Structural Engineering and Mechanics, *Vol. 54, No. 4 (2015) 725-740*
DOI: http://dx.doi.org/10.12989/sem.2015.54.4.725

Critical buckling load optimization of the axially graded layered uniform columns

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(Received March 5, 2015, Revised March 25, 2015, Accepted March 30, 2015)

Abstract. This study presents critical buckling load optimization of the axially graded layered uniform columns. In the first place, characteristic equations for the critical buckling loads for all boundary conditions are obtained using the transfer matrix method. Then, for each case, square of this equation is taken as a fitness function together with constraints. Due to explicitly unavailable objective function for the critical buckling loads as a function of segment length and volume fraction of the materials, especially for the column structures with higher segment numbers, initially, prescribed value is assumed for it and then the design variables satisfying constraints are searched using Differential Evolution (DE) optimization method coupled with eigen-value routine. For constraint handling, Exterior Penalty Function formulation is adapted to the optimization cycle. Different boundary conditions are considered. The results reveal that maximum increments in the critical buckling loads are attained about 20% for cantilevered and pinned-pinned end conditions and 18% for clamped-clamped case. Finally, the strongest column structure configurations will be determined. The scientific and statistical results confirmed efficiency, reliability and robustness of the Differential Evolution optimization method and it can be used in the similar problems which especially include transcendental functions.

Keywords: axially graded; uniform column; buckling; optimization; differential evolution

1. Introduction

Nowadays, powder metallurgy has become an important area for engineering, especially in terms of production of structural members. Slender columns are, on the other hand, the structural members that are widely used in engineering areas such as mechanical, civil, marine and aerospace. These members loaded by compression may fail due to buckling (Timoshenko and Gere 1961). So, stability is the main problem and has a crucial importance for the structures. To improve their structural stability against buckling, increasing weight and/or flexural rigidity (EI) without violating economical aspects of the design routine are the first two things that can be considered. In addition, functionally graded materials are the special materials that can be characterized by the variation in composition and structure gradually over the volume in a continuous or piecewise manner, resulting in corresponding changes in the properties of the material such as elastic modulus and density. The concept for the functionally graded materials is

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http://www.techno-press.org/?journal=sem&subpage=8 ISSN: 1225-4568 (Print), 1598-6217 (Online)

to make a composite material by varying the microstructure from one material to another material with a specific gradient. So, these special materials can be used for specific applications. For the axially graded layered columns, the critical buckling load can be controlled in a desired manner. Knowing optimum column structure configuration enables designers/engineers to save time and design costs at the same time. So, optimization of such structural members is of great importance.

Keller (1960) determined the shape of strongest column for the hinged at both ends and later, Tadjbakhsh and Keller (1962) studied the same problem for a given length and volume of the column considering other types of end conditions. From then, there have been conducted extensive studies about optimization of the columns for attaining maximum buckling load in a continuous manner available in open literature. This is not the case for the column made of axially graded materials especially in a piecewise manner. The number of studies available in literature is rather limited. Coello and Farrera (1995) conducted a study dealing with optimization of axially loaded non-prismatic columns using the genetic algorithm optimization method. They formulated the optimization problem such that the objective function is to minimize the volume of a column under a given load by changing its shape, subject to both buckling and strength constraints. O'Rourke (1997) proposed a method to determine critical buckling load for non-uniform columns. A finite difference method is used to formulate the problem. He considered segment length and moment of inertia as variables but the elastic modulus was held constant throughout the parametric studies. He tabulated column configurations for cantilevered and simply-supported end conditions. Fu and Ren (1992) conducted a study dealing with optimization of axially loaded non-prismatic column using generalized reduced gradient method. This study aimed at minimizing the weight of a column under a given load by changing its shape, subject to both buckling and strength constraints and they concluded that considerable savings were achieved. Maalawi (2002) formulated buckling optimization of segmented solid and tubular columns for a given total mass and length. The optimization problem was formulated in a dimensionless form, making his model independent on specific cross-sectional shape and dimensions. The design variables are the radius of gyration and the segment length. He considered clamped-free and clamped-clamped boundary conditions. He showed that depending on the end conditions, an increment in the critical buckling load is attained. Based on the study of Maalawi (2002), Maalawi (2009) proposed an exact method for optimization of axially graded columns with clamped-free and pinned-pinned end conditions. Two different problems are considered, i.e., uniform column with axial material grading and thin wall columns with the thickness grading. Li *et al.* (2011) proposed an approach to determine the critical buckling load of cantilevered non-uniform composite columns subjected to distributed axial load and tip force. They considered two design problems for maximizing load-carrying capacity. The first one is to obtain an optimal parameter of the shape profile when the weight or volume is prescribed, and the other is to determine a suitable gradient parameter of a functionally graded composite column with uniform cross-section.

Patnaik *et al.* (2012) presented a parametric study dealing with determination of minimum mass configurations of two segmented cantilevered columns with circular cross section for a given a buckling load constraint using numerical search algorithm. The segments of the columns are made up of either with a single material (steel) or with two materials (combination of aluminum – steel, aluminum – copper and aluminum – titanium, keeping aluminum as the free end segment). Finally, they tabulated minimum mass configurations for different inertia and length ratios. They concluded that the inertia ratio has a strong effect on the minimum mass design. In addition, Patnaik *et al*. (2013) studied the similar problem and performed mass minimization of the columns using the Kuhn-Tucker method. Sujatha *et al*. (2013) conducted stability analysis of columns using combination of finite difference method and unit step function to avoid stability problems of finite difference method. Unit step function is used to satisfy the continuity at points of discontinuity. The variation of modulus of elasticity, and moment of inertia, and loading functions are written in suitable forms using unit step functions. They also concluded that the results obtained from proposed methods are in good agreement with those obtained from the pure finite difference method. Singh and Li (2009) proposed a mathematical model for obtaining critical buckling loads of uniform and non-uniform axially graded columns. They approximated the columns with spatial variation of flexural stiffness resulted from material grading and/or non-uniform shape by an equivalent column with piecewise constant geometrical and material properties which in turns resulted in transcendental eigenvalue problems solved by a numerical method based upon Newton's Eigenvalue Iteration Method. They carried out extensive parametric studies. Alkan (2015) presented optimum buckling design of axially layered graded uniform columns using "fmincon" nonlinear programming solver provided in the MATLAB's Optimization Toolbox. For all end conditions considered, he concluded that axial load carrying capacity (i.e., critical buckling load) of the axially layered graded uniform columns is increased when comparing with the case of the one segment uniform columns.

In this study, critical buckling load optimization of the axially graded layered uniform columns is performed. Objective function formulation proposed by (Alkan 2015) for attaining higher critical buckling loads is used. As an optimization tool, Differential Evolution optimization technique coupled with eigen-value routine is used. For constraint handling, Exterior Penalty Function formulation is adapted. Firstly, the characteristic equations for determining critical buckling loads for all end conditions are obtained using the transfer matrix method. Then, for each case, square of this equation is taken as an objective function along with constraints. It is very difficult to explicitly obtain the objective function for the critical buckling load as a function of the volume fraction and the segment length, especially for the column structures with higher segment numbers. Therefore, pre-specified critical buckling loads are assumed and then the design variables satisfying constraints are searched. The design variables are volume fraction and segment length. Clamped-free, clamped-clamped and pinned-pinned type boundary conditions are considered. Some conclusions are drawn and finally, the strongest column structure configuration for each case is determined.

On the other hand, to author knowledge, application of the Differential Evolution to this problem is not available in open literature. Even if its reliability and robustness are proven, some adjustments are needed to be done to efficiently solve any problem especially in terms of control parameters of DE (Das and Sugathan 2011). There is a satisfactory agreement between the results obtained in the present study and those obtained in literature. So, this conclusion and also performance experiments confirmed efficiency, reliability and robustness of the Differential Evolution optimization method. Therefore, it can be deduced from this study that this optimization method can also be used in different eigen-value problems especially having transcendental equations.

In addition, it is mentioned in literature that optimum columns for clamped-clamped end conditions with the number of three and more than three segments are symmetrical about both the quarter and mid-span points. In other words, variation of the moment of inertia or cross-section along the column length has such a trend under the assumption of constant material properties. Present results showed that similar distributions for the material properties (i.e., Young's Modulus and density) are obtained for the column structures where moment of inertia or cross-section do not change during the optimization cycles. This point is not mentioned in literature as far as optimization studies dealing with axially graded column structures in a piecewise manner are concerned.

2. Optimization problem formulation for the columns:

A general constrained minimization (or maximization) problem can be stated as follows

$$\text{Find } x = \begin{Bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{Bmatrix} \text{ which minimizes } f(\boldsymbol{x}) \text{ or maximizes } -f(\boldsymbol{x})$$

subject to $h_j(\boldsymbol{x})=0$ *for* $j=1,\ldots,m$
$g_j(x)\leq 0$ *for* $j=1,\ldots,p$

where $\boldsymbol{x}$ is an n-dimensional vector called design vector, $f(x)$, objective function, $h_j(x)$, equality constraints, $g_j(x)$, inequality constraints, m and p are known as the number of equality constraints and inequality constraints, respectively.

In this study, the objective function is to maximize the critical buckling loads and design variables are volume fraction of the materials and segment length. Equality constraints are the total mass and total length of the baseline column structure consisting of one-segment and having uniform mass and stiffness distribution. Initially, there are no inequality constraints in the optimization problem. However, inclusion of side constraints will provide linear constraints depending on the number of design variables.

2.1 Objective function

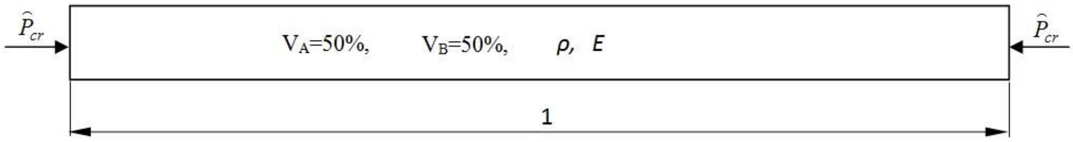
True and robust optimization routine strongly relies on the mathematical modeling of a problem. This study uses the mathematical formulation of the critical buckling load of axially graded columns proposed by (Maalawi 2009). Also, detailed information about the mathematical formulation steps can be found in (Li 2003).

For an elastic, slender, total length (L) of the axially graded non-uniform column structure which is applied axial load P and constructed from any arbitrary number of segments (N_s) having different material property (i.e., different Young's modulus and density), the following differential equation reads

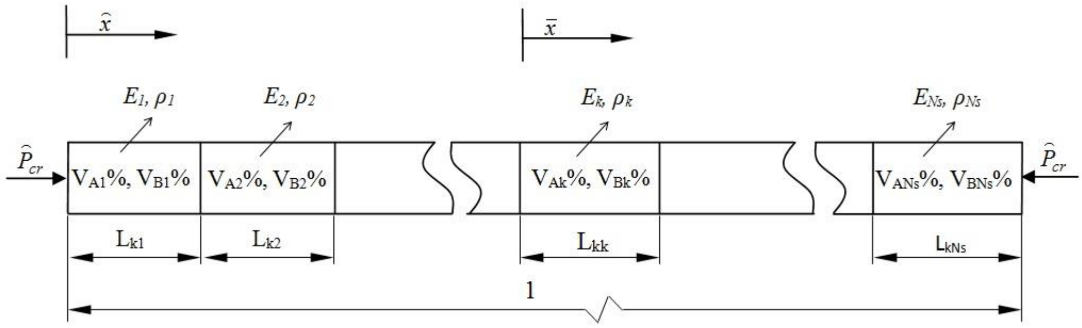
$$\frac{d^2}{dx^2}\left(E(x)\,I(x)\frac{d^2w}{dx^2}\right)+P\,\frac{d^2w}{dx^2}=0; \qquad 0<x<L \tag{1}$$

where w denotes the transverse displacement, x axial coordinate and $E(x)I(x)$, the flexural rigidity which depends on the axially graded modulus of elasticity $E(x)$ and spatially varying geometry with the moment of inertia $I(x)$ of the column. Fig. 1 shows general view of the one-segment and axially graded layered uniform column structures.

Eq. (1) must also be valid for any uniform k^{th} segment and it can be rewritten in the non-dimensional form for the present case in which cross section and moment of inertia are taken to be constant as follows



(a) A uniform column with one segment (Baseline design)



(b) A uniform axially graded layered column with N_S segments

Fig. 1 General view of the columns

$$\hat{w}'''' + P_k^{\,2}\,\hat{w}'' = 0; \quad (k = 1 \text{ to } N_s),\; 0 < \bar{x} < \hat{L}_k \tag{2}$$

where $\dfrac{\hat{P}}{(\hat{E}_k\,\hat{I}_k)} = P_k^{\,2}\ (k = 1 \text{ to } N_s)$, $\hat{x} = \dfrac{x}{L}$ refers the axial coordinate, $\hat{L}_k = \dfrac{L_k}{L}$, the length of kth segment, $\hat{w} = \dfrac{w}{L}$, transverse deflection, $\hat{E}_k = \dfrac{E_k}{E}$, modulus of elasticity, $\hat{I}_k = \dfrac{I_k}{I}$, ($\hat{I}_k = 1$ for the uniform column), moment of inertia, $\hat{P} = \dfrac{PL^2}{EI}$, the axial load, $(')$ denotes the differentiation with respect to the non-dimensional coordinate of $\hat{x}$ and $\bar{x} = \hat{x} - \hat{x}_k$. Also, it is noted that E, L and I are the baseline parameters for comparison.

The transfer matrix method can be applied herein to establish the eigenvalue equation for the buckling of the column. Finally, one can find the following relation

$$\begin{Bmatrix} \hat{w}(1) \\ \varphi(1) \\ \hat{M}(1) \\ \hat{F}(1) \end{Bmatrix} = [\mathrm{T}] \begin{Bmatrix} \hat{w}(0) \\ \varphi(0) \\ \hat{M}(0) \\ \hat{F}(0) \end{Bmatrix} \tag{3}$$

where $[T]$ is the overall transfer matrix relating the state variables at both ends of the column and given as

$$[\mathrm{T}] = [\mathrm{T}_{\mathrm{N_s}}][\mathrm{T}_{\mathrm{N_{s-1}}}][\mathrm{T}_{\mathrm{N_{s-2}}}]\ldots[\mathrm{T}_{\mathrm{N_3}}][\mathrm{T}_{\mathrm{N_2}}][\mathrm{T}_{\mathrm{N_1}}]. \tag{4}$$

Table 1 Characteristic equations for the calculation of the critical buckling loads (P_{cr}) under different boundary conditions

Type	Boundary Conditions	Characteristic Equation	Reference P_{cr} values
Clamped-free	At $\hat{x}=0, \hat{w}=\varphi=0$ and At $\hat{x}=1, \hat{F}=\hat{M}=0$	$T_{33}T_{44}- T_{34}T_{43}=0$	2.4674
Clamped-clamped (Half span)	At $\hat{x}=0, \hat{w}=\varphi=0$ and At $\hat{x}=0.5, \varphi=\hat{F}=0$	$T_{23}T_{44}- T_{24}T_{43}=0$	39.4784
Clamped-clamped (Whole span)	At $\hat{x}=0, \hat{w}=\varphi=0$ and At $\hat{x}=1, \hat{w}=\varphi=0$	$T_{13}T_{24}- T_{14}T_{23}=0$	
Pinned-pinned (Half span)	At $\hat{x}=0, \hat{w}=\hat{M}=0$ At $\hat{x}=0.5, \varphi=\hat{F}=0$	$T_{22}T_{44}- T_{24}T_{42}=0$	9.8696
Pinned-pinned (Whole span)	At $\hat{x}=0, \hat{w}=\hat{M}=0$ At $\hat{x}=1, \hat{w}=\hat{M}=0$	$T_{12}T_{34}- T_{14}T_{32}=0$	

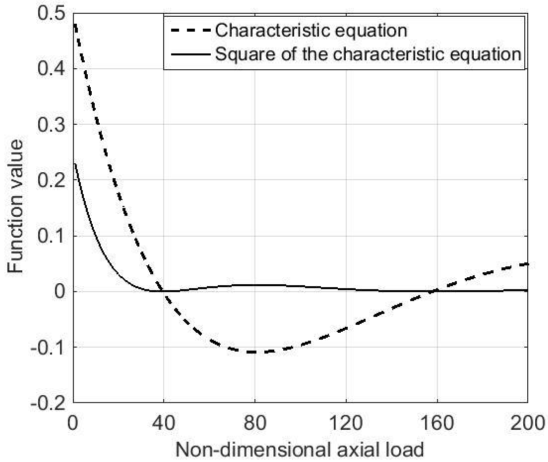


Fig. 2 The functions $[T_{23}T_{44}- T_{24}T_{43}]$ and $[T_{23}T_{44}- T_{24}T_{43}]^2$ versus non-dimensional axial loads

Finally, considering nontrivial solution and boundary conditions, characteristic equations for the critical buckling loads can be obtained and the results are tabulated in Table 1.

At this stage, objective function can be determined as taking square of the characteristic equation. As an example, for the clamped-clamped column, referring to Table 1, the objective function can be written as

$$f(V,\hat{L}_k)=[T_{23}T_{44}-T_{24}T_{43}]^2 \tag{5}$$

and this is the case for other end conditions. It is obvious that the minimum value of the function given by Eq. (5) is zero. That is to say, taking square of the characteristic equation ensures that the minimum value would be zero (Venkataraman 2002). When examining the Eq.(5) from the analytical aspect, the critical buckling load is obtained by equating the term $[T_{23}T_{44}- T_{24}T_{43}]$ to zero which is identical to $\min([T_{23}T_{44}- T_{24}T_{43}]^2)$. To clarify the idea, $[T_{23}T_{44}- T_{24}T_{43}]$ and $[T_{23}T_{44}-$

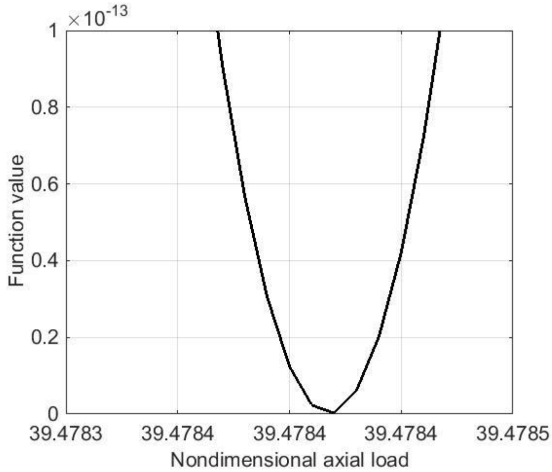


Fig. 3 The objective function, $[T_{23}T_{44}- T_{24}T_{43}]^2$, versus non-dimensional axial loads

$T_{24}T_{43}]^2$ versus non-dimensional axial loads plots are given in Fig. 2 for the clamped-clamped baseline column structure. For this case, the critical buckling load is about 39.4784. Fig. 2 is zoomed around the value 39.4784 for the function $[T_{23}T_{44}- T_{24}T_{43}]^2$ as given in Fig. 3. It is seen that the minimum value of the function, $[T_{23}T_{44}- T_{24}T_{43}]^2$ is the same as the first root of $[T_{23}T_{44}- T_{24}T_{43}]$ function.

2.2 Constraints

In many practical problems, the design variables minimizing objective functions can't be chosen arbitrarily, rather, they have to satisfy certain specified requirements called as constraints. Due to the symmetric conditions, it is possible to consider only half of the column structures for the pinned-pinned and the clamped-clamped end conditions. So, the total mass and the total column length can be reduced to half.

Non-dimensional mass of the column structure which is the first equality constraint can be written as

$$\hat{M}_s = \frac{M_s}{M} = \sum_{k=1}^{N_S} \hat{\rho}_k \hat{L}_k = 1 \qquad \text{for the whole span} \tag{6}$$

$$\hat{M}_s = \frac{M_s}{M} = \sum_{k=1}^{N_S/2} \hat{\rho}_k \hat{L}_k = 0.5 \text{ for the half span} \tag{7}$$

in which M is the total mass of the baseline column structure, M_S total mass of the axially graded uniform column. $\hat{M}_s = 1$ or $\hat{M}_s = 0.5$ means that the column being optimized has the same total mass with those of baseline column structures.

Second equality constraint of the optimization problem is the non-dimensional total length of the column and can be written as

$$\sum_{k=1}^{N_S} \hat{L}_k = 1 \quad \text{for whole span or} \tag{8}$$

$$\sum_{k=1}^{N_S/2} \hat{L}_k = 0.5 \text{ for half span.} \tag{9}$$

Also, for the realistic column design in terms of production, side constraints are present and the upper and the lower limits should be prescribed. Side constraints are $0 \le \left(V, \hat{L}_k\right) \le 1$ for the whole span and $0 \le V \le 1$ and $0 \le \hat{L}_k \le 0.5$ for the half span.

Inclusion of side constraints will provide a set of linear constraints depending on the number of design variables. Considering lower bounds, these can be written as

$$\begin{gathered} g_1(\mathbf{x}) = -\mathbf{x}(1) + (\text{lower bound} + \varepsilon) \\ g_2(\mathbf{x}) = -\mathbf{x}(2) + (\text{lower bound} + \varepsilon) \\ \cdots\cdots\cdots\cdots \\ g_{Ns}(\mathbf{x}) = -x(N_S) + (\text{lower bound} + \varepsilon) \end{gathered} \tag{10}$$

and for upper bounds, these can be written as

$$\begin{gathered} g_{Ns+1}(\mathbf{x}) = \mathbf{x}(1) - (\text{upper bound}) \\ g_{Ns+2}(\mathbf{x}) = \mathbf{x}(2) - (\text{upper bound}) \\ \cdots\cdots\cdots\cdots \\ g_{2*Ns}(\mathbf{x}) = \mathbf{x}(N_S) - (\text{upper bound}) \end{gathered} \tag{11}$$

where ε is the small number depending on the problem and x can be expanded as [V_{A1}, V_{A2},…, V_{ANs} ,L_{k1}, L_{k2},… L_{kNs}].

2.3 Optimization procedure

Before explaining the optimization procedure, it is useful to mention about the column structure considered. The column consists of different number of segments and each segment is made of two different materials and each segment has different material properties (i.e., Elastic Modulus and mass density) depending on volume fraction of the materials. Let *A* and *B* denote the different materials used. For prediction of Young's modulus and the mass density, Halpin–Tsai model is used and the following relations, under the assumption that no voids present, can be written as

$$V_A(x) + V_B(x) = 1 \tag{12}$$

$$E(x) = V_A(x)\, E_A + V_B(x)\, E_B \tag{13}$$

$$\rho(x) = V_A(x)\, \rho_A + V_B(x)\, \rho_B \tag{14}$$

On the other hand, all P_{cr} values obtained from optimization cycle are compared with those obtained from the baseline designs. The baseline design has uniform material properties and it is constructed from the same material with equal volume fraction, that is V_A=V_B=50%. So, Young's modulus and the mass density of the baseline design can be calculated as

$$E = \frac{(E_A + E_B)}{2} \tag{15}$$

$$\rho = \frac{(\rho_A + \rho_B)}{2} \tag{16}$$

In addition, for each segment of the column, the mass density and Young's modulus can be obtained using the following relations

$$\hat{\rho}_k = \frac{\rho_A V_{(A,k)} + \rho_B V_{(B,k)}}{\rho}; \quad (k = 1 \text{ to } N_s) \tag{17}$$

$$\hat{E}_k = \frac{E_A V_{(A,k)} + E_B V_{(B,k)}}{E}; \quad (k = 1 \text{ to } N_s) \tag{18}$$

It is well known that Young's modulus, mass density and segment length are the main factors affecting the critical buckling loads. P_{cr}, the fitness function, should be explicitly expressed as a function of them from the eigen-value routine. This is not the case for the present problem especially for the column structures having higher segment numbers. Therefore, P_{cr} is assumed to take predefined value depending on the end conditions. Its value is somewhat bigger than the one given in Table 1 for each end conditions. After assigning the critical buckling loads, then, V and $\hat{L}_k$ that minimize square of the characteristic equation are searched during the optimization cycle. So, the present optimization problem can be regarded as a root finding problem along with constraints. As a result, the present optimization problem can be stated as follows,

Find $\vec{x}(V, \hat{L}_k) = \begin{Bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{Bmatrix}$ which minimizes [Characteristic Equation (V, $\hat{L}_k$)]^2

subject to first equality constraint given in Eq. (6) or (7)
second equality constraint given in Eq. (8) or (9)
inequality constraints given in Eq. (10) and (11).

After stating optimization problem as shown above, at this stage, any optimization tool can be used to carry out design routine. There is no single method for solving such a constrained optimization problem. Extensive traditional and modern optimization methods have been developed for solving different types of optimization problems. Detailed information about optimization techniques can be found in the book (Rao 2009). On the other hand, parallel to developing computer technology, stochastic methods are attracting increasing popularity among researchers in engineering areas. Various heuristic optimization methods have been applied in structural engineering (Rahmanian *et al.* 2014, Ertas 2013, Yildiz 2013). Differential Evolution is the most popular method found in constrained optimization. This is due to its good and consistent performance and its simplicity (Mezura-Montes and Coello 2011). As an optimization tool, Differential Evolution is used in the present study. DE introduced by (Storn and Price 1997) is a meta-heuristics and population-based optimization method that optimizes a problem by iteratively trying to improve a candidate solution with regard to the given conditions. DE does not use the gradient of the problem being optimized and so, it does not require for fitness and constraint functions to be differentiable. DE works through three stages: Mutation, crossover and selection. In DE, a solution vector is randomly initialized at the beginning and the solution can be improved by applying mutation, crossover and selection operators. Mutation and crossover are used to generate new vectors and selection is then used to determine which new generated vectors are the

Table 2 Program parameters used in the optimization routine

Parameter	Values
Crossover probability:	0.7
Scaling factor for differential evolution operator:	0.5
Probability of choosing element from offspring in crossover:	0.8
Penalty parameter:	10^5 f or N_S=2 and 3, 10^3 for N_S=5
Number of generations :	1000
Population size:	50

best and retained for the next iteration. Detailed information can be found in (Storn and Price 1997).

In addition, in this study, Exterior Penalty Functions with static penalty factor which do not require initial feasible design variables satisfying all the constraints are adapted to the optimization cycle. A high penalty function value is assumed (de Melo 2012). To solve an optimization problem involving equality and inequality constraints, the following formulation can be used (Venter 2010).

$$\phi_k = \phi\left(\vec{x}, r_k\right) = f\left(\vec{x}\right) + r_k \sum_{j=1}^{p} <g_j\left(\vec{x}\right)>^2 + r_k \sum_{j=1}^{m} h_j\left(\vec{x}\right)^2 \tag{19}$$

where ϕ_k denotes a new function constructed by augmenting a penalty term to the objective function, r_k , penalty parameter and $g_j\left(\vec{x}\right) = \left\{max\left[0, g_j\left(\vec{x}\right)\right]\right\}^2$.

To perform the optimization routine, script files in Matlab environment is written based on the considerations outlined above. Program parameters used in the optimization routine are given in Table 2. In addition, optimization runs were terminated after a certain generation which is commonly used in evolutionary algorithms (Yildiz 2013). After preliminary tests, maximum number of generation is chosen to be 1000 for all cases considered.

3. Results and discussions

The columns are made of two different materials (*E*-glass/epoxy) denoting *A* and *B* and material properties of them are ρ_A=2.54 g/cm^3, ρ_B=1.27 g/cm^3, E_A=73 GPa and E_B=4.3 GPa (Maalawi 2009). Three different end conditions are considered: Clamped-clamped, clamped-free and pinned-pinned. Three different column structures constructing from two, three and five segments are examined. In the following tables, gain is the percentage increase in the buckling load and it is calculated by comparing the results obtained from the one-segment baseline and multi segmented columns. The terms $\left(V, \hat{L}\right)_k$ in the tables are expanded as ($V_{A1,}$ V_{A2},…, V_{ANs} ,L_{k1}, L_{k2},… L_{kNs},) for the column with N_S segments.

In Tables 3-5, design variables, objective functions, constraints (Total mass and total length of the columns), buckling load and gain are tabulated for three different segment numbers. It is observed that for all cases, parallel to an increase in the segment number, axial load carrying capacity of the column (i.e., critical buckling load) increase and at the same time, the gain increases. That is to say, the more segment number is, the higher buckling loads are attained. The maximum non-dimensional buckling load attained for the cantilevered column structure with five segments is 2.9620 which represents 20.0454 % optimization gain. It is 46.6060 (18.0544% optimization gain) for the clamped-clamped columns with five segments and 11.8553 (corresponding to 20.1194% gain) for the pinned-pinned columns with five segments. At the same time, care must be taken for the increased manufacturing costs. So, there should be balance between initial design requirements and column's configuration.

On the other hand, as mentioned in (Singh and Li 2009), the piecewise distribution of flexural rigidity (EI) can be obtained by changing the geometry (i.e., moment of inertia, I) or by axially graded material property (i.e., modulus of elasticity, E) along column length. If one obtains the same distribution of E and I, then these two cases can be regarded as to be identical. In the present study, the second case was considered and constant moment of inertia is assumed as made in (Maalawi 2009). The results showed that there is a good agreement between the results obtained from the study carried out by (Maalawi 2009) and present study considering cantilevered and pinned-pinned end conditions.

In addition, as far as practical optimized shape or material distribution along the column length is concerned, the variation of the flexural rigidity is found to be in such a way that higher flexural rigidity is obtained in the segments near to fixed end and lower values found in the segments near to free end for the clamped-free columns. For pinned-pinned case, it is pointed out that for the most economical form and increasing stability, removing a portion of the materials from the ends (lowered E values for present case) and increasing cross-section (increasing E) over the middle section are the suitable choices (Timoshenko and Gere 1961). These phenomena are observed in the tables. As seen in Eq. (13), larger V_{Ak} values (k=1,2,…,N_S) means higher E values since the elastic modulus of Material A is higher than the Material B. Also, smaller V_{Ak} values mean lower E values.

Moreover, it is shown that optimum columns for clamped-clamped case with the number of three and more than three segments are doubly symmetrical, i.e., symmetrical about both the quarter and mid-span points. So, for this case, it is possible to deal only with one-fourth of the total number of the design variables, which reduces computational time. Performing many independent runs for two segments showed that there are two possible column configurations as seen in Table 4. These are obtained by changing the values of the volume fraction and segment length of the first and second segments with each other. These observations are also mentioned in (Maalawi 2002) in which varying moment of inertia and constant elastic modulus are assumed.

Table 3 Numerical values for the cantilevered axially graded layered columns

N_S	Design Variables $\left(V,\hat{L}\right)_k$; k=1,2,..., N_S N_S: Segment Number	Objective function	Constraints		Critical Buckling Load ($\hat{P}_{cr}$)	Gain (%)
			Total Mass	Total Length		
2	(0.6386, 0.1911, 0.6904, 0.3096)	0.0000	1.0000	1.000	2.8220	14.3714
3	(0.6892, 0.3679, 0.0850, 0.5595, 0.2721, 0.1684)	0.0000	1.0000	1.000	2.9126	18.0433
5	(0.7343, 0.5576, 0.3713, 0.1974, 0.0500, 0.4098, 0.2069, 0.1496, 0.1120, 0.1217)	0.0000	1.0000	1.000	2.9620	20.0454

Table 4 Numerical values for the clamped-clamped axially graded layered columns

N_S	Design Variables $(V,\hat{L})_k$; k=1,2,..., N_S N_S: Segment Number	Objective function	Constraints		Critical Buckling Load ($\widehat{P}_{cr}$)	Gain (%)
			Total Mass	Total Length		
2	(0.4231, 0.6999, 0.3611, 0.1389) or (0.6999, 0.4231, 0.1389, 0.3611)	0.0000	0.5000	0.5000	41.0081	3.8748
3	(0.6303, 0.1734, 0.6303, 0.1787, 0.1426, 0.1787)	0.0000	0.5000	0.5000	45.1208	14.2924
5	(0.6784, 0.3519, 0.0660, 0.3519, 0.6784, 0.1463, 0.0662, 0.0749, 0.0662, 0.1463)	0.0000	0.5000	0.4999	46.6060	18.0544

Table 5 Numerical values for the pinned-pinned axially graded layered columns

N_S	Design Variables $(V,\hat{L})_k$; k=1,2,..., N_S N_S: Segment Number	Objective function	Constraints		Critical Buckling Load ($\widehat{P}_{cr}$)	Gain (%)
			Total Mass	Total Length		
2	(0.1886, 0.6390, 0.1543, 0.3457)	0.0000	0.5000	0.5000	11.2868	14.3592
3	(0.0868, 0.3690, 0.6890, 0.0845, 0.1363, 0.2792)	0.0000	0.5000	0.5000	11.6498	18.0372
5	(0.0500, 0.1957, 0.3683, 0.5531, 0.7311, 0.0545, 0.0584, 0.0776, 0.1066, 0.2029)	0.0000	0.5000	0.5000	11.8553	20.1194

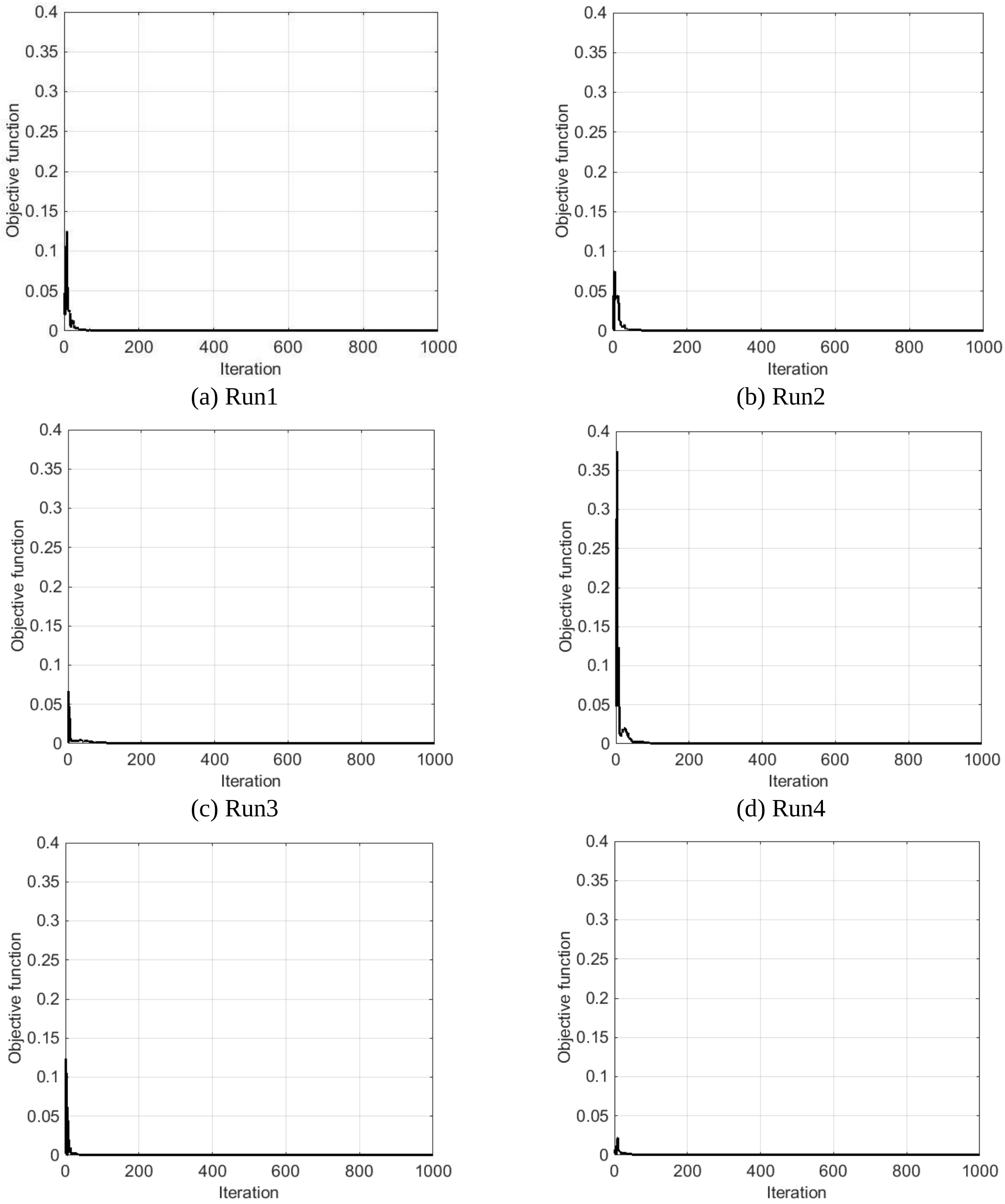
Table 6 Experimental results of 10 independent runs for the clamped-clamped column with five segments

Clamped-clamped end conditions	Iteration	Function Evaluation	Best	Mean	Worst	Std
Two segments	1000	50000	4.94E-14	4.94E-14	4.94E-14	0
Three segment	1000	50000	1.42E-14	1.47E-04	4.88E-4	2.36E-04
Five segments	1000	50000	1.94E-07	2.20E-06	1.02E-05	4.24E-06

As a performance analysis, worst, mean and best solutions found in 10 independent runs for the clamped-clamped columns are given in Table 6. The standard deviation (Std) of the optimal values is also reported in the same table. For the case of two segments, the standard deviation is zero, which means data values obtained from all runs are equal to each other. For the remain two cases, the standard deviation values close to zero, which also indicated that DE is able to produce, to some extent, high accuracy results.

In addition to considerations mentioned above, objective function versus iteration plots for clamped-clamped case are given in Fig. 4. Due to the idea of keeping the length of the paper within the meaningful levels, other cases are not presented here. For the sake of comparison, the range of the objective function in all plots is set to be the same as [0, 0.4]. It is seen that initially, up to a certain iteration (or function evaluation), the curves fluctuated and then the fitness function values converge to a value closer to zero.

All facts mentioned above including statistical results prove the reliability, efficiency and robustness of the Differential Evolution optimization method for the present problem. Also, it can be concluded that this optimization methods can be used in different eigen-value problems especially having transcendental equations, i.e., maximizing natural frequencies of the structural members like beams.

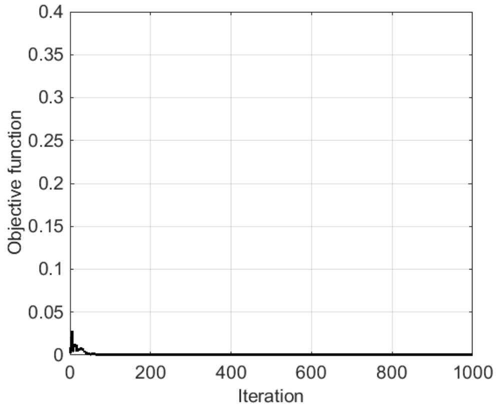


(a) Run1 (b) Run2

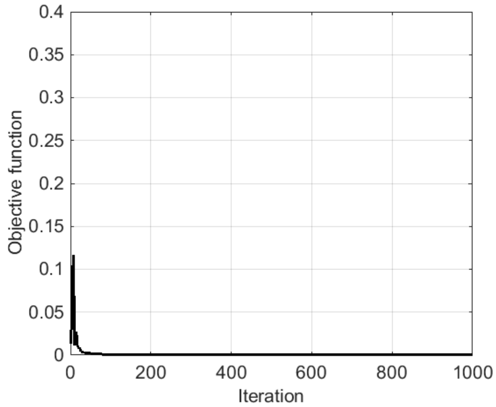
(c) Run3 (d) Run4

(e)Run5 (f) Run6

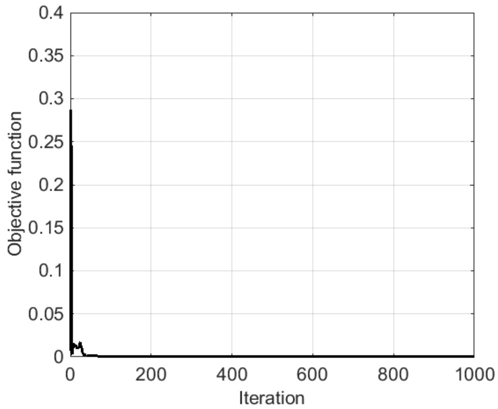
Fig. 4 Objective function versus iteration plots obtained from the 10 independent runs for clamped-clamped columns with five segments



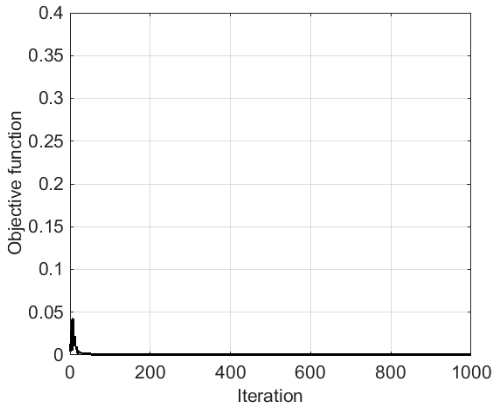
(g) Run7



(h) Run8



(i) Run9



(j) Run10

Fig. 4 Continued

5. Conclusions

In this study, critical buckling load optimization of the axially graded layered uniform columns is presented. The following results can be drawn:

• It is confirmed that for all boundary conditions, an increase in the segment number results in an increase in the critical buckling load. So, the maximum non-dimensional buckling load occurred in the case of the column structures with five segments. At the same time, it shouldn't be forgotten that the manufacturing cost will increase. Therefore, designer or engineer should determine suitable strongest column's configurations according to his/her initial design requirements.

• It is concluded that the maximum non-dimensional buckling load attained for the cantilevered column structure is 2.9620 which represents 20.0454 % optimization gain. It is 46.6060 (18.0544% optimization gain) for the clamped-clamped columns and 11.8553 (corresponding to 20.1194% gain) for the pinned-pinned columns.

• Literature survey indicates that optimum columns for clamped-clamped end conditions with the number of three and more than three segments are symmetrical about both the quarter and mid-span points. That is to say, variation of the moment of inertia or cross-section along the column length (i.e., Stepped column) has such a trend under the assumption of constant material properties. On the other hand, the present results reveal that similar distributions for the material properties (i.e., Young's Modulus and density) are obtained for the column structures where moment of inertia or cross-section do not change during the optimization cycles (Uniform axially graded layered column). To author knowledge, this point is not mentioned in literature as far as optimization studies dealing with axially graded column structures in a piecewise manner are concerned.

• There is a satisfactory agreement between the results obtained in the present study and those obtained in literature. So, this conclusion and also performance experiments confirmed efficiency, reliability and robustness of the Differential Evolution optimization method. Therefore, it can be deduced from this study that Differential Evolution optimization method can also be used in different eigen-value problems especially having transcendental equations, i.e., maximizing natural frequencies of the structural members like beams.

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