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# Buckling of axial compressed cylindrical shells with stepwise variable thickness

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**Abstract.** This paper focuses on an analytical research on the critical buckling load of cylindrical shells with stepwise variable wall thickness under axial compression. An arctan function is established to describe the thickness variation along the axial direction of this kind of cylindrical shells accurately. By using the methods of separation of variables, small parameter perturbation and Fourier series expansion, analytical formulas of the critical buckling load of cylindrical shells with arbitrary axisymmetric thickness variation under axial compression are derived. The analysis is based on the thin shell theory. Analytic results show that the critical buckling load of the uniform shell with constant thickness obtained from this paper is identical with the classical solution. Two important cases of thickness variation pattern are also investigated with these analytical formulas and the results coincide well with those obtained from other authors. The cylindrical shells with stepwise variable wall thickness, which are widely used in actual engineering, are studied by this method and the analytical formulas of critical buckling load under axial compression are obtained. Furthermore, an example is presented to illustrate the effects of each strake's length and thickness on the critical buckling load.

**Keywords:** cylindrical shells; tanks; stepwise variable wall thickness; axial compression; critical buckling load

## 1. Introduction

The cylindrical shells with stepwise variable wall thickness are widely used for the vertical containers, such as petroleum storage tanks, tower vessels and silos. The increase of internal hydrostatic pressure in the shell is addressed by increasing the wall thickness progressively from top to bottom. The equal strength design criterion can be satisfied and the utilization rate of material has been effectively improved in this structure. The cylindrical shells used for large petroleum storage tanks, which are built up from a number of individual strakes of constant thickness, are studied in this paper. They are treated as thin-walled structures in the theoretical study since the thickness-diameter ratios of these cylindrical vessels are generally very small (Chen *et al*. 2011, Yu *et al*. 2012). The simplified mechanical model of a cylindrical shell with stepwise variable wall thickness subjected to uniform axial compression $P$ is shown in Fig. 1. Affected by the combined influences of internal hydrostatic pressure and external environment,

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buckling failure or even overall instability is likely to occur to the cylindrical shell when the axial compression $P$ exceeds the critical buckling load. Therefore, predicting the critical buckling load of axial compressed cylindrical shells accurately has very important significance for the design and application of large oil storage tanks.

In recent years, the study on the buckling of axial compressed cylindrical shells with variable thickness has attracted the attention of some researchers. Elishakoff *et al*. (1992) pioneered the investigation of the influence of axisymmetric thickness variation and initial geometric imperfection on the buckling of shells. The solution obtained was composed of two terms that the first being associated with the shell of constant thickness, whereas the second incorporated the effects of the thickness variation. Based on a system of linearized governing differential equations of cylindrical shells with variable thickness, Koiter *et al*. (1994) derived the asymptotic formulas of critical buckling load of cylindrical shells whose thickness vary trigonometrically in axial direction by a hybrid perturbation-weighted residuals method. The results indicated that this pattern of thickness variation could provoke buckling load reduction significantly. Following the same method, Li *et al*. (1995) dealt with the effect of the same thickness variation on the axial buckling of composite cylindrical shells, and the relation between the buckling load reduction factor and the thickness variation parameter was established, which confirmed the previous conclusions of isotropic cylindrical shells (Koiter *et al*. 1994). By means of Koiter's energy criterion of elastic stability, Li *et al*. (1997) studied the combined effect of small axisymmetric thickness variation and axisymmetric initial imperfection on the buckling behavior of composite cylindrical shells. An asymptotic formula that related the thickness variation parameter and initial imperfection amplitude to the critical buckling load of the axial compressed composite cylindrical shells was obtained, which was a generalization and extension of former investigations. Based on the nonlinear large deflection theory of cylindrical shells as well as the Donnell assumptions, Huang and Han (2009) conducted the nonlinear and postbuckling analyses for axial compressed functionally graded cylindrical shells by using the Ritz energy method and the nonlinear strain-displacement relations of large deformation. The linear and nonlinear critical buckling load were both derived and numerical results showed that dimensional parameters and external thermal environment had significant effects on the buckling behavior. Chen *et al*. (2012) presented asymptotic formulas of the critical buckling load of axial compressed cylindrical shells with arbitrary axisymmetric thickness imperfections by the perturbation method. Meanwhile, researches on the buckling of cylindrical shells subjected to external pressure or combined external pressure and axial compression had already been done by some researchers (Shen and Chen 1991, Guggenberger 1995, Vodenitcharova and Ansourian 1996, Gusic *et al*. 2000, Mirfakhraei and Redekop 2002, Khamlichi *et al*. 2004, Aghajari *et al*. 2006, Nguyen *et al*. 2009, Khazaeinejad *et al*. 2010, Aghajari *et al*. 2011, Yang *et al*. 2012). However, the studies on the critical buckling load of cylindrical shells in the previous literatures were mainly focused on some simple thickness variations which were symmetrical about the center of shells in the axial direction, such as the trigonometric function thickness variation pattern and the local thickness variation pattern (Yang *et al*. 2012). Since the previous analytical methods can't be applied to the cylindrical shells with stepwise thickness variation which are most widely used in engineering practice, a new method is established in this paper to analyze the critical buckling load of cylindrical shells with stepwise thickness variation under axial compression.

The theoretic analysis is quite difficult for the cylindrical shells with stepwise thickness variation because the wall thickness, as shown in Fig. 1, increases discontinuously from top to bottom. An arctan function is then established in this paper to simplify the wall thickness in order

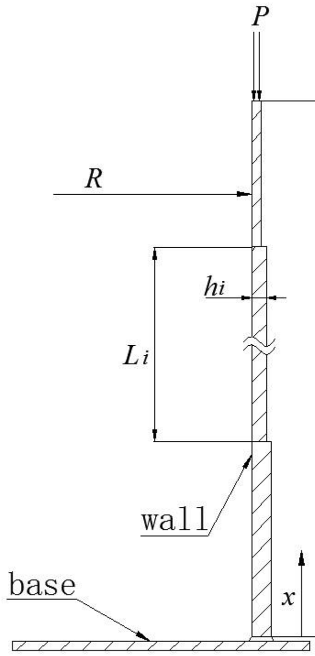


Fig. 1 Diagram of the cylindrical shell with stepwise thickness variation

to make it vary continuously in the axial direction. Based on the system of linearized governing differential equations, analytical formulas of the critical buckling load of cylindrical shells with arbitrary axisymmetric thickness variation under axial compression are derived by the means of separation of variables, small parameter perturbation and Fourier series expansion. These formulas can make up for the deficiency of the former ones that are mainly applied to the cylindrical shells with simple thickness variations. The asymptotic solution of critical buckling load of axial compressed cylindrical shells with stepwise thickness variation is obtained by this method and a cylindrical shell with three strakes is presented to illustrate the effects of each strake's length and thickness on the critical buckling load.

## 2. Establishment of the thickness function

As shown in Fig. 1, the thickness function of the cylindrical shells with axisymmetric stepwise thickness variation can be expressed as follows

$$h(x)=\begin{cases} h_1 & 0 \le x < L_1 \\ h_2 & L_1 \le x < L_2 \\ \cdots & \cdots \\ h_n & L_{n-1} \le x < L_n \end{cases} \tag{1}$$

where $L_i$ and $h_i$ ($1 \le i \le n$) represent the length and thickness of the $i$-th strake respectively from bottom to top.

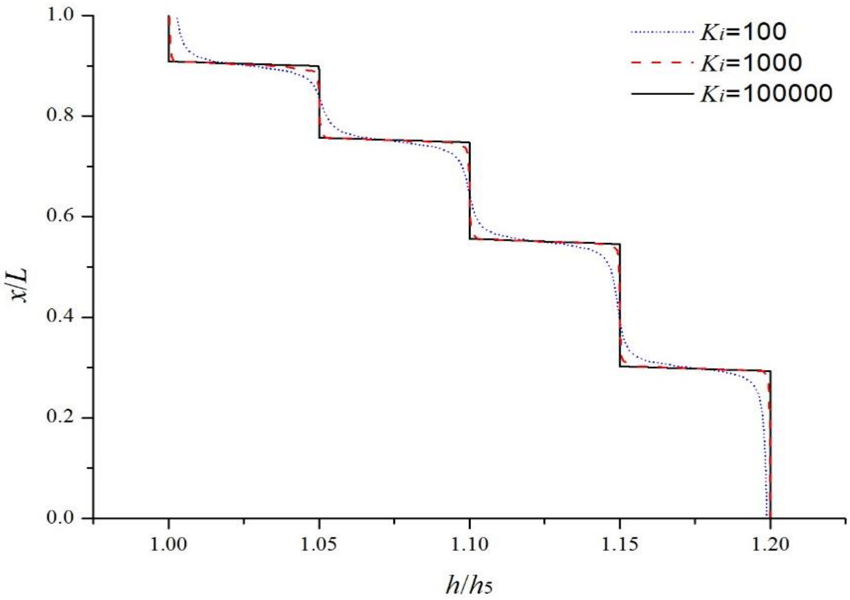


Fig. 2 Approximate thickness description of the cylindrical shell with five strakes

Obviously, the subsection function thickness variation pattern in Eq. (1) brings great difficulties to analysis and solution. In order to solve this, an arctan function is established to describe the thickness variation approximately

$$h(x)=\frac{h_1+h_n}{2}-\frac{2}{\pi}\sum_{i=1}^{n-1}\frac{h_i-h_{i+1}}{2}\tan^{-1}K_i\left(x-\sum_{j=1}^{i}L_j\right) \tag{2}$$

in which $K_i$ represent a series of auxiliary parameters that tend to be infinitely great.

As an example, a cylindrical shell with five strakes is considered here and Eq. (2) is applied to describe its thickness variation. The length of each strake is assumed as

$$L_1=0.3L,\ L_2=0.25L,\ L_3=0.2L,\ L_4=0.15L,\ L_5=0.1L \tag{3}$$

and the relationship of each strake's thickness is assumed as

$$h_1=1.2h_5,\ h_2=1.15h_5,\ h_3=1.1h_5,\ h_4=1.05h_5 \tag{4}$$

The schematic diagram of the cylindrical shell's thickness variation that expressed by Eq. (2) is shown in Fig. 2. It is obvious that as the auxiliary parameters $K_i$ become larger gradually, the thickness variation tends to be the stepwise pattern. When $K_i$ reach 100000, the wall thickness curve in Fig. 2 can describe the thickness variation of the cylindrical shell quite accurately.

## 3. Basic equations

The geometry and coordinate of the cylindrical shell are shown in Fig. 3. The linear governing

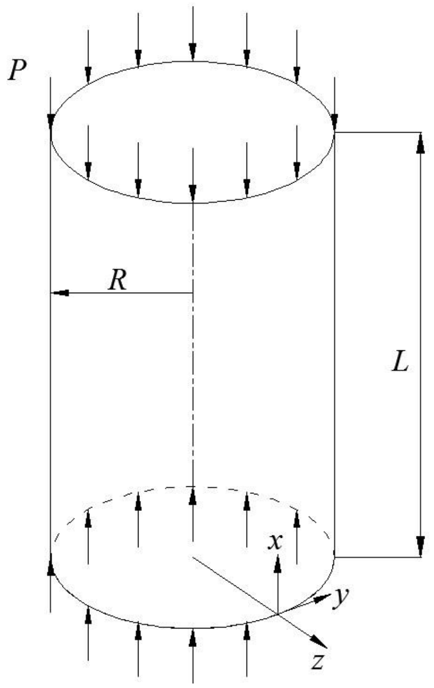


Fig. 3 Geometry and coordinate of the axial compressed cylindrical shell

differential equations of the non-uniform cylindrical shell derived by Koiter *et al*. (1994) are applied in this paper

$$h^2\left(\frac{\partial^4 F}{\partial x^4}+2\frac{\partial^4 F}{\partial x^2\partial y^2}+\frac{\partial^4 F}{\partial y^4}\right)-2h\frac{\partial h}{\partial x}\left(\frac{\partial^3 F}{\partial x^3}-v\frac{\partial^3 F}{\partial x\partial y^2}\right)-2(1+v)h\frac{\partial h}{\partial x}\frac{\partial^3 F}{\partial x\partial y^2}+\left[2\left(\frac{\partial h}{\partial x}\right)^2-h\frac{\partial^2 h}{\partial x^2}\right]\left(\frac{\partial^2 F}{\partial x^2}-v\frac{\partial^2 F}{\partial y^2}\right)=\frac{Eh^3}{R}\frac{\partial^2 w}{\partial x^2} \tag{5a}$$

$$\frac{Eh^3}{12(1-v^2)}\left(\frac{\partial^4 W}{\partial x^4}+2\frac{\partial^4 W}{\partial x^2\partial y^2}+\frac{\partial^4 W}{\partial y^4}\right)+\frac{6Eh^2}{12(1-v^2)}\frac{\partial h}{\partial x}\left(\frac{\partial^3 W}{\partial x^3}+\frac{\partial^3 W}{\partial x\partial y^2}\right)+\frac{E}{12(1-v^2)}\left[6h\left(\frac{\partial h}{\partial x}\right)^2+3h^2\frac{\partial^2 h}{\partial x^2}\right]\left(\frac{\partial^2 W}{\partial x^2}+v\frac{\partial^2 W}{\partial y^2}\right)+\frac{1}{R}\frac{\partial^2 F}{\partial x^2}+P\frac{\partial^2 W}{\partial x^2}=0 \tag{5b}$$

where $F$ and $W$ represent the Airy stress function and radial displacement, positive outward, respectively; $v$ is the Poisson’s ratio; $E$ is the modulus of elasticity; $P$ represents the uniform axial pressure; $h$ is the thickness function that only changes in the axial direction.

To make solutions more general, the following non-dimensional parameters are introduced

$$f=\frac{F}{D_0}\quad H=\frac{h}{h_0}\quad \omega=\frac{W}{h_0}\quad \xi=\frac{x}{L}\quad \eta=\frac{y}{L} \tag{6}$$

where $h_0$ is the nominal thickness of the shell; $D_0 = Eh_0^3/12(1-v^2)$, is the flexural rigidity of the cylindrical shell with nominal thickness.

The governing differential equations can be transformed in the nondimensional form by substituting Eq. (6) into Eqs. (5a)-(5b)

$$H^2\left(\frac{\partial^4 f}{\partial\xi^4}+2\frac{\partial^4 f}{\partial\xi^2\partial\eta^2}+\frac{\partial^4 f}{\partial\eta^4}\right)-2H\frac{\partial H}{\partial\xi}\left(\frac{\partial^3 f}{\partial\xi^3}-v\frac{\partial^3 f}{\partial\xi\partial\eta^2}\right)-2(1+v)H\frac{\partial H}{\partial\xi}\frac{\partial^3 f}{\partial\xi\partial\eta^2}$$
$$+\left[2\left(\frac{\partial H}{\partial\xi}\right)^2-H\frac{\partial^2 H}{\partial\xi^2}\right]\left(\frac{\partial^2 f}{\partial\xi^2}-v\frac{\partial^2 f}{\partial\eta^2}\right)=\frac{12(1-v^2)L^2}{Rh_0}H^3\frac{\partial^2\omega}{\partial\xi^2} \tag{7a}$$

$$H^3\left(\frac{\partial^4\omega}{\partial\xi^4}+2\frac{\partial^4\omega}{\partial\xi^2\partial\eta^2}+\frac{\partial^4\omega}{\partial\eta^4}\right)+6H^2\frac{\partial H}{\partial\xi}\left(\frac{\partial^3\omega}{\partial\xi^3}+\frac{\partial^3\omega}{\partial\xi\partial\eta^2}\right)$$
$$+\left[6H\left(\frac{\partial H}{\partial\xi}\right)^2+3H^2\frac{\partial^2 H}{\partial\xi^2}\right]\left(\frac{\partial^2\omega}{\partial\xi^2}+v\frac{\partial^2\omega}{\partial\eta^2}\right)+\frac{L^2}{Rh_0}\frac{\partial^2 f}{\partial\xi^2}+\frac{PL^2}{D_0}\frac{\partial^2\omega}{\partial\xi^2}=0 \tag{7b}$$

The case of simply-supported boundary conditions of the cylindrical shell is considered here, then $f$ and $\omega$ should satisfy (Li *et al*. 1995)

$$f\big|_{\xi=0,1}=0 \qquad \omega\big|_{\xi=0,1}=0 \qquad \left.\frac{\partial^2 f}{\partial\xi^2}\right|_{\xi=0,1}=0 \qquad \left.\frac{\partial^2\omega}{\partial\xi^2}\right|_{\xi=0,1}=0 \tag{8}$$

In view of the separation of variables, solution of stress function and radial displacement can be chosen as satisfying the boundary conditions as follows

$$f(\xi,\eta)=\bar{f}(\xi)\cos\frac{nL}{R}\eta \tag{9a}$$

$$\omega(\xi,\eta)=\bar{\omega}(\xi)\cos\frac{nL}{R}\eta \tag{9b}$$

where $n$ denotes the number of circumferential buckling waves.

Substituting Eqs. (9a)-(9b) into Eqs. (7a)-(7b), the ordinary differential equations as follows are obtained

$$H^2\bar{f}^{(4)}-2HH'\bar{f}'''+\left[2H'^2-HH''-2\left(\frac{nL}{R}\right)^2H^2\right]\bar{f}''+2HH'\left(\frac{nL}{R}\right)^2\bar{f}'$$
$$+\left[\left(\frac{nL}{R}\right)^4H^2+\left(2H'^2-HH''\right)v\left(\frac{nL}{R}\right)^2\right]\bar{f}=\frac{12(1-v^2)L^2}{Rh_0}H^3\bar{\omega}'' \tag{10a}$$

$$H^3\bar{\omega}^{(4)}+6H^2H'\bar{\omega}'''+\left[\frac{PL^2}{D_0}+6HH'^2+3H^2H''-2\left(\frac{nL}{R}\right)^2H^3\right]\bar{\omega}''-6\left(\frac{nL}{R}\right)^2H^2H'\bar{\omega}'$$
$$+\left[\left(\frac{nL}{R}\right)^4H^3-v\left(\frac{nL}{R}\right)^2\left(6HH'^2+3H^2H''\right)\right]\bar{\omega}+\frac{L^2}{Rh_0}\bar{f}''=0 \tag{10b}$$

## 4. Solution of critical buckling load

The shell thickness that varies in the axial direction arbitrarily will be assumed as the following form

$$h = h_0 + \varepsilon h_1 \tag{11}$$

where $\varepsilon$($0 \le \varepsilon \le 1$) is the nondimensional parameter indicating the magnitude of the thickness variation; $h_1$ denotes unknown thickness variation function that determines the thickness of the shell.

Eq. (11) can be written in the nondimensional form

$$H = 1 + \varepsilon \frac{h_1}{h_0} = 1 + \varepsilon H_1 \tag{12}$$

$\bar{f}(\xi)$ and $\bar{\omega}(\xi)$ can also be expressed in term of $\varepsilon$ as follows (Koiter *et al*. 1994, Li *et al*. 1995, Chen *et al*. 2012)

$$\bar{f}(\xi) = \bar{f}_0(\xi) + \varepsilon \bar{f}_1(\xi) + \varepsilon^2 \bar{f}_2(\xi) + \varepsilon^3 \bar{f}_3(\xi) + \ldots = \bar{f}_0(\xi) + \sum_{n=1}^{\infty} \varepsilon^n \bar{f}_n(\xi) \tag{13a}$$

$$\bar{\omega}(\xi) = \bar{\omega}_0(\xi) + \varepsilon \bar{\omega}_1(\xi) + \varepsilon^2 \bar{\omega}_2(\xi) + \varepsilon^3 \bar{\omega}_3(\xi) + \ldots = \bar{\omega}_0(\xi) + \sum_{n=1}^{\infty} \varepsilon^n \bar{\omega}_n(\xi) \tag{13b}$$

where $\bar{f}_0(\xi)$ and $\bar{\omega}_0(\xi)$ represent the solutions of the cylindrical shell with constant thickness $h_0$; $\bar{f}_n(\xi)$ and $\bar{\omega}_n(\xi)$ represent the variation of stress function and radial displacement caused by the unknown thickness variation function $h_1$.

Due to the arbitrariness of the parameter $\varepsilon$, $\bar{f}_n(\xi)$ and $\bar{\omega}_n(\xi)$ ($n \ge 0$) must satisfy the following boundary conditions

$$\bar{f}_n(0,1) = 0 \qquad \bar{\omega}_n(0,1) = 0 \qquad \bar{f}_n''(0,1) = 0 \qquad \bar{\omega}_n''(0,1) = 0 \tag{14}$$

The critical buckling load of the cylindrical shell with thickness variation that described in Eq. (11) is assumed to be the following form

$$P = P_0 + \varepsilon P_1 + \varepsilon^2 P_2 + \varepsilon^3 P_3 + \ldots = P_0 + \sum_{n=1}^{\infty} \varepsilon^n P_n \tag{15}$$

where $P_0$ denotes the critical buckling load of the cylindrical shell with constant thickness $h_0$; $P_n$ are correction terms of critical buckling load corresponding to the unknown thickness variation function $h_1$.

Substituting Eqs. (12)-(13b) and Eq. (15) into Eqs. (10a)-(10b) respectively, collecting the like terms in $\varepsilon^0$

$$\Psi_1(\bar{f}_0) = 4c^2 r^2 \bar{\omega}_0'' \tag{16a}$$

$$\Psi_2(\bar{\omega}_0) = -r^2 \bar{f}_0'' \tag{16b}$$

and then collecting the like terms in $\varepsilon^n$

$$\Psi_1\left(\bar{f}_n\right) = 4c^2 r^2 \bar{\omega}_n'' + T_n \tag{17a}$$

$$\Psi_2\left(\bar{\omega}_n\right) = -r^2 \bar{f}_n'' - \frac{P_n L^2}{D_0}\bar{\omega}_0'' + Q_n \tag{17b}$$

the differential operators $\Psi_1$ and $\Psi_2$ are defined as follows

$$\Psi_1\left(f\right) = f^{(4)} - 2N^2 f'' + N^4 f \tag{18a}$$

$$\Psi_2\left(\omega\right) = \omega^{(4)} + \left(\frac{P_0 L^2}{D_0} - 2N^2\right)\omega'' + N^4\omega \tag{18b}$$

where

$$N = \frac{nL}{R} \qquad r = \frac{L}{\sqrt{Rh_0}} \qquad c = \sqrt{3\left(1-v^2\right)} \tag{19}$$

the first term of $T_n$ and $Q_n$ are provided here

$$\begin{aligned} T_1 = &-2H_1\bar{f}_0^{(4)} + 2H_1'\bar{f}_0''' + \left(H_1'' + 4N^2H_1\right)\bar{f}_0'' - 2N^2H_1'\bar{f}_0' \\ &-\left(2N^4H_1 - vN^2H_1''\right)\bar{f}_0 + 12c^2r^2H_1\bar{\omega}_0'' \end{aligned} \tag{20a}$$

$$\begin{aligned} Q_1 = &-3H_1\bar{\omega}_0^{(4)} - 6H_1'\bar{\omega}_0''' - \left(3H_1'' - 6N^2H_1\right)\bar{\omega}_0'' + 6N^2H_1'\bar{\omega}_0' \\ &-\left(3N^4H_1 - 3vN^2H_1''\right)\bar{\omega}_0 \end{aligned} \tag{20b}$$

Firstly, the solutions of Eqs. (16a)-(16b) satisfying the boundary conditions are sought in the following form

$$\bar{f}_0\left(\xi\right) = A_{0m}\sin\frac{pL}{R}\xi = A_{0m}\sin\left(2m+1\right)\pi\xi = A_{0m}\sin\alpha_m\xi \tag{21a}$$

$$\bar{\omega}_0\left(\xi\right) = B_{0m}\sin\frac{pL}{R}\xi = B_{0m}\sin\left(2m+1\right)\pi\xi = B_{0m}\sin\alpha_m\xi \tag{21b}$$

where $p=(2m+1)\pi R/L$ is the number of half-waves along the shell length at buckling; $\alpha_m=(2m+1)\pi$; $m$ is an integer; $A_{0m}$ and $B_{0m}$ are unknown constants.

Substituting Eqs. (21a)-(21b) into Eq. (16a) yields

$$\left(\alpha_m^2 + N^2\right)^2 A_{0m} = -4c^2r^2\alpha_m^2 B_{0m} \tag{22}$$

Substituting Eqs. (21a)-(21b) into Eq. (16b) yields

$$\frac{P_0L^2}{D_0} = \frac{\left(\alpha_m^2 + N^2\right)^2}{\alpha_m^2} + \frac{4c^2r^4\alpha_m^2}{\left(\alpha_m^2 + N^2\right)^2} \tag{23}$$

According to the inequality theory, $P_0$ reaches minimum when it satisfies

$$\frac{\left(\alpha_m^2+N^2\right)^2}{\alpha_m^2}=\frac{4c^2r^4\alpha_m^2}{\left(\alpha_m^2+N^2\right)^2} \tag{24}$$

From Eqs. (23)-(24), the critical buckling load is obtained

$$P_{0cr}=\frac{4cr^2D_0}{L^2}=\frac{Eh_0^2}{R\sqrt{3\left(1-v^2\right)}} \tag{25}$$

Eq. (25) is exactly the elastic classical critical buckling load of the axial compressed cylindrical shell with constant thickness $h_0$ (Timoshenko 1961).

Then the solutions of Eqs. (17a)-(17b) satisfying the boundary conditions are sought in the Fourier series form

$$\bar{f}_n(\xi)=\sum_{k=0}^{\infty}A_{nk}\sin(2k+1)\pi\xi=\sum_{k=0}^{\infty}A_{nk}\sin\alpha_k\xi \tag{26a}$$

$$\bar{\omega}_n(\xi)=\sum_{k=0}^{\infty}B_{nk}\sin(2k+1)\pi\xi=\sum_{k=0}^{\infty}B_{nk}\sin\alpha_k\xi \tag{26b}$$

where $k$ is an integer; $A_{nk}$ and $B_{nk}$ are unknown constants.

Substituting Eqs. (26a)-(26b) into Eq. (17a) -(17b) yields

$$\sum_{k=0}^{\infty}\left(\alpha_k{}^2+N^2\right)^2A_{nk}\sin\alpha_k\xi=-4c^2r^2\sum_{k=0}^{\infty}\alpha_k^2B_{nk}\sin\alpha_k\xi+T_n \tag{27a}$$

$$\sum_{k=0}^{\infty}\left[\left(\alpha_k^2+N^2\right)^2-\frac{P_0L^2}{D_0}\alpha_k^2\right]B_{nk}\sin\alpha_k\xi=\sum_{k=0}^{\infty}r^2\alpha_k^2A_{nk}\sin\alpha_k\xi+\frac{P_nL^2}{D_0}\alpha_m^2B_{0m}\sin\alpha_m\xi+Q_n \tag{27b}$$

Multiplying the both sides of Eqs. (27a)-(27b) by sin$\alpha_m\xi$ respectively, then integrating $\xi$ from 0 to 1

$$\left(\alpha_m^2+N^2\right)^2A_{nm}=-4c^2r^2\alpha_m^2B_{nm}+2\int_0^1T_n\sin\alpha_m\xi d\xi \tag{28a}$$

$$\left[\left(\alpha_m^2+N^2\right)^2-\frac{P_0L^2}{D_0}\alpha_m^2\right]B_{nm}=r^2\alpha_m^2A_{nm}+\frac{P_nL^2}{D_0}\alpha_m^2B_{0m}+2\int_0^1Q_n\sin\alpha_m\xi d\xi \tag{28b}$$

The parameter $A_{nm}$ can be eliminated by combining Eqs. (28a)-(28b)

$$\left[\left(\alpha_m^2+N^2\right)^2+\frac{4c^2r^4\alpha_m^2}{\left(\alpha_m^2+N^2\right)^2}-\frac{P_0L^2}{D_0}\alpha_m^2\right]B_{nm}=\frac{P_nL^2}{D_0}\alpha_m^2B_{0m}+2\int_0^1Q_n\sin\alpha_m\xi d\xi+\frac{2r^2\alpha_m^2\int_0^1T_n\sin\alpha_m\xi d\xi}{\left(\alpha_m^2+N^2\right)^2} \tag{29}$$

Considering Eq. (23), the left side of Eq. (29) becomes zero, then Eq. (29) can be transformed as

$$\frac{P_n L^2}{D_0} = -\frac{2}{B_{0m}}\left(\frac{\int_0^1 Q_n \sin\alpha_m\xi d\xi}{\alpha_m^2} + \frac{r^2\int_0^1 T_n \sin\alpha_m\xi d\xi}{\left(\alpha_m^2+N^2\right)^2}\right) \tag{30}$$

Thus, the asymptotic formula of the critical buckling load of axial compressed cylindrical shell with arbitrary axisymmetric thickness variation is obtained as follows

$$\begin{aligned} P_{cr} &= P_{0cr} + \sum_{n=1}^{\infty}\varepsilon^n P_n \\ &= \frac{4cr^2 D_0}{L^2} - \frac{2D_0}{L^2 B_{0m}}\sum_{n=1}^{\infty}\varepsilon^n\left(\frac{\int_0^1 Q_n \sin\alpha_m\xi d\xi}{\alpha_m^2} + \frac{r^2\int_0^1 T_n \sin\alpha_m\xi d\xi}{\left(\alpha_m^2+N^2\right)^2}\right) \end{aligned} \tag{31}$$

## 4. Comparison and discussion

The following thickness variation form will be discussed here

$$h(x) = h_0\left(1+\varepsilon\cos\frac{2p}{R}\pi\right) \qquad \left(x\in[0,L]\right) \tag{32}$$

where $h(x)$ represents the actual thickness of the shell that varies in the axial direction; other parameters are the same with previous descriptions.

The critical buckling load of axial compressed cylindrical shell with such thickness variation has been researched by Koiter *et al*. (1994), Li *et al*. (1995), Li *et al*. (1997). For the purpose of verification of the method established in this paper, results derived through this method are compared with previous ones.

Eq. (32) can be written in the non-dimensional form as follows

$$H(\xi) = 1+\varepsilon\cos 2\alpha_m\xi \qquad \left(\xi\in[0,1]\right) \tag{33a}$$

$$H_1 = \cos 2\alpha_m\xi \tag{33b}$$

Substituting Eqs. (33a)-(33b) into Eqs. (20a)-(20b) yields

$$\begin{aligned} T_1 =& \left[4\alpha_m^4 - 2\left(\alpha_m^2+N^2\right)^2 - 4v\alpha_m^2N^2\right]A_{0m}\sin\alpha_m\xi\cos 2\alpha_m\xi \\ & -12c^2r^2\alpha_m^2B_{0m}\sin\alpha_m\xi\cos 2\alpha_m\xi + 4\alpha_m^2\left(\alpha_m^2+N^2\right)A_{0m}\cos\alpha_m\xi\sin 2\alpha_m\xi \end{aligned} \tag{34a}$$

$$\begin{aligned} Q_1 =& -3\left[\left(\alpha_m^2+N^2\right)^2 + 4\alpha_m^2\left(\alpha_m^2+vN^2\right)\right]B_{0m}\sin\alpha_m\xi\cos 2\alpha_m\xi \\ & -12\alpha_m^2\left(\alpha_m^2+N^2\right)B_{0m}\cos\alpha_m\xi\sin 2\alpha_m\xi \end{aligned} \tag{34b}$$

Substituting Eqs. (34a)-(34b) and Eq. (22) into Eq. (30) yields

$$\frac{P_1 L^2}{D_0} = \left[\left(\alpha_m^2 + N^2\right)^2 + 2(1+v)\alpha_m^2 N^2\right]\frac{4c^2 r^4 \alpha_m^2}{\left(\alpha_m^2 + N^2\right)^4} - \frac{3\left(\alpha_m^2 + N^2\right)^2}{2\alpha_m^2} + 6(1-v)N^2 - \frac{6c^2 r^4 \alpha_m^2}{\left(\alpha_m^2 + N^2\right)^2} \tag{35}$$

The nondimensional critical buckling load reduction factor due to the thickness variation is defined as

$$\lambda = \frac{P_{cr}}{P_{0cr}} = 1 + \frac{P_1}{P_{0cr}}\varepsilon + O\left(\varepsilon^2\right) \tag{36}$$

Now two classical buckling modes will be investigated in this paper:

*Case A*

The classical critical buckling load corresponding to the buckling mode at the top of the Koiter semi-circle (Koiter *et al*. 1994) is studied here. In this case, the buckling wave numbers $p$ and $n$ can be expressed as follows

$$p = n = \frac{p_0}{2} \qquad p_0^2 = 2c\frac{R}{h_0} \tag{37}$$

then the parameters satisfy the following form

$$\alpha_m^2 = N^2 = \frac{1}{2}cr^2 \tag{38}$$

Substituting Eq. (38) into Eq. (35) yields

$$P_1 = -\frac{vEh_0^2}{2R\sqrt{3\left(1-v^2\right)}} \tag{39}$$

Substituting Eq. (39) into Eq. (36) yields

$$\lambda = 1 - \frac{v}{2}\varepsilon + O\left(\varepsilon^2\right) \tag{40}$$

Eq. (40) coincides with the result obtained by Koiter *et al*. (1994).

*Case B*

The axisymmetric buckling mode is then researched. In this case, the buckling wave numbers $p$ and $n$ can be expressed as follows

$$n = 0 \qquad p = p_0 \qquad p_0^2 = 2c\frac{R}{h_0} \tag{41}$$

then the parameters satisfy

$$N = 0 \qquad \alpha_m^2 = 2cr^2 \tag{42}$$

Substituting Eq. (42) into Eq. (35) yields

$$P_1 = -\frac{Eh_0^2}{R\sqrt{3\left(1-v^2\right)}} \tag{43}$$

Substituting Eq. (43) into Eq. (36) yields

$$\lambda = 1 - \varepsilon + O\left(\varepsilon^2\right) \tag{44}$$

Eq. (44) again coincides with the result obtained by Koiter *et al*. (1994), Li *et al*. (1995), Li *et al*. (1997).

The above analysis shows that the nondimensional critical buckling load reduction factors derived in this paper coincide well with the known results, which demonstrates the rationality and accuracy of this method.

## 5. Application

Now the research on the critical buckling load of axial compressed cylindrical shell with stepwise thickness variation that introduced in Section 2 will be conducted by the method presented in this paper.

Supposing that

$$h_0 = \frac{h_1 + h_n}{2} \qquad \varepsilon = \frac{2}{\pi}\frac{h_1 - h_n}{h_1 + h_n} \tag{45}$$

Substituting Eq. (45) into Eq. (2) and taking into account Eq. (12), the nondimensional form of the thickness functions are obtained as follows

$$H\left(\xi\right) = 1 - \varepsilon \sum_{i=1}^{n-1} \frac{h_i - h_{i+1}}{h_1 - h_n} \tan^{-1} K_i \left( \xi L - \sum_{j=1}^{i} L_j \right) \tag{46a}$$

$$H_1 = -\sum_{i=1}^{n-1} \frac{h_i - h_{i+1}}{h_1 - h_n} \tan^{-1} K_i \left( \xi L - \sum_{j=1}^{i} L_j \right) \tag{46b}$$

Substituting Eqs. (46a)-(46b) into Eqs. (20a)-(20b) yields

$$T_1 = \left[ 12c^2 r^2 \alpha_m^2 \sum_{i=1}^{n-1} \frac{h_i - h_{i+1}}{h_1 - h_n} \tan^{-1} K_i \left( \xi L - \sum_{j=1}^{i} L_j \right) \right] B_{0m} \sin \alpha_m \xi$$
$$+ \left[ 2\left(\alpha_m^2 + N^2\right)^2 \sum_{i=1}^{n-1} \frac{h_i - h_{i+1}}{h_1 - h_n} \tan^{-1} K_i \left( \xi L - \sum_{j=1}^{i} L_j \right) \right] A_{0m} \sin \alpha_m \xi$$

$$
\begin{aligned}
&+\left[2\alpha_m\left(\alpha_m^2+N^2\right)\sum_{i=1}^{n-1}\frac{h_i-h_{i+1}}{h_1-h_n}\frac{LK_i}{1+K_i^2\left(\xi L-\sum_{j=1}^{i}L_j\right)^2}\right]A_{0m}\cos\alpha_m\xi \\
&+\left[\left(vN^2-\alpha_m^2\right)\sum_{i=1}^{n-1}\frac{h_i-h_{i+1}}{h_1-h_n}\frac{2L^2K_i^3\left(\xi L-\sum_{j=1}^{i}L_j\right)}{\left[1+K_i^2\left(\xi L-\sum_{j=1}^{i}L_j\right)^2\right]^2}\right]A_{0m}\sin\alpha_m\xi
\end{aligned}
\tag{47a}
$$

$$
\begin{aligned}
Q_1=&\,3\left[\left(\alpha_m^2+N^2\right)^2\sum_{i=1}^{n-1}\frac{h_i-h_{i+1}}{h_1-h_n}\tan^{-1}K_i\left(\xi L-\sum_{j=1}^{i}L_j\right)\right]B_{0m}\sin\alpha_m\xi \\
&+3\left[\left(vN^2+\alpha_m^2\right)\sum_{i=1}^{n-1}\frac{h_i-h_{i+1}}{h_1-h_n}\frac{2L^2K_i^3\left(\xi L-\sum_{j=1}^{i}L_j\right)}{\left[1+K_i^2\left(\xi L-\sum_{j=1}^{i}L_j\right)^2\right]^2}\right]B_{0m}\sin\alpha_m\xi \\
&-\left[6\alpha_m\left(\alpha_m^2+N^2\right)\sum_{i=1}^{n-1}\frac{h_i-h_{i+1}}{h_1-h_n}\frac{LK_i}{1+K_i^2\left(\xi L-\sum_{j=1}^{i}L_j\right)^2}\right]B_{0m}\cos\alpha_m\xi
\end{aligned}
\tag{47b}
$$

Since $K_i$ are infinitely great parameters, the terms that tend to be infinitely small in $T_1$ and $Q_1$ can be eliminated. Substituting Eqs. (47a)-(47b) and Eq. (22) into Eq. (30) yields

$$
\frac{P_1L^2}{D_0}=-\left[\frac{3\pi\left(\alpha_m^2+N^2\right)^2}{2\alpha_m^2}+\frac{2\pi c^2\alpha_m^2r^4}{\left(\alpha_m^2+N^2\right)^2}\right]\sum_{i=1}^{n-1}\frac{h_i-h_{i+1}}{h_1-h_n}\left[1-\frac{2\sum_{j=1}^{i}L_j}{L}+\frac{\sin(2\alpha_m\frac{\sum_{j=1}^{i}L_j}{L})}{\alpha_m}\right]
\tag{48}
$$

Taking into account Eq. (24), when $m$=1, $P_1$ in Eq. (48) will become minimum, then it can be transformed as

$$\frac{P_1 L^2}{D_0} = -4\pi c r^2 \sum_{i=1}^{n-1} \frac{h_i - h_{i+1}}{h_1 - h_n} \left[ 1 - \frac{2\sum_{j=1}^{i} L_j}{L} + \frac{\sin\left(2\pi \frac{\sum_{j=1}^{i} L_j}{L}\right)}{\pi} \right] \tag{49}$$

Substituting relevant parameters into Eq. (31) yields

$$P_{cr} = \frac{E\left(h_1 + h_n\right)^2}{4\sqrt{3\left(1-v^2\right)}R} - \frac{E\left(h_1^2 - h_n^2\right)}{2\sqrt{3\left(1-v^2\right)}R} \sum_{i=1}^{n-1} \frac{h_i - h_{i+1}}{h_1 - h_n} \left[ 1 - \frac{2\sum_{j=1}^{i} L_j}{L} + \frac{\sin\left(2\pi \frac{\sum_{j=1}^{i} L_j}{L}\right)}{\pi} \right] + O\left(\varepsilon^2\right) \tag{50}$$

Eq. (50) is the asymptotic formula of critical buckling load of axial compressed cylindrical shells with stepwise thickness variation. It is obvious that when the thickness of the shell is constant, i.e., $h_i(1 \le i \le v) = h_0$, Eq. (50) degenerates into Eq. (25), which is the elastic classical critical buckling load.

Substituting Eq. (50) into Eq. (36) yields

$$\lambda = 1 - \frac{2}{h_1 + h_n} \sum_{i=1}^{n-1} \left(h_i - h_{i+1}\right) \left[ 1 - \frac{2\sum_{j=1}^{i} L_j}{L} + \frac{\sin\left(2\pi \frac{\sum_{j=1}^{i} L_j}{L}\right)}{\pi} \right] + O\left(\varepsilon^2\right) \tag{51}$$

Here, a cylindrical shell with three strakes is presented to illustrate the effects of each strake's length and thickness on the nondimensional critical buckling load reduction factor. Two cases will be discussed here:

### *Case A*

Supposing that $h_3=0.6h_1$, $L_3=0.2L$, in this case, the relation between the critical buckling load reduction factor $\lambda$ and the thickness ratio $h_2/h_1$ has been studied. The result is shown in Fig. 4.

### *Case B*

Supposing that $h_2=0.8h_1$, $L_2=0.3L$, in this case, the relation between the critical buckling load reduction factor $\lambda$ and the thickness ratio $h_3/h_1$ has been studied. The result is shown in Fig. 5.

From Figs. 4-5, it can be concluded that

- When the total length of the shell is constant, the nondimensional critical buckling load reduction factor will increase along with the length increase of the lowest strake (i.e., the thickest one). Thus the stability of the cylindrical shell with stepwise thickness variation under axial compression can be improved by increasing the length of the lowest strake appropriately.

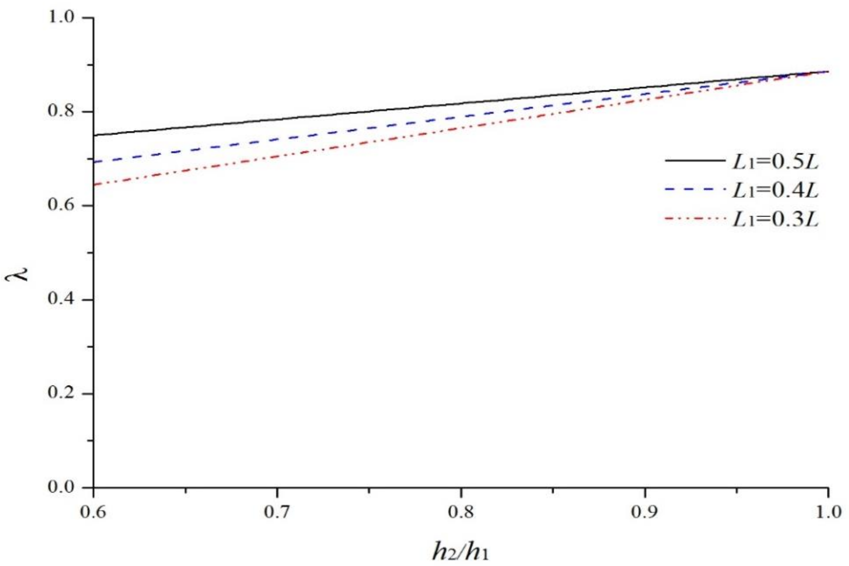


Fig. 4 Relation between $\lambda$ and $h_2/h_1$ in case A

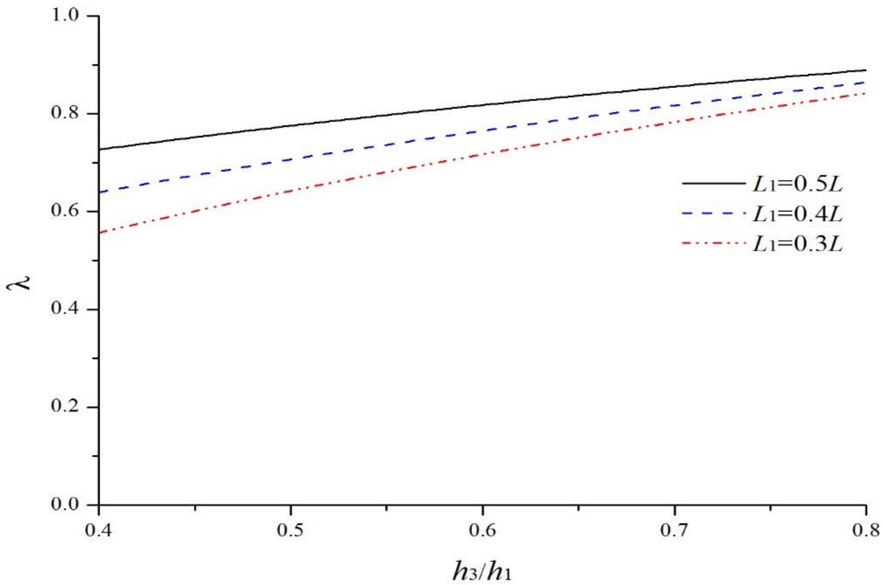


Fig. 5 Relation between $\lambda$ and $h_3/h_1$ in case B

• When the length of each strake are all constant, the nondimensional critical buckling load reduction factor will increase along with the increase of $h_2$ and $h_3$. Therefore increasing the thickness of upper strakes can also help to improve the stability of the cylindrical shell.

• When the length of each strake are all constant, the variation of $h_3$ has greater influence on the nondimensional critical buckling load reduction factor than $h_2$, which indicates that the critical buckling load is more sensitive to the upper (i.e., the thinner) strakes. It demonstrates that a reasonable selection of the thickness of upper strakes has very important significance to prevent the buckling of axial compressed cylindrical shells with stepwise thickness variation.

## 6. Conclusions

The critical buckling load of axial compressed cylindrical shells with stepwise wall thickness variation has been researched in this paper. Some appealing merits can be summarized as follows

- An arctan function is established to simplify the wall thickness of the cylindrical shell in order to make it vary continuously in the axial direction. It will contribute to the theoretic analyses on the cylindrical shells with stepwise wall thickness variation in other studies.
- The analytical formulas of critical buckling load of cylindrical shells with arbitrary axisymmetric thickness variation under axial compression are derived by using the methods of separation of variables, small parameter perturbation and Fourier series expansion. These formulas break through the limitation in previous research that the thickness variations are always symmetrical about the center of shells in the axial direction.
- The asymptotic solution of critical buckling load of axial compressed cylindrical shells with stepwise thickness variation is obtained by the method presented in this paper. A cylindrical shell with three strakes is presented to illustrate the importance of each strake's length and thickness on preventing the axial compressed buckling. Results show that the stability of the cylindrical shell can be improved by increasing the length of the lowest strake or the thickness of upper strakes properly.

The results obtained in this paper will provide reference for the design and application of cylindrical shells with stepwise wall thickness variation, such as large petroleum storage tanks or other vertical vessels.

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