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Analytical and finite element solution of a receding contact problem

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Abstract. In this paper, a receding contact problem for an elastic layer resting on two quarter planes is considered. The layer is pressed by a stamp and distributed loads. It is assumed that the contact surfaces are frictionless and only compressive traction can be transmitted through the contact surfaces. In addition the effect of body forces are neglected. Firstly, the problem is solved analytically based on theory of elasticity. In this solution, the problem is reduced into a system of singular integral equations in which contact areas and contact stresses are unknowns using boundary conditions and integral transform techniques. This system is solved numerically using Gauss-Jacobi integral formulation. Secondly, two dimensional finite element analysis of the problem is carried out using ANSYS. The dimensionless quantities for the contact areas and the contact pressures are calculated under various distributed load conditions using both solutions. It is concluded that the position and the magnitude of the distributed load have an important role on the contact area and contact pressure distribution between layer and quarter plane contact surface. The analytic results are verified by comparison with finite element results.

Keywords: receding contact; quarter plane; Gauss-Jacobi; finite element method; ANSYS

1. Introduction

Problems involving the contact of two separate bodies pressed against each other have been widely studied by many researchers. Although the contact area increases after the application of the load in many cases, there are others where the contact area becomes smaller. This kind of problem is called receding in literature. In other words, a contact can be named receding if the contact area in the loaded configuration is contained within the initial contact area (Johnson 1985).

The receding contact problem has been studied for more than four decades by many researchers both numerically and analytically. The latest numerical studies on this topic were based on either finite element method (Chan and Tuba 1971, Francavilla and Zienkiewicz 1975, Jing and Liao 1990) or boundary element method (Anderson 1982, Garrido *et al*. 1991, Garrido and Lorenzana 1998, Paris *et al*. 1992, 1995).

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Among the analytical studies on receding contact, the followings are recorded in literature. Keer *et al*. (1972) solved the smooth receding contact problem between an elastic layer and a half space when two bodies are pressed against each other by considering both plane and axisymmetric cases. The same problem was solved treating the layer as a simple beam by Gladwell (1976). The frictionless contact problem for an elastic layer resting on two quarter planes and loaded compressively was solved by Erdogan and Ratwani (1974). Civelek and Erdogan (1974) investigated the general axisymmetric double frictionless contact problem for an elastic layer resting on a half space and pressed by an elastic stamp. The smooth receding contact problem for an elastic layer pressed against a half space by a frictionless semi-infinite elastic was examined by Gecit (1986).

Aksogan *et al*. (1996) studied a contact problem for an elastic layer supported by two elastic quarter planes both symmetrical loading and axisymmetric loading (1997). Comez *et al*. (2004) solved double receding contact problem for two elastic layers having different elastic constants and heights and pressed by a rigid stamp. A receding contact plane problem for a functionally graded layer pressed against a homogeneous half space was analyzed by El-Borgi *et al*. (2006). Kahya *et al*. (2007) considered a frictionless receding contact problem between an anisotropic elastic layer and an anisotropic elastic half plane, when the two bodies are pressed together by means of a rigid circular stamp.

Rhimi *et al*. (2009) considered the axisymmetric problem of a frictionless receding contact between an elastic functionally graded layer and a homogeneous half-space when the two bodies are pressed together and double receding contact between a rigid stamp of axisymmetric profile, an elastic functionally graded layer and a homogeneous half space (2011). Chen and Chen (2012) studied the contact behaviors of a graded layer resting on a homogeneous half space and pressed by a rigid stamp. Comez (2013) considered a contact problem for a functionally graded layer loaded by means of a rigid stamp and supported by a Winkler foundation. A continuous contact problem for two elastic layers resting on an elastic half-infinite plane and loaded by means of a rigid stamp was solved by Oner and Birinci (2014). Yaylaci and Birinci (2013) studied a receding contact problem of two elastic layers supported by two elastic quarter planes. A comparative study of numerical and analytical solution of the same receding contact problem conducted by Yaylaci *et al*. (2014).

When the literature is researched, it can be seen that there are not enough studies about receding contact problems including quarter planes. Additionally, although there exist extensive studies on analytical and numerical solution of contact problems in literature, comparison of these two methods has not been explored completely. Also, in distinction to previous papers, the layer is pressed by both a rigid stamp and distributed loads, simultaneously. As a result, the aim of this paper is to present a comparative study of a receding contact problem using analytical method and FEM and compare the solutions obtained from these methods with each other.

2. Analytical solution

Consider an elastic layer of thickness h resting on two quarter planes, subjected to a concentrated load P by means of a rigid circular stamp and uniformly distributed loads as shown Fig. 1. It is assumed that the contact surfaces are frictionless and only compressive traction can be transmitted through the contact surfaces. In addition the effect of body forces are neglected.

x=0 plane is assumed to be the plane of symmetry with respect to external loads as well as geometry. Clearly, it is sufficient to consider one half (i.e., x>0) of the medium only.

The displacement and stress expressions for the layer are obtained using Fourier integral transform technique as fallows (Çömez *et al*. 2004)

$$u(x,y)=\frac{2}{\pi}\int_0^\infty\left[(A_1+A_2y)e^{-\xi y}+(A_3+A_4y)e^{\xi y}\right]\sin\xi x d\xi \tag{1.1}$$

$$v(x,y)=\frac{2}{\pi}\int_0^\infty\left\{\left[A_1+(\frac{\kappa}{\xi}+y)A_2\right]e^{-\xi y}+\left[-A_3+(\frac{\kappa}{\xi}-y)A_4\right]e^{\xi y}\right\}\cos\xi x d\xi \tag{1.2}$$

$$\frac{1}{2G}\sigma_x(x,y)=\frac{2}{\pi}\int_0^\infty\left\{\left[\xi(A_1+A_2y)-\frac{3-\kappa}{2}A_2\right]e^{-\xi y}+\left[\xi(A_3+A_4y)+\frac{3-\kappa}{2}A_4\right]e^{\xi y}\right\}\cos\xi x dx \tag{2.1}$$

$$\frac{1}{2G}\sigma_y(x,y)=\frac{2}{\pi}\int_0^\infty\left\{-\left[\xi(A_1+A_2y)+\frac{\kappa+1}{2}A_2\right]e^{-\xi y}+\left[-\xi(A_3+A_4y)+\frac{\kappa+1}{2}A_4\right]e^{\xi y}\right\}\cos\xi x dx \tag{2.2}$$

$$\frac{1}{2G}\tau_{xy}(x,y)=\frac{2}{\pi}\int_0^\infty\left\{-\left[\xi(A_1+A_2y)+\frac{\kappa-1}{2}A_2\right]e^{-\xi y}+\left[\xi(A_3+A_4y)-\frac{\kappa-1}{2}A_4\right]e^{\xi y}\right\}\sin\xi x dx \tag{2.3}$$

where $u(x,y)$ and $v(x,y)$ are the displacement components in x and y respectively, $\sigma_x(x,y)$, $\sigma_y(x,y)$ and $\tau_{xy}(x,y)$ are the stress components of the layer and $\kappa=3-4\upsilon$ for plain strain. A_i (i=1,2,3,4) are the unknown coefficients for the layer which will be determined from boundary conditions of the problem.

The displacement and stress expressions for the quarter plane in polar coordinates (r, θ) are obtained using Mellin integral transform technique as fallows (Çakıroğlu 2011)

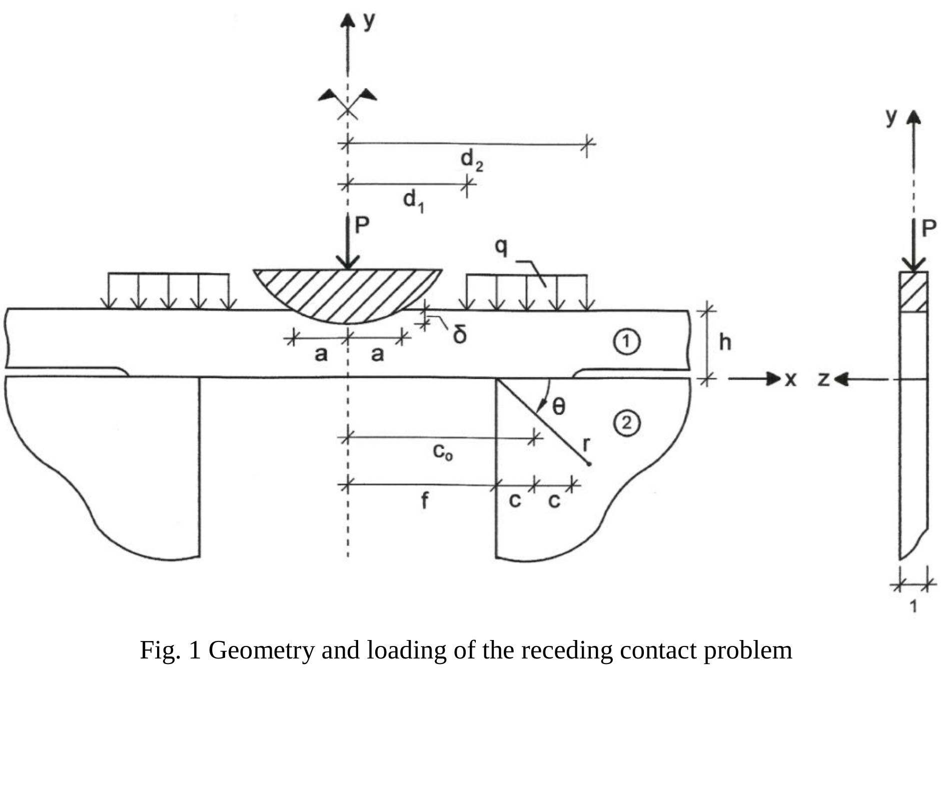

Fig. 1 Geometry and loading of the receding contact problem

$$2G\left(r^2\frac{\partial u_\theta(r,\theta)}{\partial r}\right)^M = is(s+1)\left(B_1e^{is\theta}-B_2e^{-is\theta}\right)+$$

$$i\left[(s+2)(s+1)+(1-\upsilon)(-4s-4)\right]\left[B_3e^{i(s+2)\theta}-B_4e^{-i(s+2)\theta}\right] \quad (3)$$

$$\left(r^2\sigma_r(r,\theta)\right)^M = \left(\frac{\partial^2}{\partial\theta^2}-s\right)(B_1e^{is\theta}+B_2e^{-is\theta}+B_3e^{i(s+2)\theta}+B_4e^{-i(s+2)\theta}) \quad (4.1)$$

$$\left(r^2\sigma_\theta(r,\theta)\right)^M = s(s+1)(B_1e^{is\theta}+B_2e^{-is\theta}+B_3e^{i(s+2)\theta}+B_4e^{-i(s+2)\theta}) \quad (4.2)$$

$$\left(r^2\tau_{r\theta}(r,\theta)\right)^M = (s+1)\frac{\partial}{\partial\theta}(B_1e^{is\theta}+B_2e^{-is\theta}+B_3e^{i(s+2)\theta}+B_4e^{-i(s+2)\theta}) \quad (4.3)$$

where $(f)^M$ shows Mellin integral transform of function f, $u_\theta(r,\theta)$ is the displacement component in θ and $\sigma_r(r,\theta)$, $\sigma_\theta(r,\theta)$ and $\tau_{r\theta}(r,\theta)$ are the stress components of the quarter plane. B_i (i=1,2,3,4) are the unknown coefficients for the quarter plane which will be determined from boundary conditions of the problem.

The receding contact problem described above must be solved under the following boundary conditions

$$\sigma_{y1}(x,h) = \begin{cases} -p_1(x) & 0 \le x < a \\ -q \quad , & d1 \le x \le d2 \\ 0 & other \end{cases} \quad (5.1)$$

$$\tau_{xy1}(x,h) = 0 \qquad 0 \le x < \infty \quad (5.2)$$

$$\sigma_{y1}(x,0) = \begin{cases} -p_2(x) \quad , & c_0 - c \le x \le c_0 + c \\ 0 & other \end{cases} \quad (5.3)$$

$$\tau_{xy1}(x,0) = 0 \ , \qquad 0 \le x < \infty \quad (5.4)$$

$$\sigma_{\theta 2}(r,0) = \begin{cases} p_2(r) \quad , & 0 < r < 2c \\ 0 & other \end{cases} \quad (6.1)$$

$$\tau_{r\theta 2}(r,0) = 0 \ , \qquad 0 \le r < \infty \quad (6.2)$$

$$\sigma_{\theta 2}(r,\frac{\pi}{2}) = 0 \ , \qquad 0 \le r < \infty \quad (6.3)$$

$$\tau_{r\theta 2}(r,\frac{\pi}{2}) = 0 \ , \qquad 0 \le r < \infty \quad (6.4)$$

$$\frac{\partial v_1(x,h)}{\partial x} = \frac{F(x)}{\partial x} \ , \qquad -a < x < a \quad (7.1)$$

$$\frac{\partial v_1(x,0)}{\partial x} = \frac{\partial u_{\theta_2}(r,0)}{\partial r} , \qquad c_0 - c < x < c_0 + c, \ 0 \le r < 2c \quad (7.2)$$

where index 1 and 2 show layer related components and quarter plane related components, respectively, $p_1(x)$ is the unknown contact pressures along the layer-stamp contact surface, $p_2(x)$ and $p_2(r)$ are the unknown contact pressures along the layer-quarter plane contact surface, q is the magnitude of the uniformly distributed load and $F(x)$ is a known function obtained the equation giving the profile of the rigid stamp.

Equilibrium conditions for the problem may be expressed as follows.

$$\int_{-a}^{a} p_1(x)dx = P \tag{8.1}$$

$$\int_{c_0-c}^{c_0+c} p_2(x)dx = \frac{P}{2} + Q \tag{8.2}$$

where Q shows the total distributed load magnitude.

$$Q = q(d_2 - d_1)\,. \tag{8.3}$$

The Fourier cosine transform of the unknown contact pressures $p_1(x)$ and $p_2(x)$ and the Mellin transform of the unknown contact pressures $p_2(r)$ can be written as follows.

$$p_1(\xi) = \int_0^\infty p_1(x)\cos\xi x dx = \int_0^a p_1(t_1)\cos\xi t_1 dt_1 \tag{9.1}$$

$$p_2(\xi) = \int_0^\infty p_2(x)\cos\xi x dx = \int_{c_0-c}^{c_0+c} p_2(t_2)\cos\xi t_2 dt_2 \tag{9.2}$$

$$p_2(s) = M\left[p_2(r); r\to s+2\right] = \int_0^\infty p_2(r) r^{s+1} dr = \int_0^{2c} p_2(\tau)\tau^{s+1} d\tau \tag{10}$$

After using integral transform techniques on stress boundary conditions (Eqs. (5)-(6)), unknown coefficients A_i and B_i (i=1,2,3,4) are determined in terms of $p_1(\xi)$, $p_2(\xi)$ and $p_2(s)$. By using displacement boundary conditions (Eq. (7)) after converting x to x_1 in Eq. (7.1) and x to x_2 in Eq. (7.2) and considering the symmetry condition $p_1(x)=p_1(-x)$, one can obtain following singular integral equations after some routine manipulations

$$\int_{-a}^{a}\left[-\frac{1}{t_1-x_1} + M_{11}(x_1,t_1)\right]p_1(t_1)dt_1 + \int_{c_0-c}^{c_0+c} p_2(t_2)M_{12}(x_1,t_2)dt_2 = \frac{4\pi G_1}{\kappa_1+1}\frac{x_1}{R} - qM_{13}(x_1) \tag{11.1}$$

$$\int_{-a}^{a} p_1(t_1)M_{21}(x_2,t_1)dt_1 + \int_{c_0-c}^{c_0+c} p_2(t_2)\left\{-\frac{1}{t_2+x_2} + \frac{1}{t_2-x_2} + M_{22}(x_2,t_2) + \right.$$

$$\left. + \frac{G_1}{G_2}\frac{K_2+1}{K_1+1}\frac{1}{x_2-(c_0-c)}\left[\frac{1}{\log\left(\dfrac{x_2-(c_0-c)}{t_2-(c_0-c)}\right)} + M_1 - \frac{\pi^2}{\pi^2-4}\right]\right\}dt_2 = -qM_{23}(x_2) \tag{11.2}$$

where

$$M_{11}(x_1,t_1)=\int_0^\infty \frac{-2}{\Delta_A}\left\{e^{-2\xi h}\left[1+2\xi h+2\xi^2h^2\right]-e^{-4\xi h}\right\}\sin\xi(t_1-x_1)\,d\xi \tag{12.1}$$

$$M_{12}(x_1,t_2)=\int_0^\infty \frac{4}{\Delta_A}\left\{e^{-\xi h}\left[-1-\xi h\right]-e^{-3\xi h}(1-\xi h)\right\}\sin\xi x_1\cos\xi t_2\,d\xi \tag{12.2}$$

$$M_{13}(x_1)=\int_0^\infty \frac{2}{\xi\Delta_A}\left\{1+e^{-2\xi h}(4\xi h)-e^{-4\xi h}\right\}\sin\xi x_1\left[\sin\xi d_2-\sin\xi d_1\right]d\xi \tag{12.3}$$

$$M_{21}(x_2,t_1)=\int_0^\infty \frac{-2}{\Delta_A}\left\{e^{-\xi h}(1+\xi h)+e^{-3\xi h}(-1+\xi h)\right\}\sin\xi(t_1-x_2)\,d\xi \tag{13.1}$$

$$M_{22}(x_2,t_2)=\int_0^\infty \frac{4}{\Delta_A}\left\{e^{-2\xi h}(-1-2\xi h-2\xi^2h^2)+e^{-4\xi h}\right\}\sin\xi x_2\cos\xi t_2\,d\xi \tag{13.2}$$

$$M_{23}(x_2)=\int_0^\infty \frac{4}{\xi\Delta_A}\left\{e^{-\xi h}(1+\xi h)+e^{-3\xi h}(-1+\xi h)\right\}\sin\xi x_2\cos\xi t_1\,d\xi \tag{13.3}$$

$$M_1=\int_0^\infty\left[\frac{\sinh\pi y}{-2y^2-1+\cosh\pi y}-1\right]\sin\left[\log\left(\frac{x_2-(c_0-c)}{t_2-(c_0-c)}\right)y\right]dy \tag{13.4}$$

The expression Δ_A in Eqs. (11)-(12) is defined as follows.

$$\Delta_A=e^{-4\xi h}-e^{-2\xi h}\left(4\xi^2h^2+2\right)+1 \tag{14}$$

Thus, the solution of the problem is reduced into the solution of a system of singular integral equation which may be solved numerically using Gauss-Jacobi integration formulation described in Erdoğan *et al*. (1972).

Following dimensionless quantities can be introduced in order to simplify the numerical solution.

$$t_1=ar_1\ ,\quad dt_1=adr_1 \tag{15.1}$$

$$x_1=as_1 \tag{15.2}$$

$$t_2=cr_2+c_0\ ,\ dt_2=cdr_2 \tag{15.3}$$

$$x_2=cs_2+c_0 \tag{15.4}$$

$$z=\xi h\ ,\ dz=hd\xi \tag{15.5}$$

$$\phi_1(r_1)=\frac{h}{P}p_1(t_1) \tag{15.6}$$

$$\phi_2(r_2)=\frac{h}{P}p_2(t_2) \tag{15.7}$$

Substituting given these dimensionless quantities in Eq. (11) and the equilibrium conditions Eq. (8), following expressions are obtained.

$$\int_{-1}^{1}\phi_1\left[-\frac{1}{r_1-s_1}+\frac{a}{h}K_{11}(r_1,s_1)\right]dr_1+\int_{-1}^{1}\phi_2\,\frac{c}{h}K_{12}(r_2,s_1)\,dr_2=\frac{4\pi}{\kappa_1+1}\frac{G_1}{P/h}\frac{a/h}{R/h}s_1-\frac{q}{P/h}K_{13}(s_1) \tag{16.1}$$

$$\int_{-1}^{1}\phi_1\,\frac{a}{h}K_{21}(r_1,s_2)\,dr_1+\int_{-1}^{1}\phi_2\left[-\frac{1}{r_2+s_2+2\frac{c_0/h}{c/h}}+\frac{1}{r_2-s_2}+\frac{c}{h}K_{22}(r_2,s_2)+\right.$$
$$\left.+\frac{G_1}{G_2}\frac{\kappa_2+1}{\kappa_1+1}\frac{c}{h}K_3(r_2,s_2)\right]dr_2=-\frac{q}{P/h}K_{23}(s_2) \tag{16.2}$$

$$\frac{a}{h}\int_{-1}^{1}\phi_1 dr_1=1 \tag{17.1}$$

$$\frac{c}{h}\int_{-1}^{1}\phi_2 dr_2=0.5+\frac{Q}{P} \tag{17.2}$$

To insure smooth contact at both end points (a, $-a$) for layer-stamp contact surface, i.e., $\phi_1(\pm1)=0$, the index of the first integral equation (Eq. (16.1)) is "1" (Erdogan *et al.* 1973). One may notice that layer-quarter plane contact surface has a smooth contact at the right end (c_0+c) and stress singularity at the edge of the quarter plane (c_0-c), i.e., $\phi_2(-1)=\infty$, $\phi_2(1)=0$. Hence, the solution may be sought as follows.

$$\phi_1(r_{1i})=w_1(r_{1i})g_1(r_{1i}) \qquad (-1\le r\le 1),\ (i=1,\dots,N) \tag{18.1}$$

$$w_1(r_{1i})=(1-r_{1i}^2)^{0.5} \qquad (-1\le r\le 1),\ (i=1,\dots,N) \tag{18.2}$$

$$\phi_2(r_{2i})=w_2(r_{2i})g_2(r_{2i}) \qquad (-1\le r\le 1),\ (i=1,\dots,N) \tag{19.1}$$

$$w_2(r_{2i})=(1-r_{2i})^{0.5}(1+r_{2i})^{\beta_2} \qquad (-1\le r\le 1),\ (i=1,\dots,N) \tag{19.2}$$

where $g(r)$ is bounded in the interval ($-1\le r\le 1$), and β_2 can be obtain as follows.

$$\frac{G_2(1+\kappa_1)}{G_1(1+\kappa_2)}\left(2\lambda^2-1+\cos\pi\lambda\right)\cos\pi\lambda-\sin^2\pi\lambda=0 \tag{20.1}$$

$$\beta_2=\lambda-1 \tag{20.2}$$

where λ is the minimum positive root of Eq. (20.1).

Using appropriate Gauss–Jacobi integration formulas, Eqs. (16)-(17) are replaced as

$$\sum_{i=1}^{N}W_{1i}^{N}g_1(r_{1i})\left[-\frac{1}{r_{1i}-s_{1k}}+\frac{a}{h}K_{11}(r_{1i},s_{1k})\right]+\sum_{i=1}^{N}W_{2i}^{N}g_2(r_{2i})\frac{c}{h}K_{12}(r_{2i},s_{1k})$$
$$=\frac{4\pi}{\kappa_1+1}\frac{G_1}{P/h}\frac{a/h}{R/h}s_{1k}-\frac{q}{P/h}K_{13}(s_{1k}) \qquad (k=1,\dots,N+1) \tag{21.1}$$

$$\sum_{i=1}^{N} W_{1i}^{N} g_1(r_{1i}) \frac{a}{h} K_{21}(r_{1i}, s_{2k}) + \sum_{i=1}^{N} W_{2i}^{N} g_2(r_{2i}) \left[-\frac{1}{r_{2i} + s_{2k} + 2\frac{c_0/h}{c/h}} + \frac{1}{r_{2i} - s_{2k}} + \frac{c}{h} K_{22}(r_{2i}, s_{2k}) \right.$$

$$\left. \frac{G_1}{G_2} \frac{\kappa_2 + 1}{\kappa_1 + 1} \frac{c}{h} K_3(r_{2i}, s_{2k}) \right] = -\frac{q}{P/h} K_{23}(s_{2k}) \qquad (k = 1, \dots, N) \qquad (21.2)$$

$$\frac{a}{h} \sum_{i=1}^{N} W_{1i}(r_{1i}) g_1(r_{1i}) = 1 \qquad (22.1)$$

$$\frac{c}{h} \sum_{1}^{N} W_{2i}(r_{2i}) g_2(r_{2i}) = 0.5 + \frac{Q}{P} \qquad (22.2)$$

where r_{1i}, s_{1k}, r_{2i} and s_{2k} are the roots of the related Jacobi polynomials, W_{1i}^{N} and W_{2i}^{N} are weighting constants

$$r_{1i} = \cos\left(\frac{i\pi}{N+1} \right) \qquad (i = 1, \dots, N) \qquad (23.1)$$

$$s_{1k} = \cos\left(\frac{\pi}{2} \frac{2k-1}{N+1} \right) \qquad (k = 1, \dots, N+1) \qquad (23.2)$$

$$W_{1i}^{N} = \pi \frac{1 - r_{1i}^{\ 2}}{N+1} \qquad (i = 1, \dots, N) \qquad (24)$$

$$P_N^{(\alpha_2, \beta_2)}(r_{2i}) = 0 \qquad (i = 1, \dots, N) \qquad (25.1)$$

$$P_N^{(\alpha_2 - 1, \beta_2 + 1)}(s_{2k}) = 0 \qquad (k = 1, \dots, N) \qquad (25.2)$$

$$W_{2i}^{N} = -\frac{2N + 2 + \alpha_2 + \beta_2}{(N+1)!(N + 1 + \alpha_2 + \beta_2)} \frac{\Gamma(N + 1 + \alpha_2)\Gamma(N + 1 + \beta_2)}{\Gamma(N + 1 + \alpha_2 + \beta_2)} x \frac{2^{\alpha_2 + \beta_2}}{P_N^{(\alpha_2, \beta_2)}(r_{2i}) P_{N+1}^{(\alpha_2, \beta_2)}(r_{2i})} \qquad (26)$$

Note that the system of algebraic equations (Eq. (21)) are consist of ($2N$+1) equations in total for ($2N$) unknowns namely g_{1i} and g_{2i} (i=1,...,N). Since ($N/2$+1)th equation of Eq. (21.1) corresponds to consistency condition and automatically satisfied, it can be removed. Thus, the solution of the problem is reduced into the solution of the system consists of *2N* equations for *2N* unknowns. Note that the system is highly nonlinear in *a* and *c*, and an iterative procedure have to be used in order to determine these unknowns. In this procedure, firstly a prediction for unknown *a* and *c* is made and then new values are chosen repeatedly until the value of *a* and *c* satisfy equilibrium conditions (Eq. (22)).

3. The finite element solution

The finite element method (FEM) is a numerical technique used for finding approximate solutions of partial differential and integral equations. The method works by assuming a continuous function for the solution and obtaining the parameters that govern this function which minimizes the error in the solution. For many engineering problems analytical solutions are not

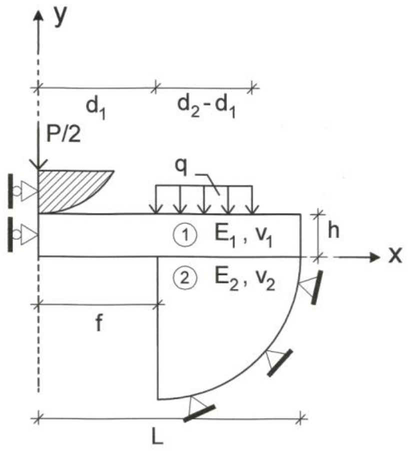

Fig. 2 The geometry for the analysis

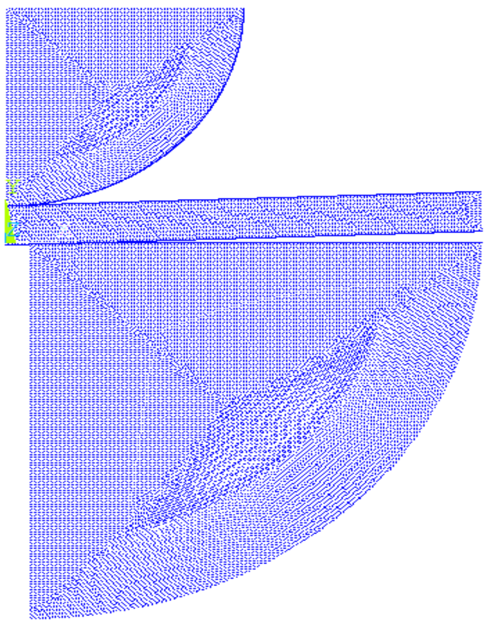

Fig. 3 Deformed geometry for the preliminary analysis

suitable because of the complexity of the material properties, the boundary conditions o the structure itself. The basis of the finite element method is the representation of a body or a structure by an assemblage of subdivisions called finite elements. The Finite Element Method translates partial differential equation problems into a set of linear algebraic equations.

$$[K]\{q\}=\{F\} \tag{27}$$

where $[K]$ is the global stiffness matrix, $\{q\}$ the structural nodal displacement vector and $\{F\}$ is the vector of structural nodal loads (Delpero *et al*. 2010).

Table 1 Variation of the contact areas (*a/h*) and (*c/h*) with (*Q/P*), (G_1/G_2=0.5, K_1=K_2=2, *R/h*=250, $G_1/(P/h)$=1000, d_1/h=0.5, d_2/h=1, *f/h*=0.5)

PARAMETER	*Q/P*=0		*Q/P*=0.5		*Q/P*=1.0		*Q/P*=2.0	
	a/h	*c/h*	*a/h*	*c/h*	*a/h*	*c/h*	*a/h*	*c/h*
Analytical	0.398	0.3187	0.3814	0.4375	0.3703	0.4756	0.3554	0.5059
FEM	0.4	0.3175	0.38	0.4375	0.37	0.475	0.355	0.505
Difference (%)	0.5	0.38	0.37	0	0.08	0.13	0.11	0.718

Table 2 Variation of the contact areas (*a/h*) and (*c/h*) with (d_1/h), (G_1/G_2=0.5, K_1=K_2=2, *R/h*=250, *Q/P*=0.5, $G_1/(P/h)$=1000, d_2/h=d_1/h+0.5, *f/h*=0.5)

PARAMETER	d_1/h=0.5		d_1/h=1.0		d_1/h=1.5		d_1/h=.0	
	a/h	*c/h*	*a/h*	*c/h*	*a/h*	*c/h*	*a/h*	*c/h*
Analytical	0.3814	0.4375	0.3884	0.6425	0.3866	0.85	0.3864	1.2
FEM	0.38	0.4375	0.385	0.65	0.3825	0.8625	0.3815	1.22
Difference (%)	0.37	0	0.88	1.17	1.06	1.47	1.26	1.67

The receding contact problem has been studied by the finite element method (FEM), using a commercial package program ANSYS. The contact is considered as a two-dimensional problem and the material of the layer and quarter plane is assumed elastic and isotropic. In the analyses, geometric properties are taken as *L*=1 m (length of the layer in *x* direction), *h*=10 cm (thickness of the layer in *y* direction) and *R*=0.5 m (radius of the stamp) and material properties are taken as E_1=25000 MPa, v_1=0.25, E_2=50000 MPa and v_2=0.25. Other parameters are chosen such that $\frac{G_1/(P/h)}{R/h}$, *Q/P*, d_1/h, d_2/h and *f/h* ratios are compatible with analytical values.

Concentrated load and distributed load acting on the layer in the negative *y* direction. Plane strain finite elements are used for the meshing of the entire geometry. Frictionless surface-to-surface contact elements are used to model the interaction between the contact surfaces and Augmented Lagrangian method is used as the contact algorithm. In the preliminary analysis is meshed with 256977 elements, 510414 nodes and the contacting line is meshed with 246 elements. The geometry and the applied load are shown schematically in Fig. 2 and the deformed geometry of the model assumes the form shown in Fig. 3

4. Numerical solutions

Some calculated results for contact areas and contact pressures obtained using analytical and finite element solution are shown in Tables 1-4 and Figs. 2-11. Note that all quantities are dimensionless.

Table 1 shows the variations of contact areas between layer and stamp contact surface (*a/h*) and between layer and quarter plane contact surface (*c/h*) with the ratio of the total distributed load magnitude to concentrated load (*Q/P*), i.e., load ratio, for fixed application surface. It appears that, with increasing the load ratio, the contact area between layer and stamp (*a/h*) decreases but the contact area between layer and quarter plane (*c/h*) increases.

The variations of the contact areas (*a/h*, *c/h*) with the start point of the distributed load (d_1/h)

Table 3 Variation of the contact areas (a/h) and (c/h) with (d_2/h), (G_1/G_2=0.5, K_1=K_2=2, R/h=250, Q/P=0.5, $G_1/(P/h)$=1000, d_1/h=0.5, f/h=0.5)

PARAMETER	d_2/h=1.0		d_2/h=1.5		d_2/h=2.0		d_2/h=2.5	
	a/h	c/h	a/h	c/h	a/h	c/h	a/h	c/h
Analytical	0.3814	0.4375	0.3836	0.5525	0.3838	0.675	0.3833	0.93
FEM	0.38	0.4375	0.385	0.55	0.3875	0.6875	0.3815	0.95
Difference (%)	0.37	0	0.37	0.45	0.96	1.85	0.47	2.15

Table 4 Variation of the contact areas (a/h) and (c/h) with (d_2/h) and (Q/P), (G_1/G_2=0.5, K_1=K_2=2, R/h=250, $G_1/(P/h)$=1000, d_1/h=0.5, Q/P=(d_2/h−d_1/h), f/h=0.5)

PARAMETER	d_2/h=0.75		d_2/h=1.0		d_2/h=1.5		d_2/h=2.0	
	a/h	c/h	a/h	c/h	a/h	c/h	a/h	c/h
Analytical	0.3863	0.3581	0.3814	0.4375	0.3755	0.6313	0.3717	0.85
FEM	0.385	0.3575	0.38	0.4375	0.375	0.6375	0.37	0.8625
Difference (%)	0.34	0.17	0.37	0	0.13	0.98	0.46	1.47

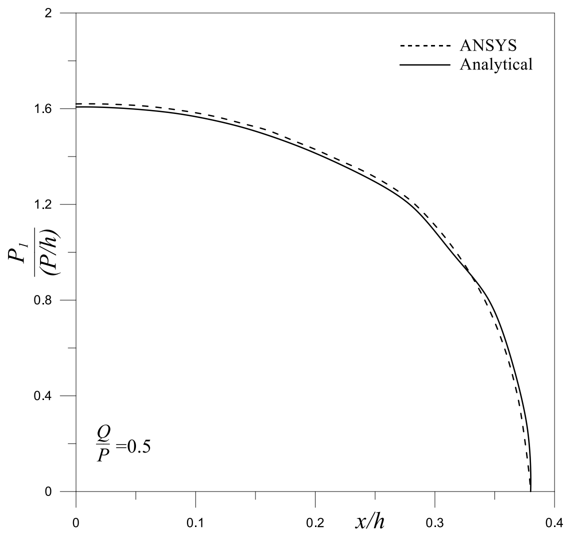

Fig. 4 Contact pressure distribution between layer and stamp ($p_1/(P/h)$) for (Q/P=0.5) (G_1/G_2=0.5, K_1=K_2=2, R/h=250, $G_1/(P/h)$=1000, d_1/h=0.5, d_2/h=1, f/h=0.5)

are given in Table 2. It can be seen that (a/h) reaches the largest value as the start point of the distributed load approaches to one (d_1/h→1). On the other hand, (c/h) increases steadily with increasing (d_1/h).

Table 3 shows the variations of contact areas (a/h, c/h) with the end point of the distributed load (d_2/h) for a constant total distributed load magnitude.

It may be observed that (a/h) reaches the largest value as the end point of the distributed load approaches to two (d_2/h→2).However, (c/h) increases continuously with increasing (d_2/h).

The variation of the contact areas (a/h, c/h) with the end point of the distributed load (d_2/h) for a constant distributed load magnitude are given in Table 4. In the event of increasing in (d_2/h), it is indicated that (a/h) decreases but (c/h c/h) increases.

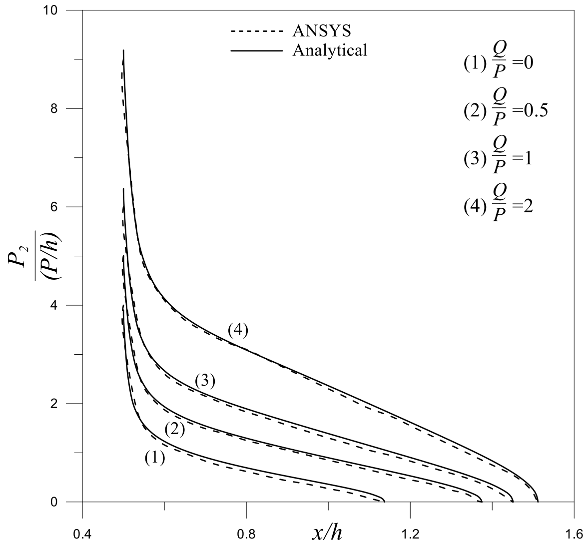

Fig. 5 Contact pressure distribution between layer and quarter plane ($p_2/(P/h)$) with (Q/P) (G_1/G_2=0.5, K_1=K_2=2, R/h=250, $G_1/(P/h)$=1000, d_1/h=0.5, d_2/h=1, f/h=0.5)

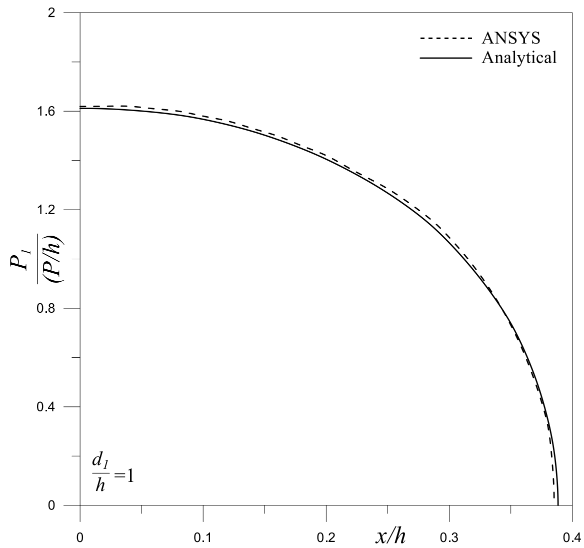

Fig. 6 Contact pressure distribution between layer and stamp ($p_1/(P/h)$) for (d_1/h=1.0) (G_1/G_2=0.5, K_1=K_2=2, R/h=250, Q/P=0.5, $G_1/(P/h)$=1000, d_2/h=1.5, f/h=0.5)

It is seen from all tables that contact areas (a/h, c/h) obtained from analytical and finite element results are close and the differences between them are less than % 2.15. In addition, (a/h) shows less than 15% change at most between maximum and minimum values among the loading cases whereas the maximum change in (c/h) is more than 250%.

Figs. 4, 6, 8 and 10 show the variation of the contact pressure distribution between layer and stamp contact surface ($p_1/(P/h)$) for various distributed load conditions. Since the obtained graphs are almost overlapped for each loading condition, only one graph is shown in the corresponding

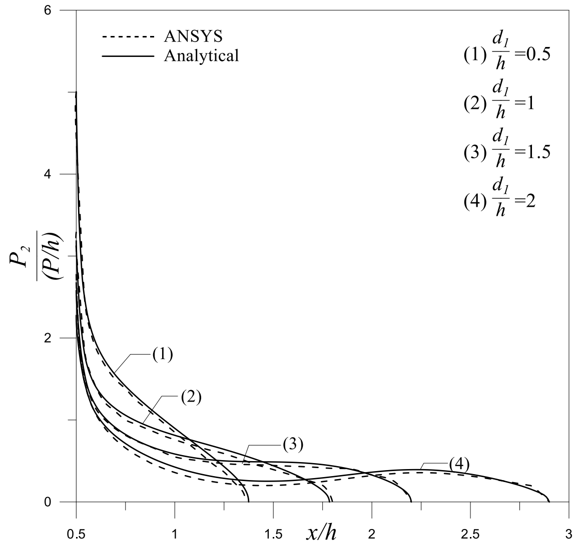

Fig. 7 Contact pressure distribution between layer and quarter plane ($p_2/(P/h)$) with (d_1/h) (G_1/G_2=0.5, K_1=K_2=2, R/h=250, Q/P=0.5, $G_1/(P/h)$=1000, d_2/h=d_1/h+0.5, f/h=0.5)

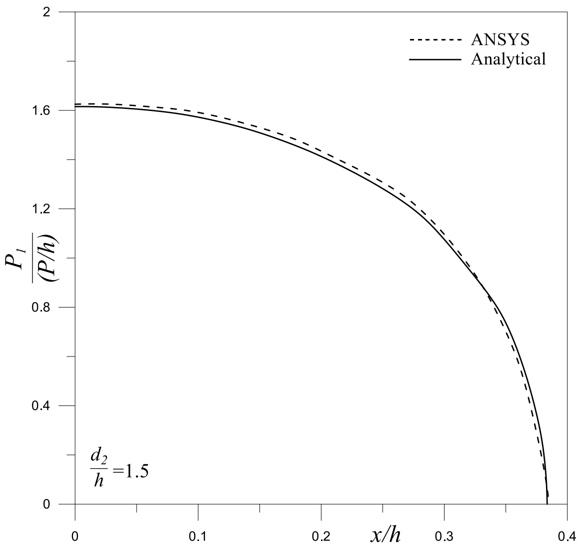

Fig. 8 Contact pressure distribution between layer and stamp ($p_1/(P/h)$) for (d_2/h=1.5) (G_1/G_2=0.5, K_1=K_2=2, R/h=250, Q/P=0.5, $G_1/(P/h)$=1000, d_1/h=0.5, d_2/h=1.5, f/h=0.5)

figure. It may be observed from these figures that the contact pressures become zero at the end of the contact and reaches its maximum value at center of the stamp.

The variation of the contact pressure distribution between layer and the quarter plane contact surface ($p_2/(P/h)$) is given in Figs. 5, 7, 9 and 11 for various distributed load conditions. It can be seen from these figures that the contact pressures become zero at the right end and go to infinity at the edge of the quarter plane. In addition, the area under graphs increases when (Q/P) increases.

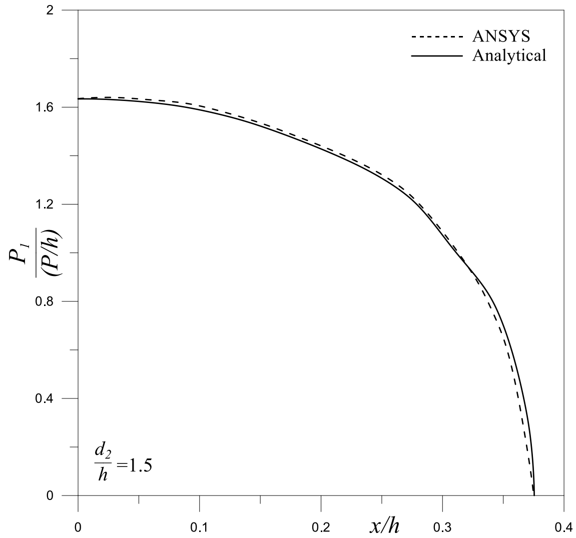

Fig. 10 Contact pressure distribution between layer and stamp ($p_1/(P/h)$) for (d_2/h=1.5) (G_1/G_2=0.5, K_1=K_2=2, R/h=250, Q/P=1.0, $G_1/(P/h)$=1000, d_1/h=0.5, d_2/h=1.5, f/h=0.5)

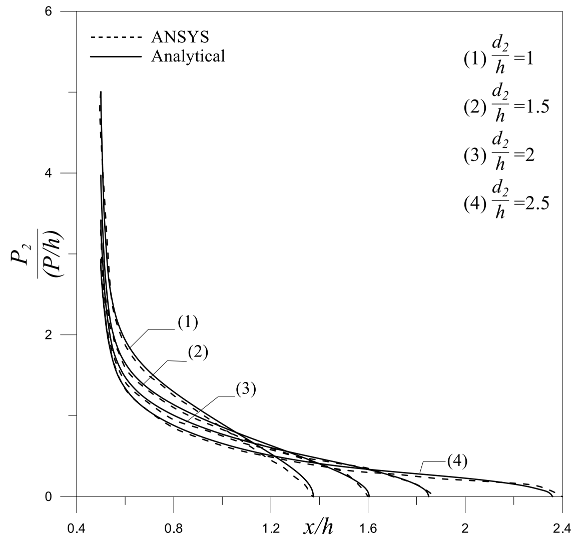

Fig. 9 Contact pressure distribution between layer and quarter plane ($p_2/(P/h)$) with (d_2/h) (G_1/G_2=0.5, K_1=K_2=2, R/h=250, Q/P=0.5, $G_1/(P/h)$=1000, d_1/h=0.5, f/h=0.5)

The compare of dimensionless contact pressures for analytical and numerical results by means of root mean square error (RMSE) between layer and stamp contact surface and between layer and quarter plane contact surface are given in Tables 5-6 respectively.

It is seen from Tables 5-6 and all figures that dimensionless contact pressures distributions obtained from analytical solution and finite element solution agree well. The contact pressures distribution between layer and stamp show similar distributions whereas contact pressures between layer and quarter plane show varieties depend on the magnitude and position of the distributed load.

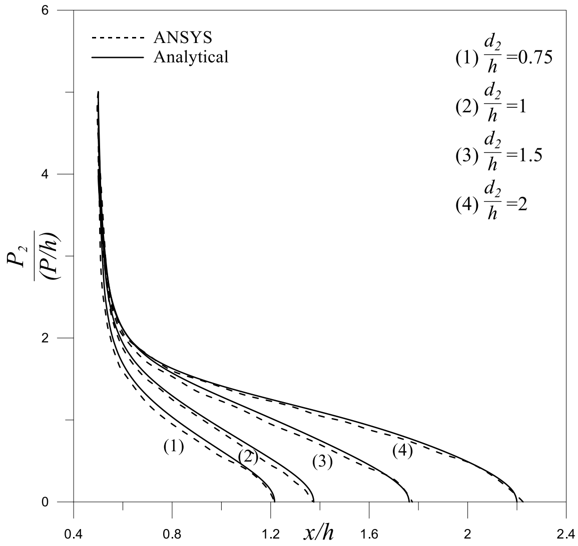

Fig. 11 Contact pressure distribution between layer and quarter plane ($p_2/(P/h)$) with (d_2/h) (G_1/G_2=0.5, $K_1=K_2$=2, R/h=250, d_1/h=0.5, $Q/P=(d_2/h-d_1/h)$, $G_1/(P/h)$=1000, f/h=0.5)

Table 5 RMSE for dimensionless contact pressures between layer and stamp contact surface

Figure	Fig. 4	Fig. 6	Fig. 8	Fig. 10
RMSE	0.0027	0.0022	0.0018	0.0033

Table 6 RMSE for dimensionless contact pressures between layer and quarter plane contact surface

Figure	Fig. 5	Fig. 7	Fig. 9	Fig. 11
Graph (1)	0.02727	0.0192	0.0234	0.0709
Graph (2)	0.0558	0.0824	0.0135	0.0124
Graph (3)	0.2642	0.0468	0.0562	0.0182
Graph (4)	0.5126	0.2642	0.0603	0.0313

5. Conclusions

The receding contact problem for an elastic layer resting on two quarter planes is investigated using both analytical method based on theory of elasticity and finite element method. Dimensionless contact areas (a/h, c/h) and contact pressure distributions ($p_1/(P/h)$, $p_2/(P/h)$) are obtained using both method between layer and stamp and between layer and quarter plane contact surfaces for various distributed load conditions. Obtained results show that

the contact area (a/h) and contact pressure ($p_1/(P/h)$) between layer and stamp contact surface do not show any significant change in case of different distributed load combinations. However, the magnitude and position of the distributed load have considerable effect on the contact area (c/h) and contact pressure ($p_2/(P/h)$) between layer and quarter plane contact surface. It is also verified that the difference between analytical solution and finite element solution carried out by ANSYS is in an acceptable range.

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