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Effects of edge crack on the vibration characteristics of delaminated beams

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Abstract. Delaminations and cracks are common failures in structures. They may significantly reduce the stiffness of the structure and affect their vibration characteristics. In the present study, an analytical solution is developed to study the effect of an edge crack on the vibration characteristics of delaminated beams. The rotational spring model, the 'free mode' and 'constrained mode' assumptions in delamination vibration are adopted. This is the first study on how an edge crack affects the vibration characteristic of delaminated beams and new nondimensional parameters are developed accordingly. The crack may occur inside or outside the delaminated area and both cases are studied. Results show that the effect of delamination length and thickness-wise location on reducing the natural frequencies is aggravated by an increasing crack depth. The location of the crack also influences the effect of delamination, but such influence is different between crack occurring inside and outside the delaminated area. The difference of natural frequencies between 'free mode' and 'constrained mode' increases then decreases as the crack moves from one side of the delaminated region to the other side, peaking at the middle. The analytical results of this study can serve as the benchmark for FEM and other numerical solutions.

Keywords: vibration; beam; delamination; crack; natural frequency

1. Introduction

Delaminations in structures may arise during manufacturing (e.g., incomplete wetting, air entrapment) or during service (e.g., low velocity impact) (Della and Shu2007). They may not be visible on the surface since they are embedded within the structures. However, the presence of delamination may significantly reduce the stiffness and strength of the structures (Tay 2003) and may affect their vibration characteristics (e.g., natural frequency and mode shape). At the same time, the effect of cracks on the dynamic behavior of structural elements such as shafts, beams and plates has been the subject of many investigations (Dimarogonas1996, Chondros and Dimarogonas 1998, Chondros *et al*. 2001). Damage detection/structural health monitoring problems are also considered in the frameworks of vibration analysis (Gounaris *et al*. 1996, Zou *et al*.2000, Sayyad and Kumar 2012). Therefore, it is of great practical importance to understand how the presence of delamination and crack affects the vibration characteristics of the structures.

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To study the influence of a through-width delamination on the free vibration of an isotropic beam, Wang *et al*. (1982) presented an analytical model using four Euler-Bernoulli beams that are joined together. They assumed that the delaminated layers deform ‘freely’ without touching each other (‘free mode’) and will have different transverse deformations. While Mujumdar and Suryanarayan (1988) assumed that the delaminated layers are in touch along their whole length all the time, but are allowed to slide over each other (‘constrained mode’). Thus, the delaminated layers are ‘constrained’ to have identical transverse deformations. The same analytical model has also been used on the study of the buckling of delaminated beams (Parlapalli and Shu 2006, Parlapalli *et al.* 2006, Parlapalli *et al*. 2008).

Analytical solutions for the vibration of beams with multiple delaminations have been presented by many researchers. An analytical formulation is proposed and studied by Lee *et al*. (2003) for the vibration analysis of composite beams with arbitrary lateral and longitudinal multiple delaminations. Saravanos and Hopkins (1996) provided the analytical solution for the damped free vibration of delaminated beams. Luo and Hanagud (2000) presented a new analytical model for composite beams with delaminations. Nonlinear interaction, due piecewise linear spring models between the delaminated sublaminates, is included. Della and Shu (2004, 2005a, b, 2006, 2009) used the ‘free mode’ and ‘constrained mode’ assumptions to study beams with several delamination configurations. Liu and Shu (2012) provided the analytical solution to the free vibration of rotating Timoshenko beams with multiple delaminations, the results of which show that the effects of rotary inertia and shear deformation, as well as rotating speed, have significant influences on delamination vibration.

Another form of damage that can lead to catastrophic failure if undetected is the cracking of the structural elements (Chondros*et al*. 1998a). The vibration of beams and bars with edge cracks or closing cracks are extensively studied (Ruotolo *et al*. 1996, Chondros *et al*. 1998b, Shifrin and Ruotolo1999, Pugno *et al*.2000, Ruotolo and Surace 2004). Birman and Simitses (2001) studied the vibration of sandwich panels and beams with matrix cracks in the facings. Kisa (2004, 2012) conducted the free vibration analysis of axially loaded beams and a cantilever composite beam with multiple cracks. The finite element and the component mode synthesis methods are used to model the problem. The cantilever composite beam is divided into several components from the crack sections. Kisa and Gurel (2005, 2007) conducted the vibration analysis of cracked cantilever composite beams as well as cracked beams with circular cross sections. The effects of edge crack on the buckling loads, natural frequencies and dynamic stability of circular curved beams are investigated numerically by Karaagac *et al*. (2011). Fallah and Mousavi (2012) developed an inverse approach for calculating the flexibility coefficient of open-side cracks in the cross-section of beams. Ibrahim *et al*. (2013) studied the effects of crack on the vibration of framed structures. Recently, the study on the vibration of cracked beams has been extended to inhomogeneous materials (Yang and Chen 2008, Yang *et al*. 2008, Kitipornchai *et al*. 2009, Ke *et al*. 2009).

To the authors’ knowledge, this is the first study on the influences of edge crack on the free vibration of delaminated beams. In the present study, an analytical solution is developed to study the effect of an edge crack on the vibration characteristics of delaminated beams and new nondimensional parameters are developed accordingly. The Euler-Bernoulli beam theory, the ‘free mode’ and ‘constrained mode’ assumptions in delamination vibration, as well as the rotational spring model are adopted. The crack may occur inside or outside the delaminated area and both cases are studied. The analytical results of this study can serve as the benchmark for FEM and other numerical solutions.

2. Formulation

Fig. 1(a) shows a beam with length L and thickness H_1. The Young's modulus of the beam is E. The beam is separated along the interface by a delamination with length a and located at a distance l from the center of the beam, the thickness of the top and bottom layer are H_3 and H_4. The beam can then be subdivided into three span-wise regions, a delamination region and two integral regions. The delamination region is comprised of two segments (delaminated layers), beam 3 and beam 4, which are joined at their ends to the integral segments, beam 2 and beam 5. The edge crack is of depth b, locating with a distance d from the left end of the beam. Each of the five beams are treated as Euler-Bernoulli beam, thus the analytical solutions are valid provided that $L_i \gg H_i$, where i=1-5.

2.1 Governing equations

The present study is carried out within the framework of small deformation theory. Let $w_i(x,t)$ denote the midplane deflection of beami. The governing equation for the free vibration of Euler-Bernoulli beam is: (i=1-5)

$$EI_i \frac{\partial^4 w}{\partial x^4} + \rho A_i \frac{\partial^2 w}{\partial t^2} = 0 \tag{1}$$

Where ρ is the mass density, A_i and EI_i are the cross-sectional area and bending stiffness of beam i, respectively.

Assuming a harmonic motion for the free vibration of the beam then

$$w_i = W_i(x)\sin(\omega t) \tag{2}$$

substituting Eq. (2) into Eq. (1), it follows that

$$EI_i W_i'''' - \omega^2 \rho A_i W_i = 0 \tag{3}$$

The generalized solution for Eq. (3) is given by

$$W_i(x) = C_i \cos\left(\alpha_i \frac{x}{L}\right) + S_i \sin\left(\alpha_i \frac{x}{L}\right) + CH_i\, cosh\left(\beta_i \frac{x}{L}\right) + SH_i\, sinh\left(\beta_i \frac{x}{L}\right) \tag{4}$$

where

$$\lambda_i^4 = {\omega^2 \rho A_i}\big/{EI_i}\, L^4 \tag{5}$$

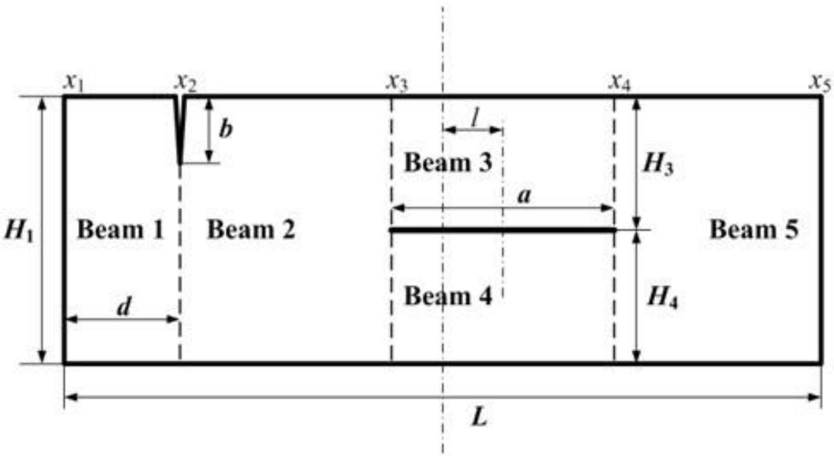
2.2 Free mode model

Eq. (3) is applied to the five interconnected sub-beams, respectively (Fig. 1(a)). The appropriate boundary conditions that can be applied at the supports, $x=x_1$ and $x=x_5$ are $W_i=0$ and $W'_i=0$ if the end of the beam is clamped, $W_i=0$ and $W''_i=0$ if hinged, $W''_i=0$ and $W'''_i=0$ if free, where i=1, 5.

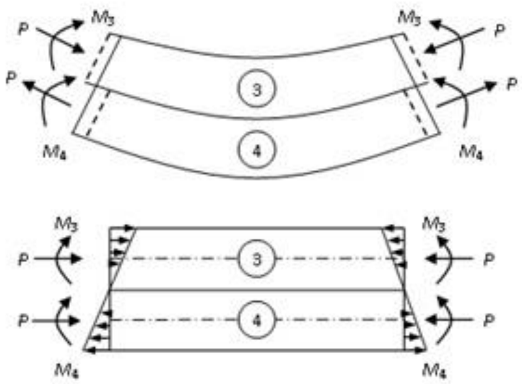
The continuity conditions for deflection and slope at $x=x_5$ are

$$W_2 = W_3 = W_4 \tag{6}$$

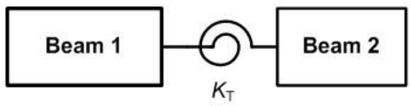
$$W_2{}' = W_3{}' = W_4{}' \tag{7}$$



(a) Beam with a single delamination and an edge crack



(b) The compatibility between the stretching/shortening of the delaminated layers and axial equilibrium



(c) Rotational spring model.

Fig. 1 Analytical modeling of delaminated beams with an edge crack

The continuity condition for shear force and bending moments at x=x_3 are

$$EI_2 W_2''' = EI_3 W_3''' + EI_4 W_4''' \tag{8}$$

$$EI_2 W_2'' = EI_3 W_3'' + EI_4 W_4'' - T_3 \left({}^{H_2}/_2 - {}^{H_3}/_2 \right) + T_4 ({}^{H_2}/_2 - {}^{H_4}/_2) \tag{9}$$

The axial force T_i can be solved from the compatibility between the stretching/shortening of the delaminated layers and axial equilibrium (Mujumdar and Suryanarayan 1988), as is shown in Fig. 1(b), thus

$$ {}^{T_4 a}/_{EA_4} - {}^{T_3 a}/_{EA_3} = (W_2'(x_3) - W_5'(x_4)) \, {}^{H_2}/_2 \tag{10}$$

$$T_2 = T_3 + T_4 = 0 \tag{11}$$

where EA_3 and EA_3 are the axial stiffness of beams 3 and 4, respectively. By substituting Eqs. (10) and (11) into Eq. (9), the continuity condition of bending moment can be expressed as

$$EI_2W_2'' + {(EH_3EH_4)H_2^2}\Big/{4a(EH_3+EH_4)}\,(W_2' - W_5') = EI_3W_3{}'' + EI_4W_4{}'' \tag{12}$$

And similarly, one can derive the continuity conditions at x=x_4 .

2.3 Constrained mode model

The ‘constrained mode’ model is simplified by the assumption that the delaminated layers are constrained to have the same transverse deformations. The delaminated beam is analyzed as four beam segments Ⅰ-Ⅳ.

For the ‘constrained mode’ model, the governing equations for beam 1, 2 and 5 are identical to Eq. (3). For beam 3 and beam 4, i.e., segment III, the governing equation is

$$(EI_3 + EI_4)W_{III}'''' - (\rho A_3 + \rho A_4)\omega^2 W_{III} = 0 \tag{13}$$

The boundary conditions for the ‘constrained mode’ are identical to the boundary conditions of the ‘free mode’. The continuity conditions for deflection, slope, shearand bending moments x=x_3 are

$$W_{II} = W_{III} \tag{14}$$

$$W_{II}{}' = W_{III}{}' \tag{15}$$

$$EI_2W_{II}''' = (EI_3 + EI_4)W_{III}''' \tag{16}$$

$$EI_2W_{II}'' + {(EH_3EH_4)H_2^2}\Big/{4a(EH_3+EH_4)}\,(W_{II}' - W_{IV}') = (EI_3 + EI_4)W_{III}'' \tag{17}$$

where ${(EH_3EH_4)H_2^2}\Big/{4a(EH_3+EH_4)}\,(W_{II}' - W_{IV}')$ represents the consideration of the compatibility between the stretching/shortening of the delaminated layers and axial equilibrium. And similarly, one can derive the continuity conditions at x=x_4.

2.4 Continuity condition at the crack

The continuity conditions of deflection, shearing force and compatibility of moment, at x=x_2, are expressed in Eq. (18) - Eq. (20), Moreover, Eq. (21) introduces a discontinuity into the rotating of the beam axis, by imposing equilibrium between transmitted bending moment and rotation of the spring representing the crack (Shifrin and Ruotolo 1999), as is shown in Fig. 1(c)

$$W_1 = W_2 \tag{18}$$

$$W_1{}''' = W_2{}''' \tag{19}$$

$$W_1{}'' = W_2{}'' \tag{20}$$

$$W_1' + cW_1'' = W_2{}' \tag{21}$$

where c = 5.346 $H_1f(\xi)$ and $f(\xi)$= 1.8624 ξ^2- 3.95 ξ^3 + 16.375 ξ^4- 37.226 ξ^5 + 76.81 ξ^6-126.9 ξ^7 + 172 ξ^8 - 143.97 ξ^9 + 66.56 ξ^{10}, in which $\xi = b/H_1$.

The total number of boundary and continuity conditions is 20, which is equal to the total

Table 1 First and second non-dimensional frequencies (λ^2) of a clamped-clamped isotropic beam without edge crack (by denoting b/H_1=0.001) but suffering a central midplane delamination

a/L	First Mode			Second Mode		
	Present Free & Constrained	Analytical Wang *et al*. (1982)	FEM Lee (2000)	Present Free & Constrained	Analytical Wang *et al*. (1982)	FEM Lee (2000)
0.0	22.37	22.39	22.36	61.67	61.67	61.61
0.1	22.37	22.37	22.36	60.81	60.76	60.74
0.2	22.36	22.35	22.35	56.00	55.97	55.95
0.3	22.24	22.23	22.23	49.00	49.00	48.97
0.4	21.83	21.83	21.82	43.89	43.87	43.86
0.5	20.89	20.88	20.88	41.52	41.45	41.50
0.6	19.30	19.29	19.28	41.04	40.93	41.01
0.7	17.23	17.23	17.22	40.82	40.72	40.80
0.8	15.05	15.05	15.05	39.07	39.01	39.04
0.9	13.00	13.00	12.99	35.39	35.38	35.38

number of unknown coefficients. A non-trivial solution exists only when the determinant of the coefficient matrix vanishes. The natural frequencies can be obtained as eigenvalues.

2.5 Introduction on nondimensional parameters

To study the effect of the location of the edge crack, non-dimensional parameter D is introduced. When the crack occurs outside the delaminated area, $D=d/(L/2-a/2)$, where d is the distance of the crack from the left end of the beam; when inside the delaminated area, $D=d/a$, where d is the distance of the crack from the left boundary of the delaminated area.

3. Results and discussions

3.1 Verification

Table 1 shows the first and second non-dimensional natural frequency λ^2 of an isotropic beam without edge crack (by denoting b/H_1=0.001) but suffering a single central midplane delamination of various lengths, compared with the analytical results of Wang *et al*. (1982) and FEM results of Lee (2000). As is shown in Table 1, the results of the current study agree well with previous published results.

Table 2 shows the comparison of the current results against the experimental results of Mujumdar and Suryanarayan (1988) and the mixed-FEM results of Ramtekkar (2009). The first and second natural frequency ω_1 (Hz) of an isotropic cantilever beam without edge crack (by denoting b/H_1=0.001) but suffering a single delamination (H_3/H_1=0.33) is tabulated here.

In Table 3, a clamped-free beam with two surface cracks (no delamination) is compared with the work of Shifrin and Ruotolo (1999) (Fig. 2), to which the configuration of the beam can be referred. Table 3 shows good agreements between the previous results and present ones.

Table 2 First and second natural frequency $\omega_{1,}\omega_2$ (Hz) of cantilever isotropic beam without edge crack (by denoting b/H_1=0.001) but suffering a single delamination, H_3/H_1=0.33

Specimen span (mm)	Dimension		First mode ω_1			Second Mode ω_2		
	$L/2$-d_a	a	Experiment (Mujumdar and Suryanarayan 1988)	Ramtekkar (2009) Unconstrained-interface model	Present Free mode	Experiment (Mujumdar and Suryanarayan 1988)	Ramtekkar (2009) Unconstrained-interface model	Present Free mode
240.0	153.5	96.0	31.6	31.11	32.25	172.10	156.24	159.93
250.0	104.0	148.0	31.7	31.51	31.88	190.50	197.23	198.81
175.0	133.0	88.5	56.9	56.35	57.89	339.30	244.42	248.83
200.0	122.0	106.0	46.6	46.34	47.26	291.00	258.95	260.71
155.0	122.0	61.0	69.1	68.82	69.82	363.70	249.20	281.82

Table 3 Normalizedfirst naturalfrequency (ω/ω_0) of a clamped-clamped isotropic Euler-Bernoulli beam with two edge cracks, with a clamped-free boundary condition, without delamination

Depth of the second crack b_2/H_1 (mm)	Distance of the second crackfrom the clamped end			
	0.2 m		0.5 m	
	Shifrin and Ruotolo (1999)	Present	Shifrin and Ruotolo (1999)	Present
0.1	0.888	0.892	0.93	0.932
0.2	0.757	0.76	0.914	0.916
0.3	0.536	0.539	0.883	0.885

3.2 Crack occurs outside the delaminated area

As is shown in Fig. 1(a), the edge crack occurs outside the delaminated area, with a distance d from the left end of the beam, of depth b. The boundary condition is clamped-clamped. The delamination is of length, locating at the middle of the beam, with H_3/H_1=0.5.

Fig. 2 shows the normalized natural frequency λ^2/λ_d^2 versus the non-dimensional distance $D=d/(L/2-a/2)$, considering different length of delamination, for a clamped-clamped beam with a single delamination and a crack occurring outside the delaminated area. λ_d^2 is the nondimensional natural frequency of an undelaminated beam.

It can be concluded that the difference of natural frequency between a/L=0.1 and 0.2 is bigger when the depth of crack increases, indicating that the effect of delamination on reducing natural frequency is aggravated by an increasing depth of the edge crack. Such difference increases then decreases as the edge crack moves towards the left boundary of the delamination.

One can also see from Fig. 2 that the differences of natural frequencies between b/H_1=0.1 and 0.2 as well as 0.2 and 0.3 decrease to zero then increases as the crack moves from the left to the right.

3.3 Crack occurs inside the delaminated area

As is shown in Fig. 3, the edge crack occurs inside the delaminated area, with a distance d from the left boundary of the delamination, of depthb. The delamination is of length, locating at the middle of the beam. The boundary condition is clamped-clamped.

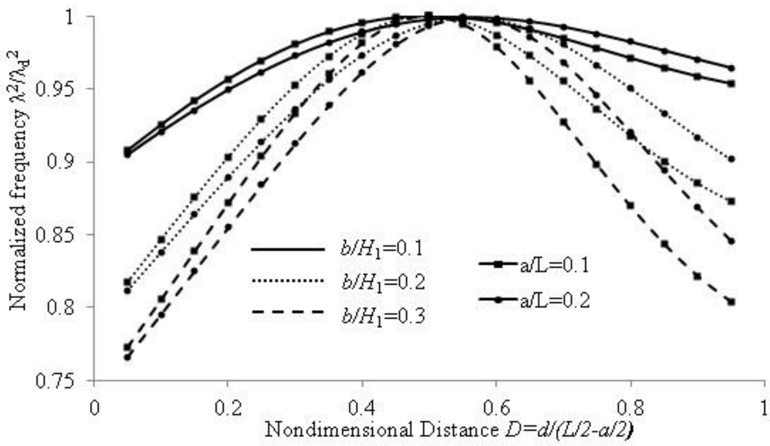


Fig. 2 Influence of the depth and location of the crack on the 'free mode' natural frequencies λ^2/λ_d^2 for a homogeneous clamped-clamped beam with a single central delamination of various lengths and the crack locates outside the delaminated area

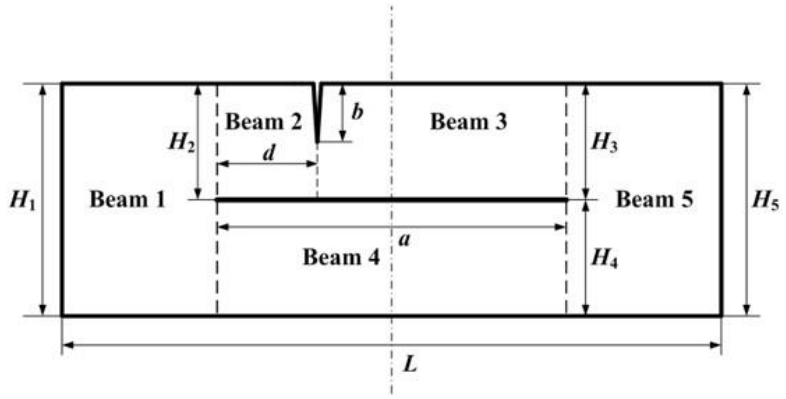


Fig. 3 Cracked beam with a single delamination and the crack locates inside the delaminated area

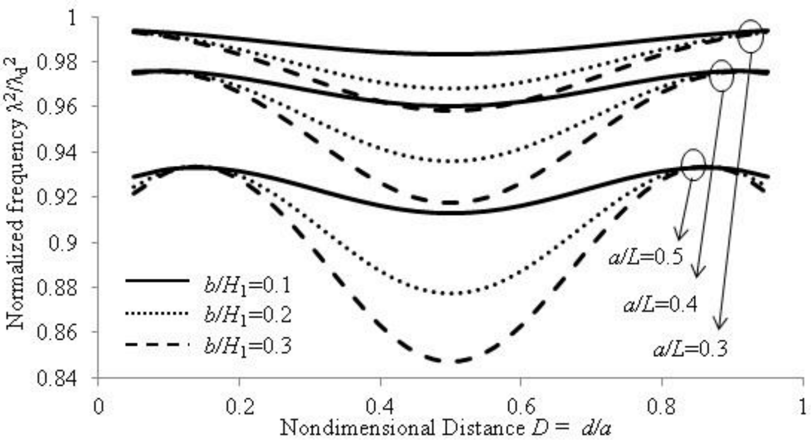


Fig. 4 Influence of the depth and location of the crack on the 'free mode' natural frequencies λ^2/λ_d^2 for a homogeneous clamped-clamped beam with a single central delamination of various lengths

Table 4 Non-dimensional normalized fundamental frequency λ^2/λ_d^2 of a clamped-clamped isotropic Euler-Bernoulli beam with a single delamination and one edge crack, the crack occurs inside the delaminated area

D	b/H_1	*a/L*=0.5			*a/L*=0.4			*a/L*=0.3		
		0.1	0.2	0.3	0.1	0.2	0.3	0.1	0.2	0.3
0.1	0.933	0.932	0.931	0.976	0.976	0.976	0.993	0.992	0.991	0.1
0.2	0.932	0.929	0.928	0.973	0.969	0.966	0.990	0.986	0.983	0.2
0.3	0.924	0.910	0.899	0.967	0.955	0.947	0.987	0.978	0.972	0.3
0.4	0.916	0.887	0.863	0.962	0.942	0.926	0.984	0.971	0.962	0.4
0.5	0.913	0.877	0.847	0.960	0.936	0.917	0.983	0.968	0.958	0.5
0.6	0.916	0.887	0.863	0.962	0.942	0.926	0.984	0.971	0.962	0.6
0.7	0.924	0.910	0.899	0.967	0.955	0.947	0.987	0.978	0.972	0.7
0.8	0.932	0.929	0.928	0.973	0.969	0.966	0.990	0.986	0.983	0.8
0.9	0.933	0.932	0.931	0.976	0.976	0.976	0.993	0.992	0.991	0.9

Fig. 4 shows the normalized natural frequency λ^2/λ_d^2, of a clamped-clamped beam with a single delamination and a crack occurring inside the delaminated area, versus the nondimensional distance $D=d/a$, considering different lengths of delamination, with H_3/H_1=0.5.

At midpoint when D=0.5, the drop of natural frequency from 0.913 when b/H_1=0.1 to 0.847 when b/H_1=0.3, for delamination length a/L=0.5. The drop is only from 0.960 to 0.917 for a/L=0.4, and 0.983 to 0.958 for a/L=0.3. It indicates that when the edge crack occurs inside the delaminated area, the effect of crack on aggravating the influence of delamination is bigger when the beam suffers a longer delamination.

It can also be observed from Fig. 4 that the differences of natural frequencies between crack depth b/H_1=0.1 and 0.2, 0.2 and 0.3 increase as the crack moves from left towards the middle, then decreases as it continues to the right. Such difference is bigger when the delamination length a/L increases from 0.3 to 0.5. The difference of natural frequencies between delamination lengths a/L=0.3 and 0.4, 0.4 and 0.5, is bigger with a deeper crack. For the same crack depth b, the difference peaks when the crack occurs at the middle of the delamination. The results of Fig. 4 arealso shown in Table 4 for the convenience of serving as a benchmark for other numerical results.

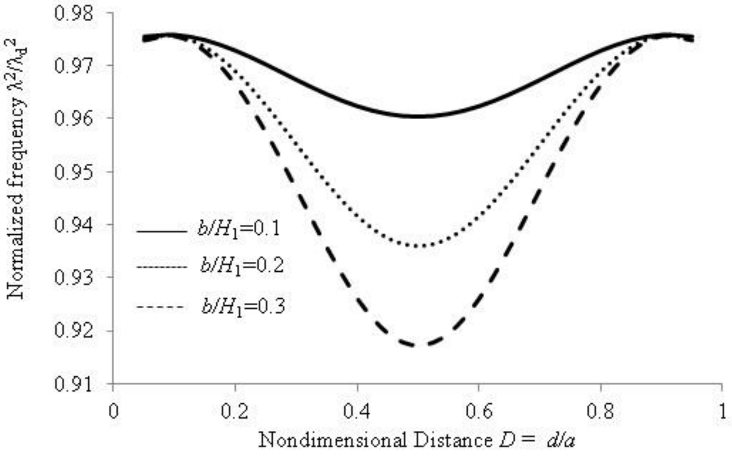
Fig. 5 shows the normalized natural frequency λ^2/λ_d^2, of a clamped-clamped beam with a single delamination and a crack occurring inside the delaminated area, versus the non-dimensional distance $D=d/a$, considering different thickness-wise locations, with a/L=0.4.

When comparing the results between Fig. 5 (a), (b) and (c), it can be concluded that the differences of natural frequencies between crack depth b/H_1=0.1 and 0.2, 0.2 and 0.3 decreases when the delamination moves towards the surface. The differences of natural frequencies between the thickness-wise location H_3/H_1=0.3 and 0.4, 0.4 and 0.5 are bigger with a deeper crack.

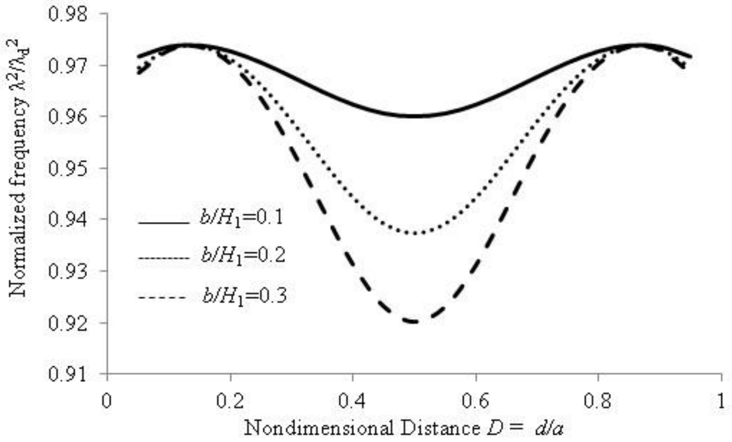
Fig. 6 shows the normalized natural frequency λ^2 / λ_d^2, of a clamped-clamped beam versus the non-dimensional distance $D=d/a$, adopting both 'free mode' and 'constrained mode' assumptions. There is a single mid-plane central delamination and a crack occurring inside the delaminated area.

The difference of natural frequencies between 'free mode' and 'constrained mode' increases then decreases as D increases, peaking at the middle of the beam. It can also be observed from Fig. 6 that such difference is bigger with a deeper crack.

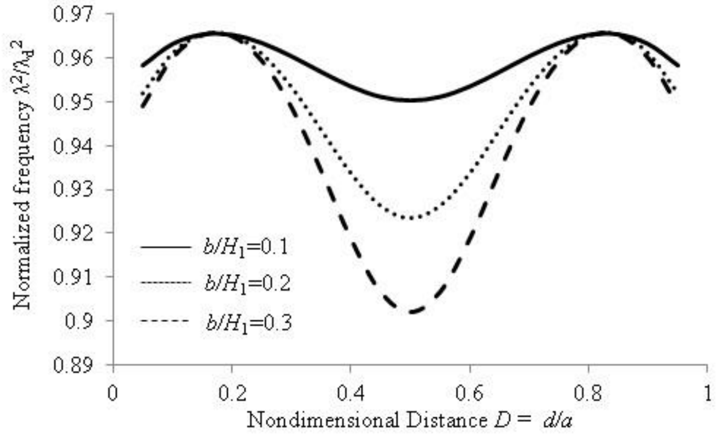
When comparing the result among Fig. 6 (a), (b) and (c), one can see that the difference of natural frequencies between 'free mode' and 'constrained mode' is bigger with a longer delamination, when $0.25<D<0.75$; bigger with a shorter delamination when $0<D<0.25$ and $0.75<D<1$.



(a) H_3/H_1=0.5

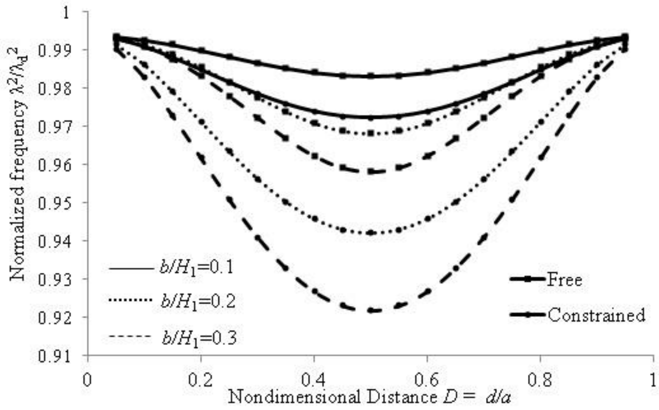


(b) H_3/H_1=0.4

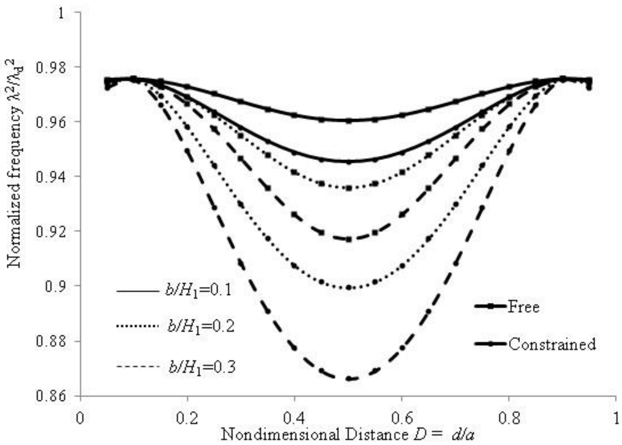


(c) H_3/H_1=0.3

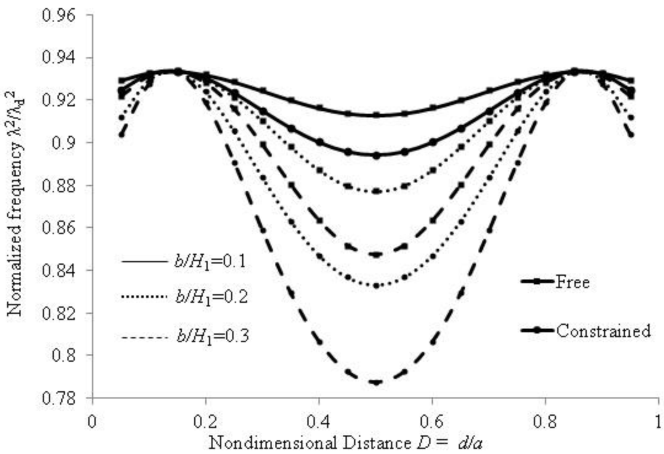
Fig. 5 Influence of the depth and location of the crack on the 'free mode' natural frequencies for a homogeneous clamped-clamped beam with a single delamination of various thickness-wise locations



(a) a/L=0.3



(b) a/L=0.4



(c) a/L=0.5

Fig. 6 Influence of the crack on 'free mode' and 'constrained mode'natural frequencies λ^2/λ_d^2 for an isotropic homogeneous clamped-clamped beam with a single central delamination, considering different delamination length

4. Conclusions

In the present study, an analytical solution to the vibration of cracked beams with delaminations is developed. We study how the location and depth of the crack influence the effects delamination on natural frequencies and mode shapes. Both cases of crack occurring outside and inside the delaminated area are thoroughly investigated. The comparison of results between 'free mode' and 'constrained mode' of delamination vibration assumptions is conducted. Based on the theoretical investigations the following conclusions may be drawn.

• The effect of delamination on reducing the natural frequency is aggravated by an increasing depth of the edge crack. Such effect becomes bigger then smaller as the crack moves towards the boundary of the delaminated region (when crack occurs outside the delaminated area.)

• When the edge crack occurs inside the delaminated area, the effect of crack on aggravating the influence of delamination is bigger when the beam suffers a longer delamination. Such effect peaks when the crack occurs at the middle of the delaminated region.

• When the edge crack occurs inside the delaminated area, the effect of crack depth on natural frequency is smaller when the delamination moves towards the surface. Also the effect of the delamination thickness-wise location on natural frequency is bigger when the crack is deeper.

• The difference of natural frequencies between 'free mode' and 'constrained mode' increases then decreases as the crack moves from one side of the delaminated region to the other side, peaking at the middle.

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