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# Material distribution optimization of 2D heterogeneous cylinder under thermo-mechanical loading

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**Abstract.** In this paper optimization of volume fraction distribution in a thick hollow cylinder with finite length made of two-dimensional functionally graded material (2D-FGM) and subjected to steady state thermal and mechanical loadings is considered. The finite element method with graded material properties within each element (graded finite elements) is used to model the structure. Volume fractions of constituent materials on a finite number of design points are taken as design variables and the volume fractions at any arbitrary point in the cylinder are obtained via cubic spline interpolation functions. The objective function selected as having the normalized effective stress equal to one at all points that leads to a uniform stress distribution in the structure. Genetic Algorithm jointed with interior penalty-function method for implementing constraints is effectively employed to find the global solution of the optimization problem. Obtained results indicates that by using the uniform distribution of normalized effective stress as objective function, considerably more efficient usage of materials can be achieved compared with the power law volume fraction distribution. Also considering uniform distribution of safety factor as design criteria instead of minimizing peak effective stress affects remarkably the optimum volume fractions.

**Keywords:** 2D heterogeneous; volume fraction optimization; thermo-mechanical stresses; genetic algorithm

## 1. Introduction

In recent years, the composition of several different materials is often used in structural components in order to optimize the responses of structures subjected to thermal and mechanical loads. Functionally graded materials (FGMs) are suitable to achieve this purpose. The mechanical properties of FGMs vary continuously between several different materials. This idea was used for the first time by Japanese researcher Koizumi (1993), leads to the concept of FGMs. These materials are expected to be used for thermal applications and high rate thermal and mechanical loadings. In the application of FGM cylindrical structures to aerospace, nuclear and automobile industries, analysis of thermal and mechanical stresses are of great importance. Analytical and computational studies of stresses, displacements and temperature in cylindrical structures made of FGM have been carried out by a lot of researchers.

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In most of the previous studies the volume fraction and properties of the FGMs are one-dimensional dependent and properties vary continuously from one surface to the other with a prescribed function, while in advanced machine elements temperature and load distributions may change in two or three directions. Therefore, if the FGM has two-dimensional dependent material properties, more effective material resistance can be obtained. Based on this fact 2D-FGMs whose material properties are bi-directionally dependent are introduced. Recently a few authors investigated 2D-FGMs. Aboudi and Pindera (1996) studied thermo-elastic/plastic theory for the response of materials functionally graded in two directions. Nemat-Alla (2003) investigated the reduction of thermal stresses by developing 2D-FGMs. They considered a FGM plate under transient thermal loading and solved the governing equations using finite element method. Asgari and Akhlaghi (2010, 2009, 2011) considered thermal stresses and transient heat conduction in a 2D FGM thick hollow cylinder. Finite elements method are used in numerous studies in order to model the variation of material properties in FGMs. Conventional finite element formulations use a single material property to each element such that the property field is constant within an individual element. But using this method for FGM problems leads to significant discontinuities and inaccuracies. These inaccuracies will be more significant in 2D FGM cases. Sentare and Lambros (2000), Kim and Paulino (2002) showed that graded finite elements can improve accuracy without increasing the number of degrees of freedom and decreasing the size of elements. A main problem in the design of an FGM, in addition to constituent material selection, lies in determining the optimal spatial dependence on the composition. This can be regarded as the best composition profile that accomplishes the design objectives of the materials while all constraints are satisfied (Huang *et al*. 2002). Based on this fact, attentions have been focused on the design optimization of FGMs. Cho and Ha (2002, 2009) concerned with the volume fraction optimization for minimizing steady-state thermal stresses in heat-resisting 1D FGM composites and also addressed a two-dimensional volume-fraction optimization procedure for relaxing the effective thermal stress distribution employing golden section method as optimization techniques, together with finite difference method for the sensitivity analysis. Cho and Choi (2004) introduced a yield-criteria optimization of the volume fraction distribution. In their work the objective function is defined by linearly combining the total strain energy and the peak effective stress scaled by the spatial-varying yield stress. A procedure for bi-objective optimization design of FGMs using a parametric formulation for both the geometric representation and the optimization procedure has been presented by Huang *et al*. (2002). Turteltaub (2002a, b) concerned with control and optimization of material layout of a FGM within the transient heat conduction phenomenon and thermomechanical loadings in order to determine the effective material with the goal of controlling the evolution of the corresponding field quantity. In his work the difference between the actual temperature field and a prescribed target field has been minimized in a mean squares sense in space and time. Cho and Shin (2004) applied an artificial neural network to the material composition optimization of heat-resisting FGMs based on approximation of the objective function by a back propagation ANN model. A systematic numerical technique for performing sensitivity analysis and optimization of coupled thermomechanical problem of FGMs to conduct the heat transfer analysis and structural analysis has been investigated by Chen and Tong (2005). Goupee and Vel (2006, 2007) proposed a methodology for the multi-objective optimization of material distribution of functionally graded materials with temperature-dependent material properties for steady thermomechanical loadings using the element-free Galerkin and genetic algorithm method for analysis and optimization. The effective material properties are estimated from the local volume fractions of the material constituents using the Mori-Tanaka and selfconsistent homogenization schemes. Vel and Pelletier (2007) also applied the same solution to both thin and thick functionally graded shells. Boussaa (2009) considered thermoelastic analysis of temperature-dependent FGMs under steady-state conditions and address the problem of the optimal choice of composition profile using a gradient-based algorithm to optimize a thick-walled functionally graded sphere. Na and Kim (2009, 2010) investigated FGM flat panels for volume fraction optimization by considering stress and critical temperature using a 3-D finite element model to analyze the variation of material properties and temperature field in the thickness direction. Nie *et al*. (2011) presented a technique to tailor materials for functionally graded elastic hollow cylinders and spheres with varying volume fraction only with the radius to attain through-the-thickness either a constant hoop stress or a constant in-plane shear stress. Kou *et al*. (2012) proposed a new method for the optimal design of FGM based on a feature tree procedural model instead of using explicit functional models to represent generic material heterogeneities. Takezawa *et al*. (2014) considered the topology optimization with strength and heat conduction constraints of structures using solid isotropic material with penalization (SIMP) method. Takezawa and Kitamura (2012) also discussed an application of the topology optimization method for the design of thermoelectric generators. Lee *et al*. (2014) considered steel plates with optimal material distributions achieved through a specific material topology optimization by using a CCARAT as an optimizer.

Analysis of thermo-mechanical loading of a thick hollow functionally graded cylinder with finite length can be rarely seen in literatures. Also the heat conduction in finite length cylinder is often investigated only in radial direction in these cases, while in real situations the heat conduction can be two dimensional in a finite length cylinder. On the other hand in most of cases the material gradation of FGM is one dimensional. A thick hollow cylinder with finite length made of 2D-FGM that its material properties are varied in the radial and axial directions with a power law function subjected to axisymmetric steady-state temperature loads on the inner surface, heat convection on the other surface and non-uniform internal pressure using an effective graded finite element method has been considered by the author (Asgari and Akhlaghi 2011). The effects of two-dimensional material distribution on the temperature, displacements and components of stresses as well as two-dimensional distributions of stresses through the cylinder are studied.

Based on importance of an optimum tailoring of material distribution, in this paper we consider optimization of volume fraction distribution of a thick hollow cylinder with finite length made of 2D-FGM that its material properties are varied in the radial and axial directions. Material properties are calculated by using linear rule of mixture and prescribed volume fraction in each point. The thermal and mechanical stresses in the structure under steady state loads are considered. It is subjected to axisymmetric steady-state temperature loads on the inner surface and heat convection on the other surface. Internal pressure applies to the hollow cylinder and external pressure is zero. The volume fraction distribution, thermo-mechanical loads and cylinder geometry are assumed to be axisymmetric but not uniform along the axial directions. The finite element method with graded material properties within each element (Graded FEM) is used to model the material properties variations. The optimization of the volume fractions distribution of each of constituent phases of the 2D FGM is considered as the optimization of the material distribution of structure. In such an axisymmetric structure the volume fraction can be determined independently in both radial and axial directions. In order to reduce the number of design variables, volume fractions of constituent materials on a finite number of design points are taken as design variables and the volume fractions at any arbitrary location in the cylinder are obtained via cubic spline interpolation functions. The objective function in this study selected as having the normalized

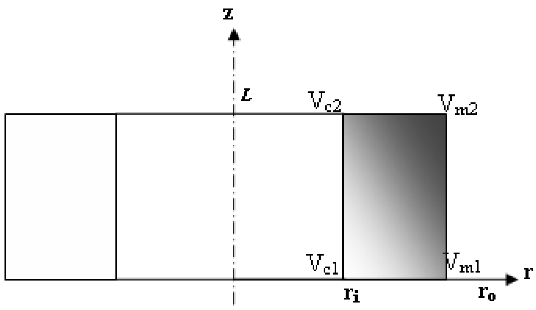



Fig. 1 Axisymmetric cylinder with two dimensional material distributions

effective stress equal to one at all points that leads to uniform distribution of safety factor in the structure. The maximum peak effective stress added as a nonlinear constraint. Genetic Algorithm jointed with interior penalty-function method for implementing constraints is effectively employed to find the global solution of the optimization problem. Obtained results indicates that by using the uniform distribution of normalized effective stress as objective function, considerably more efficient usage of materials can be achieved compared with the power law volume fraction distribution. Also considering uniform distribution of safety factor as design criteria instead of minimizing peak effective stress affects remarkably the optimum volume fractions.

The paper is arranged as follows. In Section 2 governing equations of heat transfer and thermoelasticity for the 2d-FGM thick hollow cylinder are derived via a variational formulation. The graded finite element is used for solving the governing equations effectively and considering the non-homogeneity of the materials are described in this section. Details of the optimization algorithm are given in Section 3. In Section 4, the proposed method is used to analyze and optimize the material distribution for prescribed problem and illustrative numerical results are presented and discussed. The conclusion is addressed in Section 5.

## 2. Basic equations for 2D-FGM cylinder

Consider a 2D-FGM thick hollow cylinder of internal radius $r_i$, external radius $r_o$, and finite length $L$. Because of axisymmetric geometry and loading, coordinates $r$ and $z$ are used in the analysis. 2D-FGMs are usually made by continuous gradation of three or four distinct material phases that one or two of them is/are ceramics and the others are metal alloy phases. The volume fractions of the constituents vary in a predetermined composition profile. For instance the volume fraction distribution of a 2D-FGM axisymmetric cylinder changed by a power law function is shown in Fig. 1. Needless to say, this distribution cannot create an optimum material distribution. In the present cylinder the inner surface is made of two distinct ceramics and the outer surface from two metals. $c1$, $c2$, $m1$ and $m2$ denote first ceramic, second ceramic, first metal and second metal, respectively. Material properties at each point can be obtained by using the linear rule of mixtures, in which a material property, $P$, at any arbitrary point ($r$, $z$) in the 2D-FGM cylinder is determined by linear combination of volume fractions and material properties of the basic materials as

$$P = P_{c1}V_{c1} + P_{c2}V_{c2} + P_{m1}V_{m1} + P_{m2}V_{m2} \tag{1}$$

It should be noted that Poisson's ratio is assumed to be constant through the body. This assumption is reasonable because of the small differences between the Poisson's ratios of basic materials.

### 2.1 Heat transfer equations

Without the existence of heat sources, the equation of heat conduction in axisymmetric cylindrical coordinates for the 2D-FGM cylinder is obtained as

$$1/r\,\frac{\partial}{\partial r}\left(rk_r(r,z)\frac{\partial T(r,z)}{\partial r}\right)+1/r\,\frac{\partial}{\partial z}\left(rk_z(r,z)\frac{\partial T(r,z)}{\partial z}\right)=0\,, \tag{2}$$

where $k_r$ and $k_z$ are thermal conductivity in the radial and axial directions of the FGM, respectively that vary in two directions. Distribution of conduction coefficient is

$$k(r,z)=k_{c1}V_{c1}(r,z)+k_{c2}V_{c2}(r,z)+k_{m1}V_{m1}(r,z)+k_{m2}V_{m2}(r,z)\,. \tag{3}$$

The thermal boundary conditions are as follow

Temperature at inner radius: $T(r_i,z)=T_i(z),$

heat convection at the outer radius: $k_r(r,z)\dfrac{\partial T(r_o,z)}{\partial r}+h_0(T-T_\infty)=0,$

heat convection at the lower edge: $k_z(r,z)\dfrac{\partial T(r,0)}{\partial z}+h_b(T-T_\infty)=0\,,$

heat convection at the upper edge: $k_z(r,z)\dfrac{\partial T(r,L)}{\partial z}+h_u(T-T_\infty)=0$ (4a-d)

where $h_0$, $h_b$, $h_u$ and $T_\infty$ are convection coefficients on the outer, lower, upper surfaces and surrounding temperature, respectively.

### 2.2 Thermoelasticity equations

For evaluation of thermal stresses due to temperature gradient the classical thermoelasticity is used. Based on this theory the equilibrium and strain-displacement equations are the same as elasticity problems for homogenous material (Boressi 1999). But constitutive stress-strain-temperature relations are

$$\sigma_{rr}=\frac{E(r,z)}{(1+\upsilon)(1-2\upsilon)}\left[(1-\upsilon)\varepsilon_{rr}+\upsilon(\varepsilon_{\theta\theta}+\varepsilon_{zz})\right]-\frac{E(r,z)\alpha(r,z)}{1-2\upsilon}T(r,z)\,,$$

$$\sigma_{\theta\theta}=\frac{E(r,z)}{(1+\upsilon)(1-2\upsilon)}\left[(1-\upsilon)\varepsilon_{\theta\theta}+\upsilon(\varepsilon_{rr}+\varepsilon_{zz})\right]-\frac{E(r,z)\alpha(r,z)}{1-2\upsilon}T(r,z)$$

$$\sigma_{zz}=\frac{E(r,z)}{(1+\upsilon)(1-2\upsilon)}\left[(1-\upsilon)\varepsilon_{zz}+\upsilon(\varepsilon_{rr}+\varepsilon_{\theta\theta})\right]-\frac{E(r,z)\alpha(r,z)}{1-2\upsilon}T(r,z)\,,$$

$$\sigma_{rz}=\frac{E(r,z)}{1+\upsilon}\varepsilon_{rz}\,. \tag{5a-d}$$

Where $E(r, z)$ and $\alpha(r, z)$ are modulus of elasticity and thermal expansion coefficient, respectively that are functions of position. $E(r, z)$ is determined at each point through the element using distribution function of this property based on the linear rule of mixtures as

$$E(r,z) = E_{c1}V_{c1}(r,z) + E_{c2}V_{c2}(r,z) + E_{m1}V_{m1}(r,z) + E_{m2}V_{m2}(r,z) \tag{6}$$

Also the thermal expansion is

$$\alpha(r,z) = \alpha_{c1}V_{c1}(r,z) + \alpha_{c2}V_{c2}(r,z) + \alpha_{m1}V_{m1}(r,z) + \alpha_{m2}V_{m2}(r,z)\,. \tag{7}$$

The cylinder is clamped on its two end edges and outer surface is free from pressure. Thus mechanical boundary conditions on the upper and lower edges are assumed as zero radial and axial displacements. Also on the inner surface we have internal pressure and outer surface is free from traction.

### *2.3 Graded finite element modeling*

In order to solve the governing equations the finite element modeling with graded element properties is used. For modeling a continuously non-homogeneous material, the material property function must be discretized according to the size of element mesh. This approximation can provide significant discontinuities. These artificial discontinuities can cause enormous error in the results. Effects of these discontinuities will be more considerable in the 2D-FGMs because of its 2D non-homogeneity. The use of a graded finite element has several potential advantages over the use of conventional elements in the study of 2D-FGMs (Asgari and Akhlaghi 2011). For modeling the heat transfer problem, the variational form of the present case leads to minimize the following integral

$$I = 1/2\iiint_{\bar{V}}\left[k_r(r,z)r(\frac{\partial T}{\partial r})^2 + k_z(r,z)r(\frac{\partial T}{\partial z})^2\right]dV + 1/2\iint_{\bar{S}} h_r(T^2 - 2T_\infty T)ds \tag{8}$$

where $\bar{V}$ and $\bar{S}$ are the volume of the cylinder and the boundary on which the convective heat loss is specified. Using axisymmetric ring elements with triangular cross-section and three nodes, the cylinder can be divided into the finite elements. By taking the three nodal values of temperature as the degrees of freedom and using the matrix of linear interpolation functions the functional $I$ which is the sum of elemental quantities is minimized and subjected to the boundary conditions. It should be noted that thermal conductivity in the radial and axial directions depends on $r$ and $z$ coordinates in the above integral. The graded finite element equations of heat conduction for each element can then be derived in which conduction coefficients in the radial and axial directions at each point assumed to be the same. The characteristic matrix in which the material properties are special dependent is described in reference (Asgari and Akhlaghi 2011).

For thermoelasticity problem also axisymmetric elements are used. By using the stress-strain relations the strain energy for each element and the work done by the external forces can be derived. Then By substituting equations of strain-displacement and nodal displacements field into strain energy for each element and summing up, the potential energy of the body can be expressed as

$$\pi_p = \frac{1}{2}\{\tilde{a}\}^T \left[ \sum_{e=1}^{M} \iiint_{V^e} \left[B^s(r,z)\right]^T \left[D^s(r,z)\right]\left[B^s(r,z)\right] dV^s \right] \{\tilde{a}\} - \{\tilde{a}\}^T \left[ \sum_{e=1}^{M} \iiint_{V^e} \left[B^s(r,z)\right]^T \left[D^s(r,z)\right]\{\varepsilon_T\} dV^s - \iint_{S^e} \left[N^s(r,z)\right]^T \{P_i\} dS \right] \tag{9}$$

where $\{\tilde{a}\}$ is the vector of nodal displacements of the entire structure. Now applying the principle of minimum potential energy, desired equilibrium equations of the overall structure can be obtained as

$$\left( \sum_{e=1}^{M} \left[k^s\right]^e \right) \{\tilde{a}\} = \sum_{e=1}^{M} \left( \{F_T\}^e + \{F_P\}^e \right) \tag{10}$$

or

$$\left[K^s\right] \{\tilde{a}\} = \{F_T + F_P\}. \tag{11}$$

Where the characteristic matrices are given in terms of the matrix of coefficients of elasticity in which modulus of elasticity, $E(r, z)$ is determined at each point through the element using distribution function of Eq. (6) and functions of thermal expansion coefficient given in Eq. (7). $F_T$ and $F_p$ are loads vectors caused by temperature and internal pressure. In calculating the characteristic matrices, the integral is taken over the volume of the element. On the element level, the accuracy of this numerical approximation is dependent on the compatibility of the assumed shape functions to the exact displacement field. In fact, if a set of shape functions is chosen which is perfectly compatible with the exact field solution, the finite element results will capture this solution with any level of mesh refinement (Kim and Paulino 2002).

## 3. Optimization problem

### *3.1 Volume fraction interpolation and material distribution in 2D-FGM cylinder*

It is clear that a common power law material distribution could not be the optimum one or proper for a prescribed design objective. In this study the optimization of the volume fractions distribution of each of constituent phases of the 2D FGM, $V_{c1}(r,z)$, $V_{c2}(r,z)$, $V_{m1}(r,z)$ and $V_{m2}(r,z)$ are considered as the optimization of the material distribution of structure. In such an axisymmetric structure the volume fraction distribution can be determined independently in both radial and axial directions. As some researchers mentioned a direct point-wise optimization of the volume fraction at every location will be computationally intractable (Vel and Pelletier 2007, Cho and Ha 2009). Based on this fact, volume fractions of constituent materials on a finite number of design points in the radial and axial directions are used in order to reduce the number of design variables. The volume fractions at any arbitrary point in the cylinder are obtained via cubic spline interpolation function from the volume fractions at the design points. Using such a $C^1$ continuous interpolation function instead of bilinear or linear $C^0$ continuous interpolation functions leads to obtain smoother distributions of volume fractions, material properties in elements and so stress distributions (Vel and Pelletier 2007).

A rectangular grid of nodes used to specify the volume fractions at design points. It is clear that the volume fraction nodes can be independent of the finite element analysis nodes. Because of computational effort, we use volume fraction nodes less than analysis nodes to reduce the number of optimization design variables (Lancaster and Alkauskas 1996). A total number of $(M+1)\times(N+1)$ design points in the radial and axial directions at following locations like what Vel and Pelletier (2007) used only for one direction are utilized as

$$r_m = r_i + \frac{(r_o - r_i)(m-1)}{M} \quad , \qquad m=1, 2, \ldots, M+1$$
$$z_n = \frac{L(n-1)}{N}. \qquad n=1, 2, \ldots, N+1 \qquad (12a\text{-}b)$$

Where $M$ and $N$ are the number of rectangular grid sections. The volume fractions at the design point located at $r_m$ and $z_n$ are denoted by $V_{c1}^{i,j}, V_{m1}^{i,j}, V_{c2}^{i,j}, V_{m2}^{i,j}$ where $i=1,2,\ldots,N+1$ and $j=1,2,\ldots,M+1$.

It should be noted that there is four volume fractions at each design point. So the physical constraints of the problem require that the volume fractions at each design point be in the range of zero to one. Also total summation of constituent volume fractions at all points should be equal to unity. In order to have feasible volume fractions the inequality side constraints of the problem are

$$\begin{aligned} &0 \le V_{c1}^{i,j} \le 1, \\ &0 \le V_{m1}^{i,j} \le 1, \\ &0 \le V_{c2}^{i,j} \le 1, \\ &0 \le V_{m2}^{i,j} \le 1. \end{aligned} \qquad (13)$$

And the equality linear constraints are

$$V_{c1}^{i,j} + V_{m1}^{i,j} + V_{c2}^{i,j} + V_{m2}^{i,j} = 1 \qquad (14)$$

for all design points.

In order to have feasible volume fractions in all interpolated points of the domain and also to have smooth stress distributions the range-restricted bicubic interpolation technique is used, which is based on sufficiency conditions for univariate piecewise bicubic interpolation to preserve positivity (Brodlie *et al*. 2005, Goupee and Vel 2006). A piecewise bicubic representation of the volume fraction is to be formed over a rectangular grid defined by

$$D = [r_1\ r_2 \cdots\ r_m\ ] \times [z_1\ z_2 \cdots\ z_n\ ]. \qquad (15)$$

The volume fraction at the grid points ($r_m$, $z_n$) will be denoted as $V^{(i,j)}$. The interpolated volume fraction in the domain is

$$V(r,z) = \sum_{p=0}^{1}\sum_{q=0}^{1}\begin{bmatrix} V^{(i+p,j+q)}H_{1+p}^{i}(r)H_{1+q}^{j}(z) + V_{,r}^{(i+p,j+q)}H_{3+p}^{i}(r)H_{1+q}^{j}(z) \\ +V_{,z}^{(i+p,j+q)}H_{1+p}^{i}(r)H_{3+q}^{j}(z) + V_{,rz}^{(i+p,j+q)}H_{3+p}^{i}(r)H_{3+q}^{j}(z) \end{bmatrix} \qquad (16)$$

where $V_{,r}$ represents the partial derivative of the volume fraction profile at the grid points with respect to the radial direction, $V_{,z}$ indicates the partial derivative with respect to the axial direction and $V_{,rz}$ is the mixed partial derivative. The functions $H$ in (16) are the Hermite basis functions as follow (Goupee and Vel 2006)

$$H_1^i(r) = \frac{2}{(f^i)^3}\left[r - r_i + \frac{f_k^i}{2}\right](r - r_{i+1})^2,$$

$$H_2^i(r) = -\frac{2}{(f^i)^3}\left[r - r_{i+1} + \frac{f_k^i}{2}\right](r - r_i)^2,$$

$$H_3^i(r) = \frac{2}{(f^i)^2}(r - r_{i+1})^2(r - r_i),$$

$$H_4^i(r) = \frac{2}{(f^i)^2}(r - r_i)^2(r - r_{i+1}), \tag{17a-d}$$

where $f^i = r_{i+1} - r_i$.

The Hermite basis functions for axial direction is also the same. In order to approximate the slopes and partial derivatives at volume fraction nodes a numerical three point difference method is used (Goupee and Vel 2006).

### *3.2 Defining standard optimization problem*

On the next step we should define the objective function. From the plasticity and fracture-mechanics point of view, plastification and microcracking are fundamentally affected by the magnitude of the effective stress (Cho and Ha 2009). Therefore, the peak local effective stress can be the objective function as it has been commonly selected. The von Mises effective stress in this case is

$$\sigma_{ef} = \sqrt{\frac{3\boldsymbol{\sigma}:\boldsymbol{\sigma}}{2}} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_{rr} - \sigma_{\theta\theta})^2 + (\sigma_{zz} - \sigma_{\theta\theta})^2 + (\sigma_{rr} - \sigma_{zz})^2 + 6(\sigma_{rz}{}^2)}. \tag{18}$$

But, as the yield stress of FGM is also a function of the volume fraction varying spatially with the volume fraction distribution, yielding would not necessarily occur at a point with higher peak effective stress level (Cho and Choi 2004). Thus, the normalized peak effective stress $\left(\frac{\sigma_{ef}}{\sigma_y}\right)$ becomes a more suitable criterion to optimize material tailoring for better strength of FGMs under thermo-mechanical loadings. So, the modified design criterion used in some recent cases (Cho and Choi 2004) is

$$\left\|\frac{\sigma_{ef}}{\sigma_y}\right\| \leq 1. \tag{19}$$

On the other hand, the uniformity of stress distribution in structural design is one of the engineers concerns. It is clear that having a structure with simultaneous yielding at all points is an optimum use of material without any over design. Based on this fact the objective function in this study selected as having the normalized effective stress equal to one at all points. On the other words the design objective is to have safety factor of unity in all points of the structure. So the modified objective function will be

$$F(X) = F(V^{i,j}) = \left\|1 - \frac{\sigma_{ef}(r,z)}{\sigma_y(r,z)}\right\|. \tag{20}$$

Of course, in this situation the peak effective stress should not exceed to the yield stress at none of points. So the following constraints will be added as

$$max\{\sigma_{ef}(r,z)\} \leq \sigma_y(r,z). \tag{21}$$

The yield stress in FGMs is also a function of the volume fraction and can be evaluated by rule of mixtures (Cho and Ha 2011).

The constrained volume-fraction optimization together with the finite element governing equations is formulated as follows

$$\textit{Find} \quad \boldsymbol{X} = \{x_k\} = \{V_{c1}^{i,j}, V_{m1}^{i,j}, V_{c2}^{i,j}, V_{m2}^{i,j}\} \quad k=1,\ldots, 4\times(M+1)\times(N+1) \tag{22}$$

$$\textit{Minimize} \quad F(\boldsymbol{X}) = F(V^{i,j}) = \left\| 1 - \frac{\sigma_{ef}(r,z)}{\sigma_y(r,z)} \right\| \tag{23}$$

$$\textit{Subjected to} \quad -x_k \leq 0$$

$$x_k - 1 \leq 0 \quad k=1,\ldots, 4\times(M+1)\times(N+1)$$

$$\sum_{q=1}^{4} x_{4p-q+1} - 1 = 0 \quad p=1,2,\ldots,ND$$

$$\max\{\sigma_{ef}(r,z)\} - \sigma_y(r,z) \leq 0$$

$$[K^s]\{\tilde{a}\} = \{F_T + F_P\} \tag{24}$$

where $ND=(M+1)\times(N+1)$ is the number of design points.

Minimizing the peak effective stress also is considered as objective function and the solutions will be compared.

In order to solve the above constrained optimization problem, we employ the interior penalty-function method. This method transforms the basic optimization problem into alternative formulations such that numerical solutions are sought by solving a sequence of unconstrained minimization problems. In the interior penalty function methods, a new pseudo objective ($\varphi$) is constructed by augmenting a penalty term to the objective function as (Rao 2009)

$$\varphi(\boldsymbol{X}, p_k) = F(\boldsymbol{X}) + p_k \sum_{j=1}^{ND} \left(\frac{-1}{g_j(\boldsymbol{X})}\right), \tag{25}$$

where $g_j$ are all constraints other than side conditions, and $p_k$ is a positive constant known as the *penalty parameter*. The penalty term is chosen such that its value will be small at points away from the constraint boundaries and will tend to infinity as the constraint boundaries are approached. Hence the value of the $\varphi$ function also blows up as the constraint boundaries are approached. Minimization of $\varphi(X, pk)$ is started from any feasible point $X_1$, the subsequent points generated will always lie within the feasible domain since the constraint boundaries act as barriers during the minimization process.

### 3.3 GA optimization algorithm

Classical gradient-based optimization algorithms often are not efficient for solving complex problems since they tend to find local minima instead of global solution. One of the most efficient modern optimization methods for finding the global minima is Genetic Algorithms (GA) belongs to the class of stochastic search optimization methods. The algorithm is very general and can be applied to all kinds of complicated problems like here. In addition, the methods determine global optimum solutions as opposed to the local solutions determined by a continuous variable optimization algorithm (Arora 2004).

A GA begins its search with a population of random individuals (volume fraction at each node here) and analyzes each individual of the population of designs and assigns it a fitness value related to objective function. The population of solutions is modified to a new population by applying three operators similar to natural genetic operators as reproduction, crossover and mutation. The main GA operations used in this study are as follow (Sivanandam and Deepa 2008):

*Reproduction* is the process of choosing two parents from the population for crossing. The tournament selection strategy provides selective pressure by holding a tournament competition among individuals. The best individual from the tournament is the one with the highest fitness, which is the winner of tournament competitions and the winner are then selected for reproduction. The fitness difference provides the selection pressure, which drives GA to improve the fitness of the succeeding individuals.

*Uniform Crossover* is the process of taking two parent solutions and producing from them a child. After the selection process, the population is enriched with better individuals. Selection makes clones of good strings but does not create new ones. Crossover operator is applied to the selected individual with the hope that it creates a better offspring. In Uniform crossover each gene (volume fraction at each node) in the offspring (a new design) is created by copying the corresponding gene from one of the other parent chosen according to a random generated binary crossover mask of the same length. A new crossover mask is randomly generated for each pair of parents. Offsprings therefore contain a mixture of genes from each parent.

Mutation prevents the algorithm to be trapped in a local minimum. Mutation plays the role of recovering the lost genetic materials as well as for randomly disturbing genetic information. It is an insurance policy against the irreversible loss of genetic material.

The genetic algorithm stops when one of following Convergence Criteria conditions occurs.

*Maximum generations*: the specified number of generation. No change in fitness: there is no change to the population's best fitness for a specified number of generations. Stall generations: there is no improvement in the objective function for a sequence of consecutive generations.

## 4. Numerical results and discussion

Implementation of graded finite element method for solving the governing equations of thermomechanical loading of 2D-FGM finite length cylinder has been verified in previous author's publication (Asgari and Akhlaghi 2011). A thick hollow cylinder of inner radius $r_i$=1 m, outer radius $r_o$=1.5 m, and length $L$=5 m made of FGM with two–dimensional gradation of distribution is considered. Constituent materials are two distinct ceramics and two distinct metals denoted in Table 1.

Table 1 Basic constituents of the 2D-FGM cylinder

| Constituents | Material | $E$ (Gpa) | $K$ (kg/m$^3$) | Yielding stress (MPa) | $\alpha$ $10^{-6}$/°$K$ |
|---|---|---|---|---|---|
| $m1$ | Ti6Al4V | 115 | 6 | 850 | 23 |
| $m2$ | Al 1100 | 69 | 220 | 105 | 8 |
| $c1$ | SiC | 440 | 100 | 500 | 4.3 |
| $c2$ | $SiO_2$ | 150 | 13 | 120 | 3 |

Finding of the volume fractions distribution of each of constituent phases of the 2D FGM, $V_{c1}(r,z)$, $V_{c2}(r,z)$, $V_{m1}(r,z)$ and $V_{m2}(r,z)$ are considered as the optimization of the material distribution of structure in order to finding the solution of problem described in Eqs. (22)-(24).

In the considered cylinder thermal boundary conditions temperature at inner surface applied as

$$at \quad r = r_i \quad T_i(z) = 200\sin(\frac{\pi z}{L}) \tag{26}$$

also convection coefficients and ambient temperature at other surfaces are $h_o=h_b=h_u=100$ $W/m^2K$ and $T_\infty=25$°C.

In addition to the thermal loading and boundary conditions an internal pressure exerts as

$$P_i(z) = 10^8 \sin(\frac{\pi z}{L}) \tag{27}$$

which is in ($Pa$) and the external pressure is zero. So the natural boundary conditions are assumed as

$$\sigma_{rr}(r_i, z) = P_i(z)$$

$$\sigma_{rr}(r_o, z) = \tau_{rz}(r_i, z) = \tau_{rz}(r_o, z) = 0\,. \tag{28 a-b}$$

Assuming clamped end supports, boundary conditions are

$$u(r,0) = u(r, L) = 0$$

$$w(r,0) = w(r, L) = 0\,. \tag{29 a-b}$$

A MATLAB (MathWorks 2012) code is developed in order to solve the governing equations as the optimization problem and implementing GA method. Firstly a cylinder with variation of constituent volume fractions and material properties in two directions based on power law function profile without any optimization is considered. The volume fraction of the first ceramic material is changed from 100% at the lower surface to zero at the upper surface by a power law function. This volume fraction is also changed continuously from inner surface to the outer surface. The volume fractions of the other materials change similar to the mentioned one in two directions. The function of volume fraction distribution of each material can be explained as

$$V_{c1}(r,z) = \left[1 - (\frac{r - r_i}{r_0 - r_i})^{n_r}\right]\left[1 - (\frac{z}{L})^{n_z}\right], \qquad V_{c2}(r,z) = \left[1 - (\frac{r - r_i}{r_0 - r_i})^{n_r}\right]\left[(\frac{z}{L})^{n_z}\right]$$

$$V_{m1}(r,z) = (\frac{r - r_i}{r_0 - r_i})^{n_r}\left[1 - (\frac{z}{L})^{n_z}\right], \qquad V_{m2}(r,z) = (\frac{r - r_i}{r_0 - r_i})^{n_r}(\frac{z}{L})^{n_z} \tag{30a-d}$$

where $n_r$ and $n_z$ are non-zero parameters that represent the basic constituent distributions in $r$ and $z$-directions. For instant, the volume fraction distribution of one basic material for the typical values of $n_r$=2 and $n_z$=2 is shown in the Fig. 2.

Fig. 3 shows the effective stress distribution through the cylinder for the power law exponents of material distribution profiles as $n_r$=2 and $n_z$=2 in radial and axial directions. The maximum value of effective stress is 379.5 Mpa which occurs at inner surface.

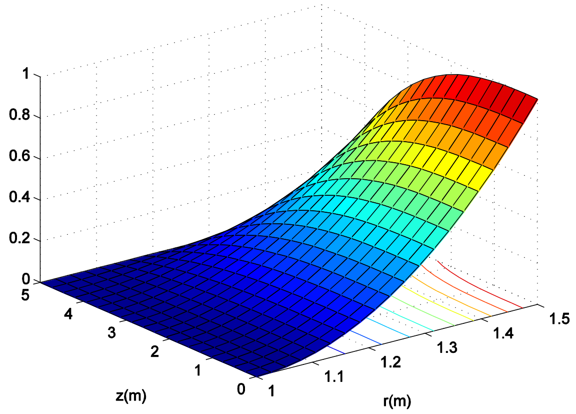



Fig. 2 Variation of conduction coefficient through the cylinder with power law exponents $n_r$=2 and $n_z$=2

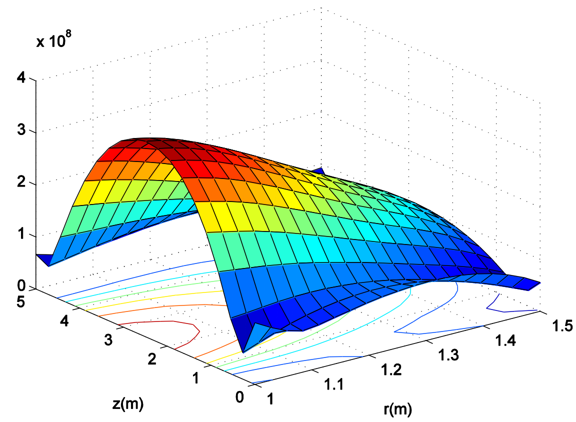



Fig. 3 Effective stress distribution through the cylinder with power law distribution profile $n_r$=2, $n_z$=2

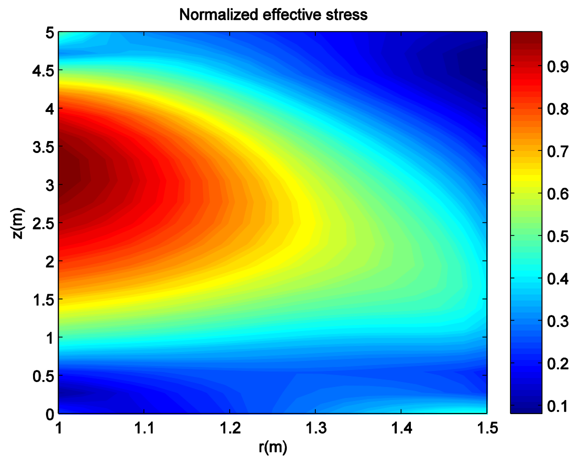



Fig. 4 distribution of normalized effective stress through the cylinder with power law distribution profile $n_r$=2, $n_z$=2

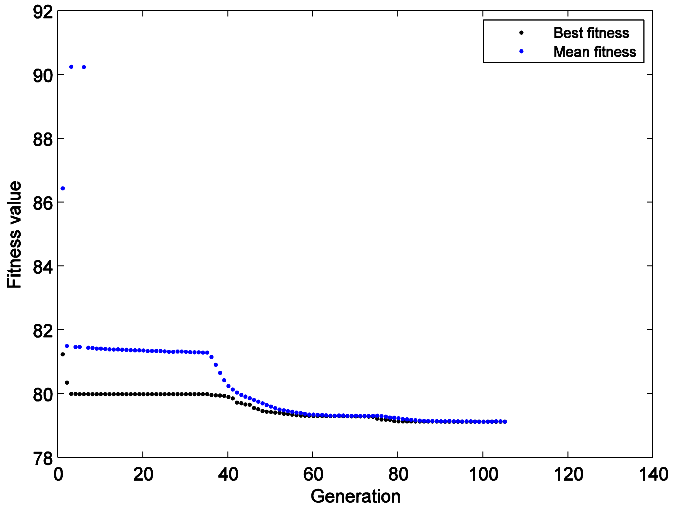



Fig. 5 Fitness value of populations during GA generations using normalized effective stress as objective function

Distribution of the normalized effective stress is also shown in Fig. 4. It is clear that the range of variation of normalized effective stress is extended from less than 0.1 up to more than 0.9. The normalized effective stress is the inverse of safety factor in the structure. Having safety factor from 1 up to 10 with non-uniform distribution is not desired at all in an optimum design because of ineffective material usage.

For optimization, the number of volume fraction design point is chosen to be 10×10 and the GA population size is 1000. Crossover fraction of 0.8 and probability of mutation of 0.1 are used as GA parameters. In each case some different runs were performed with random initial population to ensure repeatability of the final results.

The fitness value of the best individual and the average fitness of each population during the generations for the case of uniformly distributed normalized effective stress used as objective function are depicted in Fig. 5. The GA optimization process converges in 114 generations.

The contour plots of optimized volume fractions distribution of constituent materials obtained by uniformly distributed normalized effective stress objective function are illustrated in Fig. 6.

Contour plot of corresponding normalized effective stress distribution of the cylinder with optimized volume fractions is depicted in Fig. 7. It is clear compared to the initial case shown in Fig. 4 that the uniformity of safety factor distribution through the cylinder is considerably increased. On the other hand the numerical value of the safety factor varies from 0.5 up to 0.97. In other word the safety factor is between 1 up to 2 while at the most points of the cylinder it is about 0.9. It is equivalent to a safety factor near 1.1 through the structure. It means a much better material tailoring obtained.

Surface plot of the absolute effective stress distribution in this case can be seen in Fig. 8. The effective stress reaches the peak value 363.7 MPa for the optimized volume fraction distribution at the middle of the cylinder radius and length.

Material properties distribution through the cylinder with the optimized volume fraction based on uniform safety factor design criteria are depicted in Fig. 9.

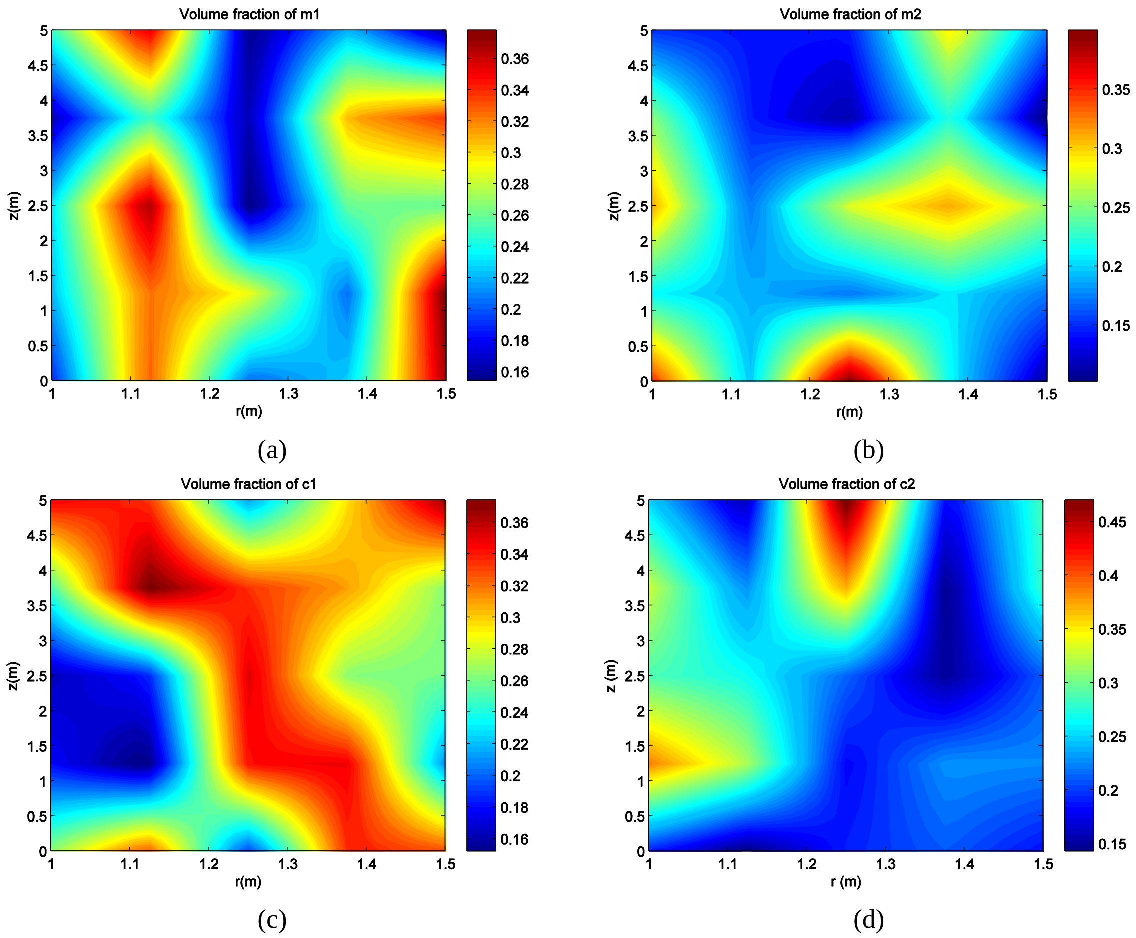



Fig. 6 Volume fraction distribution material constituent: (a) $m1$ (b) $m2$ (c) $c1$ (d) $c2$ using normalized effective stress as objective function

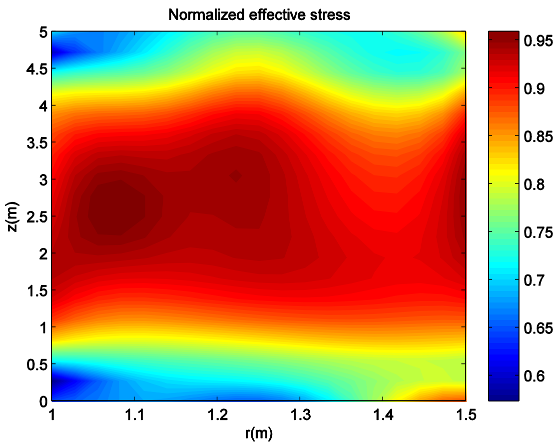



Fig. 7 distribution of normalized effective stress using uniform distribution of normalized effective stress as objective function

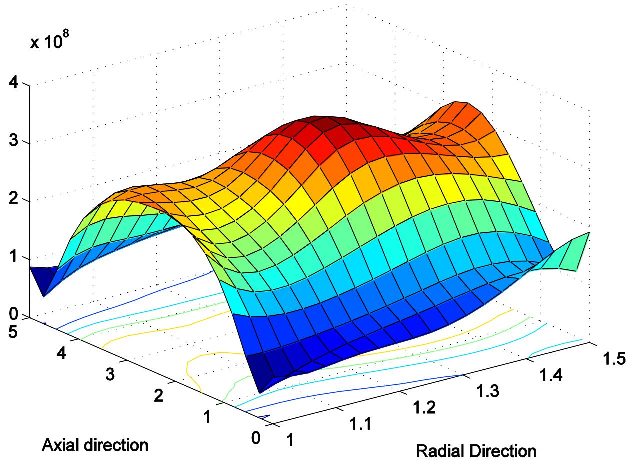
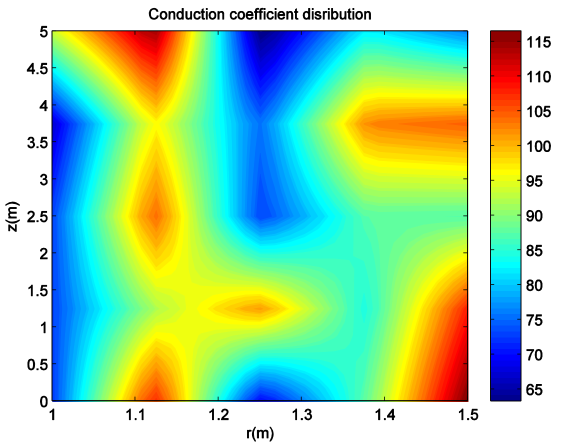
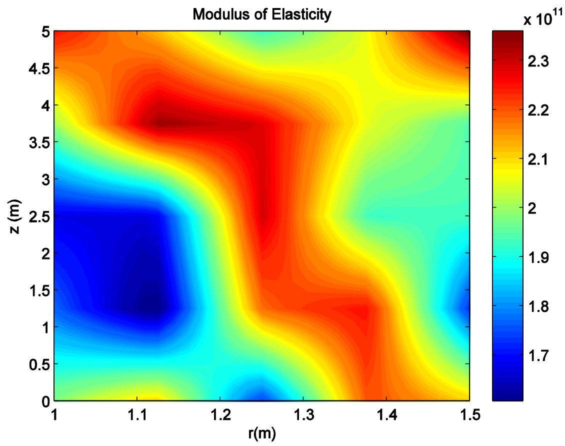



Fig. 8 Effective stress distribution using uniform distribution of normalized effective stress as objective function




(a)




(b)

Fig. 9 Material properties distribution using normalized effective stress as objective function (a) conduction coefficient (b) modulus of elasticity

In order to examine optimization results to the choice of the objective function, we carried out the volume fraction optimization by using the peak effective stress design criterion. While, the other problem conditions are kept unchanged. The fitness value of the best individual and the average fitness of each population during the generations for the case of peak effective stress is directly used as objective function are depicted in Fig. 10.

The consequent optimum volume fraction depicted in Fig. 11 shows the considerably different distribution.

Distribution of the absolute effective stress after the optimization with peak stress value objective function is shown in Fig. 12. One can observe the reduction compared to the previous distributions shown in Fig. 4 and Fig. 7. The location of high peaks also differs from the previous

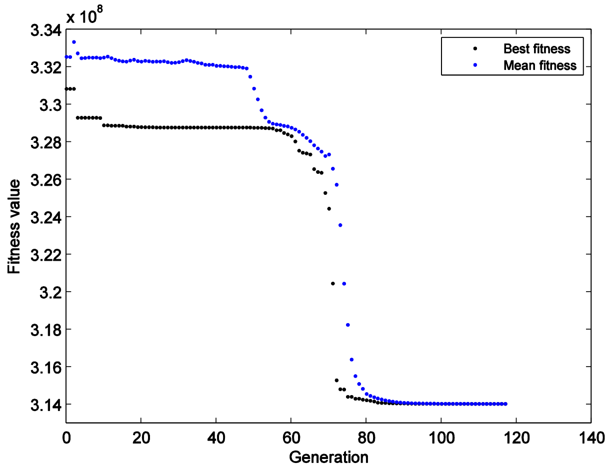



Fig. 10 Fitness value of populations during GA generations using peak effective stress as objective function

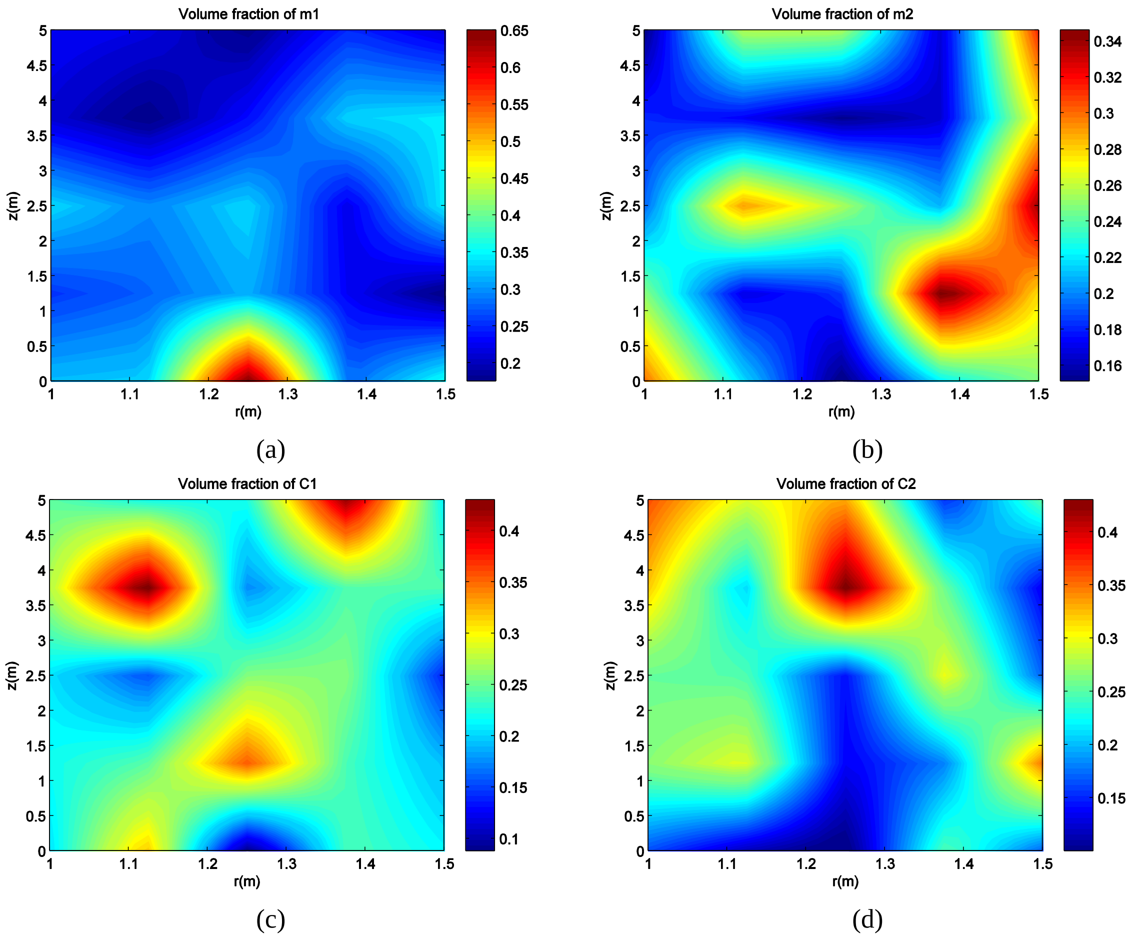



Fig. 11 Volume fraction distribution material constituent: (a) $m1$ *(b)* $m2$ *(c)* $c1$ *(d)* $c2$ using peak effective stress as objective function

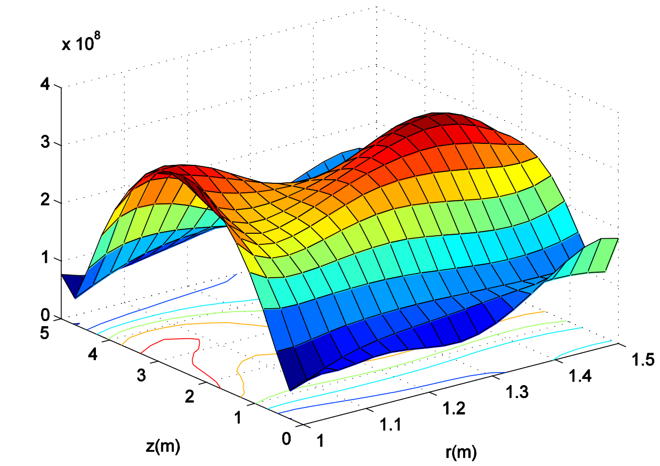



Fig. 12 Effective stress distribution using peak effective stress as objective function

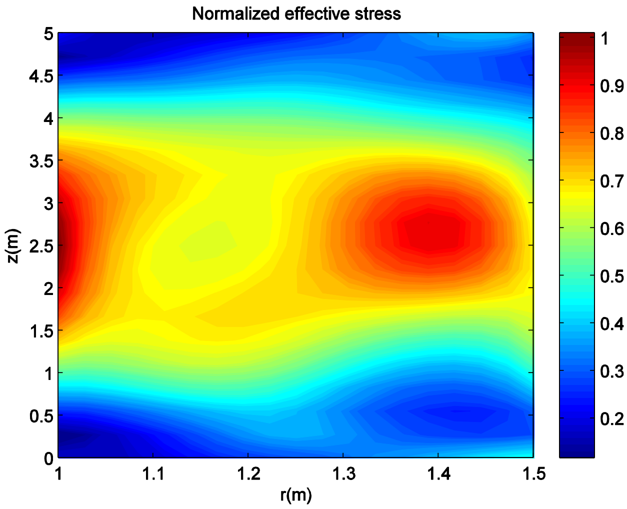



Fig. 13 Distribution of normalized effective stress using peak effective stress as objective function

cases. In the current case the peak stress occurs near two regions at the inner and outer surface at middle of cylinder length. Its magnitude is 288.2 MPa which is 25% less than 2D FGM cylinder with power law volume fraction profile. A reduction more than 20% is also achieved compared with optimized case with safety factor design criteria.

The normalized effective stress distribution of the structure is shown in Fig. 13. As shown in this figure, distribution of the normalized stress is also different from that of the uniform safety factor objective function. When compared to the previous case shown in Fig. 7, the normalized effective stress is varying in an extended range from less than 0.2 up to 1. In other words we have the safety factor of 1 up to 5 through the structure which is not optimum usage of materials.

Distribution of modulus of elasticity through the cylinder with the optimized volume fraction based on peak effective stress design criteria are depicted in Fig. 14 for instance.

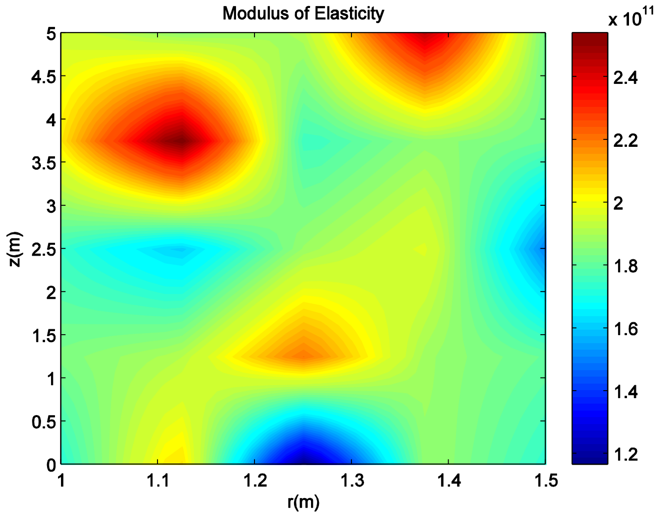



Fig. 14 Distribution of modulus of elasticity using peak effective stress as objective function

Based on the obtained results, by using the uniform distribution of normalized effective stress as objective function, considerably more efficient usage of materials can be achieved compared with the power law volume fraction distribution. It should be noted that although the manufacturing of multidimensional FGM may seem to be costly or difficult, but these material compositions can be manufactured by some existing manufacturing techniques such as the computer-controlled thermal spray method. While these technologies are relatively new, processes such as three-dimensional printing (3DP™) and Laser Engineering Net Shaping (LENS®) can also currently produce FGMs with relatively arbitrary tree-dimensional grading (Goupee and Vel 2006). With further refinement FGM manufacturing methods may provide the designers with more control of the composition profile of functionally graded components with reasonable cost.

## 5. Conclusions

Finding optimum material distribution of a 2D functionally graded cylinder with finite length under thermal and mechanical loadings has been considered. For modeling and simulation of governing equations graded finite element method was used that has some advantages to conventional finite element method. Volume fractions of constituent materials on a finite number of design points are taken as design variables and the volume fractions at any arbitrary location in the cylinder are obtained via cubic spline interpolation functions. Having a uniform stress distribution in structure and minimizing peak effective stress selected as two objective functions and results have been compared with a common FGM with power law distribution profile. Genetic Algorithm jointed with interior penalty-function method for implementing constraints is effectively employed to find the global solution of the optimization problem. Obtained results indicates that by using the uniform distribution of normalized effective stress as objective function, considerably more efficient usage of materials can be achieved compared with the power law volume fraction distribution. Also considering uniform distribution of safety factor as design criteria instead of minimizing peak effective stress affects remarkably the optimum volume fractions. Based on our results, the proposed methodology provides a framework for designing functionally graded structures with optimum material tailoring.

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