions, the analytical integrations are also possible. However, due to the presence of weight function $\alpha(\mathbf{x}, \mathbf{x}')$ in the nonlocal finite element formulation, the

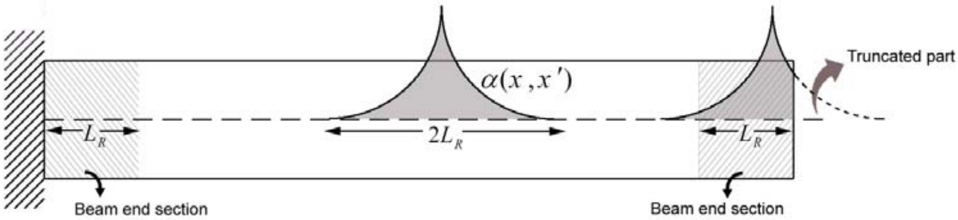


Fig. 1 Modification of kernel at beam end sections (Pisano *et al*. 2009)

numerical integration is mandatory. In this study, for the evaluation of integrations, n-point Gaussian quadrature rule has been used. For one-dimensional elements, 3 Gauss sampling points and for two-dimensional elements 3×3 points have been used.

There are a lot of discussions in the literature to correct the integral kernels, especially in the vicinity of the end supports (from infinite domain support to finite domain support). For example Challamel *et al*. (2014) have obtained the shape of kernels for simply supported-simply supported, free-free and clamped-free boundary conditions. However in this study, as can be seen in Fig. 1, the kernels near the supports are modified by truncating the part of the kernels which is located beyond the length of beam.

3. One-dimensional tension problem

For the verification study of the developed nonlocal integral finite elements formulation, a bar with length L, thickness t and uniform cross-section area S, under a uniform tension loading equal to $\bar{\sigma}$ is considered. For the subject case, no analytical solution is available. However, the numerical solution of Fredholm integral equation as outlined below is attempted in order to compare with the results obtained by the developed nonlocal integral finite elements formulation. It is noted that the similar problem is solved by Pisano and Fuschi (2003) with a different type of attenuation function which is leaded to the Fredholm integral equation of the second kind. Having taken Eq. (9) into consideration, the tension in the entire bar will be constant and equal to the applied load. The stress-strain relationship at a general point is given by the following equation

$$\bar{\sigma} = E\int_0^L \alpha(x,x')\varepsilon(x')\mathrm{d}x' \tag{9}$$

The above equation is in the form of Fredholm integral equation of the first kind (Eq. (10)) that does not have an analytical answer in general form and should be solved by numerical methods (e.g., Baker 1977, Lewis 1973). One of the methods being used to solve equations of this type over the recent years is Legendre wavelets method (Maleknejad and Sohrabi 2007). Since in this section the results of nonlocal integral method are not to be compared with those of nonlocal differential method, any desired attenuation function can be implemented. In this section, both of the nonlocal finite element method as well as Legendre wavelets numerical method are developed by implementing attenuation function as bi-exponential function. The results from both methods are compared in Fig. 2. The correlation between the results is quite satisfactory.

$$f(\mathbf{x}) = \int_a^b K(x,t)\phi(t)dt \tag{10}$$

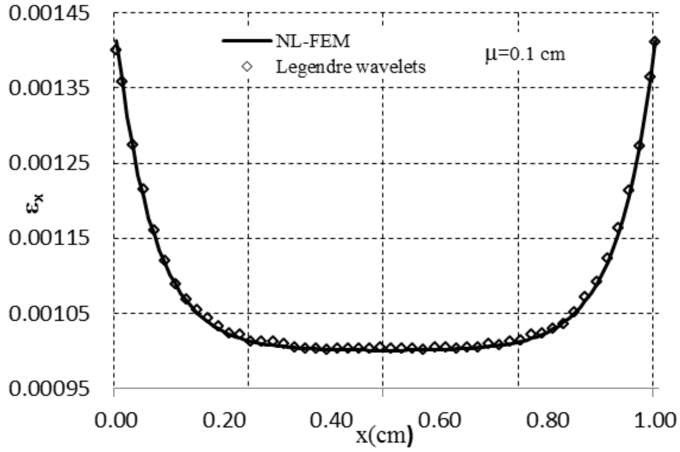


Fig. 2 Strain distribution for a bar in tension with NL-FEM (solid line) and Legendre wavelets method (points). μ is the scale coefficient that incorporates the small-scale effect and is equal to $l\tau$

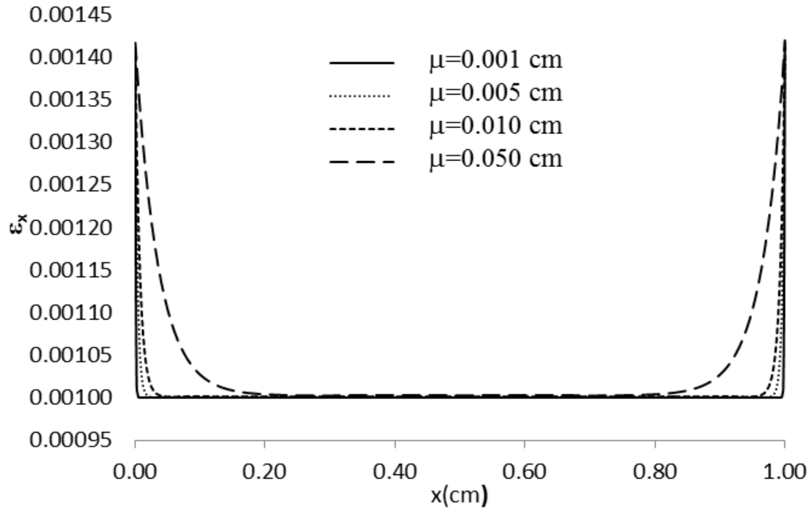


Fig. 3 Strain distribution for a bar in tension with various influence distance parameter

It is noted that for a particular case of Eq. (10) in which the integration limits are infinite and $K(x,t)$ is only a function of the difference of its arguments, namely $K(x,t)=K(x-t)$, Fredholm integral equation of the first kind will have an analytical solution. In this case a constant $f(\mathbf{x})$ function would result in a constant $\phi(t)$ solution. In other words, if the bar under consideration is of infinite length, the nonlocal effects will disappear. Therefore, the structural boundaries have resulted in the nonlocal characteristics of the bar.

In Fig. 3 Strain distribution for a bar with various L_R has been shown. As it could be seen, with increasing L_R, greater area of the bar is affected by nonlocal effects. Maximum strain occurring in boundaries is constant and does not depend on the amount of μ due to kernel function being normal. On the other hand, the strain is constant in area far from boundaries and is almost equal to its local value. Next sections will show that in the study of a beam under bending, as thickness of beam is much less than the other dimensions, considering nonlocal effects along thickness is considerably important so that not considering such effects may causes significant error in answers.

4. Beam bending analysis using nonlocal integral elasticity

In this section, the bending of a beam is investigated by using nonlocal integral finite element method based on either classical beam theory or the three-dimensional elasticity theory. A variety of boundary conditions and loadings are considered. The obtained results are compared with the corresponding ones from nonlocal differential theory.

For completeness and the future reference, a note on the nonlocal differential theory is given below. The constitutive law according to nonlocal differential theory for the beam is given by

$$\sigma_{xx} - \mu^2 \frac{\mathrm{d}^2\sigma_{xx}}{dx^2} = E\varepsilon_{xx} \tag{11}$$

Where μ is the scale coefficient that incorporates the small-scale effect and is equal to $l\tau$ or e_0a. The nonlocal bending moment is obtained through multiplication of Eq. (11) by zdA and integration over area A.

$$M - \mu^2 \frac{\mathrm{d}^2M}{dx^2} = -EI\frac{\mathrm{d}^2\mathrm{w}}{dx^2} \tag{12}$$

Where M is the second moment of area and w is the transverse displacement. On combining Eq. (12) and the field equilibrium equation, the governing equation for the bending of nonlocal Euler-Bernoulli beam is given by (Ghannadpour *et al.* 2014)

$$EI\frac{\mathrm{d}^4\mathrm{w}}{dx^2} - \mu^2 \frac{\mathrm{d}^2\mathrm{q}}{dx^2} + q = 0 \tag{13}$$

This equation can be solved by different numerical methods such as Ritz method (Ghannadpour *et al.* 2014) and General Differential Quadrature (GDQ) method (Pradhan and Phadikar 2009). Moreover, Wang *et al.* (2008) have analytically solved the Timoshenko beam bending problem based on the nonlocal differential theory. It is noted that in the nonlocal one-dimensional Eq. (13), the nonlocality effects are only governed by the variation of the ratio of scale coefficient to the length. It is obvious that in a more general theory the effects of nonlocality due to the variation of the ratio of scale coefficient to other smaller dimensions such as thickness can be taken into account.

4.1 Beam bending analysis with NL-FEM based on classical beam theory

The classical beam hypothesis implies that the displacement field has the form

$$\begin{aligned} u(x,z) &= u_0(x) - z\frac{\partial w_0}{\partial x} \\ w(x,z) &= w_0(x) \end{aligned} \tag{14}$$

Where $u(x,z)$ and $w(x,z)$ are components of displacements at a general point, whilst $u_0(x)$ and

$w_0(x)$ are similar components at the middle surface.

Using Eq. (14) in the Green's expression for nonlinear strains and neglecting lower order terms in a manner consistent with the usual Von Karman assumption gives the following expressions for strain at a general point

$$\overline{\varepsilon} = \varepsilon^0 + z\varepsilon^1 \tag{15}$$

Where

$$\varepsilon^0 = \frac{\partial u_0}{\partial x}, \qquad \varepsilon^1{}_{xx} = -\frac{\partial^2 w_0}{\partial x} \tag{16}$$

So

$$\varepsilon(\mathbf{x}) = \mathbf{B}_n^0(\mathbf{x})\mathbf{d}_{\mathrm{n}} + z\mathbf{B}_n^1(\mathbf{x})\mathbf{d}_{\mathrm{n}}; \quad \forall \mathbf{x} \in \mathrm{V}_n \tag{17}$$

Based on Eqs. (7) and (15), nonlocal total potential energy functional is given by

$$\begin{aligned}\Pi = &\frac{1}{2}bt\sum_{n=1}^{N_e}\sum_{m=1}^{N_e}\mathbf{d}_{\mathrm{n}}^{\mathrm{T}}\left(\int_{x_n}\int_{x_m}\alpha(x,x')(\mathbf{B}_n^0)^T(x)\mathbf{D}\mathbf{B}_m^0(x')\mathrm{d}x'\mathrm{d}x\right)\mathbf{d}_m \\ &+\frac{1}{2}bI_{xx}\sum_{n=1}^{N_e}\sum_{m=1}^{N_e}\mathbf{d}_{\mathrm{n}}^{\mathrm{T}}\left(\int_{x_n}\int_{x_m}\alpha(x,x')(\mathbf{B}_n^1)^T(x)\mathbf{D}\mathbf{B}_m^1(\mathbf{x}')\mathrm{d}x'\mathrm{d}x\right)\mathbf{d}_m \\ &-bt\sum_{n=1}^{N_e}\mathbf{d}_{\mathrm{n}}^{\mathrm{T}}\left(\int_{x_n}\mathbf{N}_{\mathrm{n}}^{\mathrm{T}}(x)\overline{\mathbf{b}}(x)\mathrm{dV} + \int_{S_{t(n)}}\mathbf{N}_{\mathrm{n}}^{\mathrm{T}}(x)\overline{\mathbf{t}}(x)\mathrm{d}x\right)\end{aligned} \tag{18}$$

Thus nonlocal stiffness matrices is obtained as follows

$$\begin{aligned}\mathbf{k}_{\mathrm{nm}}^{\mathrm{nonloc}} &= \mathbf{k}_{0\mathrm{nm}}^{\mathrm{nonloc}} + \mathbf{k}_{1\mathrm{nm}}^{\mathrm{nonloc}} \\ \mathbf{k}_{0\mathrm{nm}}^{\mathrm{nonloc}} &= bt\int_{x_n}\int_{x_m}\alpha(x,x')(\mathbf{B}_n^0)^T(x)\mathbf{D}\mathbf{B}_m^0(x')\mathrm{d}x'\mathrm{d}x \\ \mathbf{k}_{1\mathrm{nm}}^{\mathrm{nonloc}} &= bI_{xx}\int_{x_n}\int_{x_m}\alpha(\mathbf{x},\mathbf{x}')(\mathbf{B}_n^1)^T(\mathbf{x})\mathbf{D}\mathbf{B}_m^1(\mathbf{x}')\mathrm{d}x'\mathrm{d}x\end{aligned} \tag{19}$$

$\mathbf{k}_{0\mathrm{nm}}^{\mathrm{nonloc}}$ is zero under pure bending condition which is the case in the current study. A two-node element with 2 DOFs per node has been used. In view of Eqs. (3) and (11), an attenuation function is given by

$$\alpha(|x-x'|,\tau) = \lambda_0(2\pi\mu^2)K_0(|x-x'|/\mu) \tag{20}$$

Where

$$\lambda_0 = \frac{1}{\int_{-\infty}^{+\infty}(2\pi\mu^2)K_0(|x-x'|/\mu)} \tag{21}$$

Fig. 4 shows the effects of μ on attenuation function distribution. It should be noted that the distance of influence is usually about a few atoms (up to maximum 10 atoms (Eringen 2002)), so the large value of μ/L (for example, larger than 0.1) means that the beam can be as short as

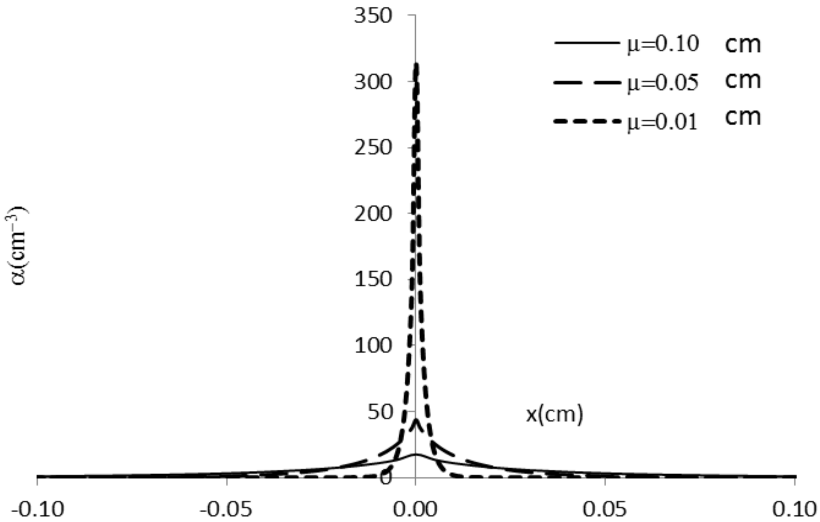


Fig. 4 Effects of μ on attenuation function distribution

Table 1 Non-dimensional maximum deflections for a clamp-clamp nanobeam with two loading conditions

	μ/L=0.1	
	$\bar{k}=1$	$\bar{q}=1$
n=number of elements	$\bar{w}_{\max}$	$\bar{w}_{\max}$
10	0.005106	0.002616
20	0.005908	0.003053
50	0.006417	0.003345
100	0.006418	0.003348
	μ/L=0.05	
	$\bar{k}=1$	$\bar{q}=1$
n=number of elements	$\bar{w}_{\max}$	$\bar{w}_{\max}$
10	0.003023	0.001522
20	0.004351	0.002201
50	0.005286	0.002688
100	0.005520	0.002814
150	0.005577	0.002846

approximately one hundred atoms. In the latter case, according to the beam theories in which a relatively small thickness to length ratio is assumed, the beam thickness will be in the order of a few atoms only. In other words, the beam is too thin to allow for either differential or integral nonlocal theories to be applicable.

A numerical code has been developed based on the proposed nonlocal integral finite element method. A convergence study is carried out for a clamp-clamp nanobeam with two scaling effect parameter μ/L of 0.1 and 0.05. A central point load parameter $\bar{k}=FL^2/EI=1$ and a distributed load parameter $\bar{q}=qL^3/EI=1$ are considered. The non-dimensional deflection $\bar{w}_{\max}$, is presented in Table 1.

Table 2 Non-dimensional maximum deflection for a nanobeam with various boundary conditions and various scaling effect parameter under uniformly distributed load (L/t=100).

μ/L	Method	Boundary conditions		
		SS-SS	CL-CL	CL-FR
0	1-D NL-FEM	0.0130	0.0026	0.1250
	(Wang *et al.* 2008)	0.0130	0.0026	0.1250
	(Ghannadpour *et al.* 2014)	0.0130	0.0026	0.1250
0.1	1-D NL-FEM	0.0132	0.0033	0.1408
	(Wang *et al.* 2008)	0.0142	0.0026	0.1200
	(Ghannadpour *et al.* 2014)	-	0.0026	-
0.2	1-D NL-FEM	0.0144	0.0052	0.1747
	(Wang *et al.* 2008)	0.0180	0.0026	0.1050
	(Ghannadpour *et al.* 2014)	0.0180	0.0026	0.1050

It is noted that for the subject problem the local finite element analysis can deliver converged results with very few elements. In the case of nonlocal finite element, however, it is seen in the above table that the convergence rate is rather slow due to the presence of attenuation function. In fact, the convergence rate depends directly on kernel shape used, and for instance, for a bilateral exponential function the convergence is achieved relatively faster than that for a Bessel function. Also, the convergence rate depends directly on scaling effect parameter value. In other words, a smaller scaling effect parameter requires more elements than those are necessary for a larger scaling parameter. It is noted that based on the problem under consideration, in the remaining of the paper different numbers of finite elements are utilized in order to achieve converged results.

In Table 2, maximum deflection for a beam with various boundary conditions, namely simply supported-simply supported (SS), clamped-clamped (CL-CL) and clamped-free (CL-FR), under a uniformly distributed load is presented. In order to compare the results with those from other references, similar values of scaling parameter are selected, although meaningless in some cases. Wang *et al.* (2008) has obtained the results based on the analytical solution of beam bending nonlocal differential equation, and Ghannadpour *et al.* (2014) has solved the corresponding nonlocal differential equation by using Ritz method.

It is seen in the table for simply supported beam (SS-SS) in both nonlocal differential and nonlocal integral theories, by increasing the scaling effect parameter, the maximum deflection increases. However, the latter increase is more pronounced in the case of differential theory. Furthermore, for the case of fully clamped beam (CL-CL), the increase in the scale factor has caused no changes in the results of differential theory, whilst for the same beam, the deflections obtained from the integral theory have increased with a considerable rate even higher than that for the simply supported case. This behavior can be explained on the following grounds. By further review of Eq. (13) corresponding to the governing nonlocal differential equation for Euler-Bernoulli beam, it is seen that μ is multiplied by the term d^2q/dx^2 only, and the boundary conditions of the beam are also obtained as that follows (Wang *et al.* 2008).

$$w = 0, M = -EI\frac{d^2w}{dx^2} + \mu^2 q \quad \text{for a simple supported end} \tag{22}$$

Table 3 Ratio of the critical buckling load parameter of the nanobeam to the corresponding local beam with various boundary conditions and various scaling effect parameter (Ghannadpour *et al.* 2014)

μ/L	Boundary conditions			
	SS-SS	CL-SS	CL-CL	CL-FR
0	1	1	1	1
0.2	0.7169	0.5532	0.3877	0.9101
1	0.0919	0.0471	0.0247	0.2883

$$w = 0,\ \frac{dw}{dx} = 0 \quad \text{for a clamped end} \tag{23}$$

$$M = -EI\frac{d^2 w}{dx^2} + \mu^2 q = 0,\ V = -\frac{dM}{dx} = 0 \quad \text{for a free end} \tag{24}$$

Therefore, for either simply supported or free boundary condition, regardless of the loading type (whether point or distributed), the nonlocal effects will appear in the bending solution due to the parameter μ^2 being present at the above boundary condition equations. On the contrary, for a clamped boundary condition since the term μ has not appeared in the corresponding boundary condition equation, the nonlocality can only come to play when the term d^2q/dx^2 in Eq. (13) is non-zero (for example when the loading is in the form of a sinusoidal distributed load). Therefore, since only a uniformly distributed load is considered in Table 2, the nonlocal effects have caused no changes in the results based on nonlocal differential theory for a clamped beam. However, it is widely known that the nonlocal effects appear due to the long range interactions between particles of material, thus for any type of loadings or boundary conditions these effects are expected to be present. The latter expectation is fulfilled by the results presented in Table 2 corresponding to the nonlocal integral theory. The above shortcoming of differential theory is somewhat removed by Challamel and Wang (2009) who presented a gradient elasticity model, in which the local and nonlocal curvature elastic relations are combined, in order to solve the bending problem of an Euler-Bernoulli cantilever nanobeam with a point load. Of course, in this model an additional parameter is added to the equations of bending that needs to be obtained experimentally.

Having seen earlier in Table 2 that the nonlocal effects are more pronounced for a clamped beam as distinct from a simply supported one, a similar behavior is also observed in Table 3 with respect to the beam buckling solution obtained by using differential theory (Ghannadpour *et al.* 2014). Given the fact that there is a direct correlation between the buckling load of a beam and its bending stiffness value, the higher variation of the buckling load in the case of fully clamped beam implies a higher change in the bending stiffness of the same beam. This is attributed to the fact that the nonlocality causes reduction in the stiffness near the geometrical boundaries. Thus, in the case of classical beam theory in which the nonlocalitly is considered only along length, the nonlocal effects will cause reduction in the stiffness at regions near to the end supports. Of course, for a stiffer support such as a clamped one, the nonlocal effects will be more pronounced.

In Table 4, maximum deflection for a beam with different boundary conditions under sinusoidal load is presented. The comparison between the current FEM results and those of Wang *et al.* (2008) is generally satisfactory.

4.2 Beam bending analysis with NL-FEM based on 3-D elasticity theory

In the previous section, nonlocality along thickness was ignored due to the application of the classical beam theory. In this section, the same beam bending problem is solved by using two-dimensional elements so that the nonlocal effects along thickness are also taken into consideration. In Fig. 4(a), a typical finite element mesh arrangement for a beam and in Fig. 4(b), a typical element is shown.

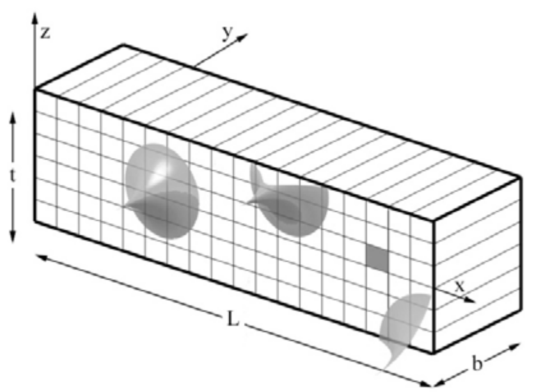
In the nonlocal finite element method, the distances between Gaussian points of a given element to the Gaussian points of all other elements, which are affected by the kernel function, must be measured. However, this task becomes less demanding when a regular mesh with equal size elements are used. In this study, 8-nodes C^0-quadratic isoparametric Serendipity elements with 2 DOFs per node and 3×3 Gauss sampling points have been used.

With introducing a natural coordinate system $\xi=(\xi,\eta)$ being $\mathbf{x}(\xi)=\mathbf{x}(\xi,\eta)=\{x(\xi,\eta)\ z(\xi,\eta)\}^T$, Eq. (8) can be given the shape (based on Pisano *et al.* 2009)

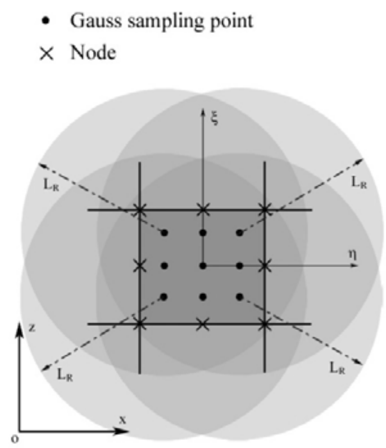
$$\mathbf{k}_{(nm)ij}^{nonloc}=\int_{-1}^{1}\int_{-1}^{1}\left[\int_{-1}^{1}\int_{-1}^{1}\alpha(|\mathbf{x}'(\boldsymbol{\xi}')-\mathbf{x}(\boldsymbol{\xi})|)\mathbf{B}_{(n)i}^{T}(\boldsymbol{\xi})\mathbf{D}\mathbf{B}_{(m)j}^{T}(\boldsymbol{\xi}')b\det\mathbf{J}(\boldsymbol{\xi}')d\xi' d\eta'\right] \times b\det\mathbf{J}(\boldsymbol{\xi})d\xi d\eta \tag{25}$$

Table 4 Non-dimensional maximum deflection for a nanobeam with various boundary conditions and various scaling effect parameter under sinusoidal load by nonlocal integral elasticity and nonlocal differential elasticity

μ/L	Method	Boundary conditions		
		SS-SS	CL-CL	CL-FR
0	1-D NL-FEM	0.0102	0.0022	0.0738
	Based on Wang *et al.* (2008)	0.0102	0.0022	0.0738
0.1	1-D NL-FEM	0.0103	0.0027	0.0839
	Based on Wang *et al.* (2008)	0.0112	0.0024	0.0706
0.2	1-D NL-FEM	0.0114	0.0043	0.1054
	Based on Wang *et al.* (2008)	0.0143	0.0030	0.0611



(a)



(b)

Fig. 5 Typical finite element mesh arrangement and typical element for a beam. (Pisano *et al.* 2009)

Table 5 Non-dimensional maximum deflection for nanobeams with various boundary conditions and various scaling effect parameter under distributed load by 2-D nonlocal integral elasticity, 1-D nonlocal integral elasticity and nonlocal differential elasticity (Wang *et al.* 2008)

μ/L	Method	Boundary conditions		
		SS-SS	CL-CL	CL-FR
0	2-D NL-FEM	0.0130	0.0026	0.1250
	1-D NL-FEM	0.0130	0.0026	0.1250
	Wang *et al.* (2008)	0.0130	0.0026	0.1250
0.01	2-D NL-FEM	0.0282	0.0037	0.2087
	1-D NL-FEM	0.0130	0.0026	0.1250
	Wang *et al.* (2008)	0.0130	0.0026	0.1250
0.02	2-D NL-FEM	0.0694	0.0144	0.3088
	1-D NL-FEM	0.0132	0.0026	0.1250
	Wang *et al.* (2008)	0.0133	0.0026	0.1262

Where $\mathbf{J}$ is the Jacobean matrix of transformation, $\mathbf{B}^T_{(n)i}(\boldsymbol{\xi}) = \mathbf{B}_{(n)i}[\mathbf{x}(\boldsymbol{\xi})]$, $\mathbf{B}^T_{(m)j}(\boldsymbol{\xi}) = \mathbf{B}_{(m)i}[\mathbf{x}'(\boldsymbol{\xi}')]$ and b is the beam width. Finally by applying the Gauss-Legendre quadrature rule for numerical integration to Eq. (25) one gets

$$\begin{aligned} k^{nonloc}_{(nm)ij} &= \sum\nolimits_{h=1}^{N_G} \sum\nolimits_{g=1}^{N_G} \sum\nolimits_{r=1}^{N_G} \sum\nolimits_{s=1}^{N_G} [\alpha(\left|x'(\xi'_r,\eta'_s) - x(\xi_h,\eta_g)\right|) \\ &\times \mathbf{B}^T_{(n)i}(\xi_h,\eta_g)\mathbf{D}\mathbf{B}^T_{(m)j}(\xi'_r,\eta'_s) \times w_h w_g w'_r w'_s b^2 \det \mathbf{J}(\xi_h,\eta_g)] \end{aligned} \tag{26}$$

Where N_G is the Gauss points number per element, $\alpha(\left|\mathbf{x}'(\xi'_r,\eta'_s) - \mathbf{x}(\xi_h,\eta_g)\right|)$ is the kernel associated to Gauss points in the global coordinate system and w_h, w_g, w'_r, w'_s are the Gauss weights.

It is obvious that a converged result can only be achieved if a sufficient number of elements are allocated to the L_R region of any given element. As a result, for problems with a small scaling effect parameter the total number of the elements is considerably more than the corresponding one in the case of local finite element.

In Table 5, the non-dimensional maximum deflections for nanobeams with various boundary conditions and different scaling effect parameter under a uniformly distributed load are presented. The results correspond to three types of analysis, i.e., the nonlocal integral two-dimensional finite element, the nonlocal integral finite element based on classical beam theory as well as the nonlocal differential elasticity analysis.

It is seen that for the beam under consideration, which is really thin (L/t=100), the results of nonlocal two-dimensional finite element indicate a profound difference with those of other methods. In other words, by increasing the scaling effect parameter for a thin beam, the kernel function of any given element through the thickness tends to be truncated by the upper and lower boundaries of the beam. Of course, this would inevitably lead to a higher flexibility of the beam if the nonlocal effects along the thickness are accounted for properly in a method such as the developed 2-D finite element analysis. When the effects of nonlocality through the thickness are to be investigated, it might be advisable to replace the scaling effect parameter μ/L by μ/t. This is because in the case of thin beams where for example the length is one hundred times bigger than the thickness, a scaling effect parameter of μ/L=0.01 would mean the nonlocality region is longitudinally confined to the two ends of the beam. However, for the same beam, the effect of nonlocality is quite significant throughout the whole thickness since μ/t=1. Thus, one might be misled if the variations of the results were only shown based on the scaling effect parameter as μ/L.

5. Conclusions

In this study, the bending behavior of a nanobeam is investigated by using 1-D and 2-D finite element method based on nonlocal integral elasticity. The results are compared with those from nonlocal differential elasticity. In the case of nonlocal integral elasticity analysis, it is seen that the convergence rate is rather slow due to the presence of attenuation function. In fact, the convergence rate of the developed finite element method directly depends on the shape of implemented kernel function. In the case of clamped boundary conditions, the differential theory is incapable of demonstrating the nonlocal effects whereas by using the nonlocal integral theory, such effects are observed. Subsequently, it is shown that in the beam bending analysis, it is necessary to have a model in which the discretization through the thickness, in addition to the lengthwise discretization, is properly accounted for. This has led to a considerably lower bending stiffness of the beam. The present study has paved the way to facilitate the analyses and design of nano size structures with general boundary conditions and loadings.

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