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Wave propagation in fibre-reinforced anisotropic thermoelastic medium subjected to gravity field

A.M. Abd-Alla[*1,3], S.M. Abo-Dahab[1,2a] and F.S. Bayones[1b]

[1] *Mathematics Department, Faculty of Science, Taif University, Saudi Arabia*
[2] *Mathematics Department, Faculty of Science, Qena, Egypt, Qena 83523, Egypt*
[3] *Mathematics Department, Faculty of Science, Sohag University, Egypt*

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Abstract. The objective of this paper is to investigate the surface waves in fibre-reinforced anisotropic thermoelastic medium subjected to gravity field. The theory of generalized surface waves has firstly developed and then it has been employed to investigate particular cases of waves, viz., Stoneley waves, Rayleigh waves and Love waves. The analytical expressions for displacement components, force stress and temperature distribution are obtained in the physical domain by using the harmonic vibrations. The wave velocity equations have been obtained in different cases. The numerical results are given and presented graphically in Green-Lindsay and Lord-Shulman theory of thermoelasticity. Comparison was made with the results obtained in the presence and absence of gravity, anisotropy, relaxation times and parameters for fibre-reinforced of the material medium. The results indicate that the effect of gravity, anisotropy, relaxation times and parameters for fibre-reinforced of the material medium are very pronounced.

Keywords: wave propagation; fibre-reinforced medium; thermal stresses; thermo-elastic medium; gravity field

1. Introduction

The dynamical problem of propagation of surface waves in a homogeneous and non-homogeneous elastic and thermoplastic media are of considerable importance in earthquake, engineering and seismology on account of the occurrence of non-homogeneities in the earth's crust, as the earth is made up of different layers. Abd-Alla *et al*. (2011) investigated the propagation of Rayleigh waves in generalized magneto-thermoelastic orthotropic material under initial stress and gravity field. Stoneley and Rayleigh waves in a non-homogeneous orthotropic elastic medium under the influence of gravity has been investigated by Abd-Alla and Ahmed (2003). Abd-Alla (1999) studied the propagation of Rayleigh waves in an elastic half-space of orthotropic material. Abd-Alla and Ahmed (1999) investigated propagation of Love waves in a non-homogeneous orthotropic elastic layer under initial stress overlying semi-infinite medium. Rayleigh waves in a magnetoelastic half-space of orthotropic material under the influence of initial

∗Corresponding author, Professor, E-mail: mohmrr@yahoo.com
[a] Ph.D.,E-mail: sdahb@yhoo.com
[b] Ph.D.,E-mail: f.s.bayones@hotmail.com

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stress and gravity field investigated by Abd-Alla, *et al*. (2004). Elnaggar and Abd-Alla (1989) studied Rayleigh waves in magneto-thermo-microelastic half-space under initial stress. Abd-Alla and Ahmed (1996) discussed Rayleigh waves in an orthotropic thermoelastic medium under gravity field and initial stress. Propagation of Rayleigh waves in a rotating orthotropic material elastic half-space under initial stress and gravity investigated by Abd-Alla *et al*. (2012). Wu and Chai (1994) studied the propagation of surface waves in anisotropic solids: theoretical calculation and experiment. Wu and Liu (1999) investigated the measurement of anisotropic elastic constants of fiber-reinforced composite plate using ultrasonic bulk wave and laser generated Lamb wave. The group velocity variation of Lamb wave in fiber reinforced composite plate studied by Rhee *et al*. (2007). Fu and Zhang (2006) investigated the continuum-mechanical modelling of kink-band formation in fibre-reinforced composites. Espinosa *et al*. (2000) discussed the modeling impact induced delamination of woven fiber reinforced composites with contact/cohesive laws. Wave propagation in materials reinforced with bi-directional fibers presented by Weitsman and Benveniste (1974). Weitsman (1972) introduced the wave propagation and energy scattering in materials reinforced by inextensible fibers. Dai and Wang (2006) considered the stress wave propagation in piezoelectric fiber reinforced laminated composites subjected to thermal shock. Ohyoshi (2000) studied the propagation of Rayleigh waves along an obliquely cut surface in a directional fiber-reinforced composite. Rogerson (1992) investigated the Penetration of impact waves in a six-ply fibre composite laminate. Weitsrian (1992) studied the reflection of harmonic waves in fiber-reinforced materials. Huang, *et al*. (1995) investigated the effect of fibre-matrix interphase on wave propagation along, and scattering from, multilayered fibres in composites. Transfer matrix approach. Fu and Zhang (2006) discussed the continuum-mechanical modelling of kink-band formation in fibre-reinforced composites. Singh and Singh (2004) investigated the reflection of plane waves at the free surface of a fibre-reinforced elastic half-space. Sengupta and Nath (2001) studied the surface waves in fibre-reinforced anisotropic elastic media. Samal and Chattaraj (2011) studied the surface wave propagation in fiber-reinforced anisotropic elastic layer between liquid saturated porous half space and uniform liquid layer. Abd-Alla and Bayones (2011) studied the effect of rotation and initial stress on generalized thermoelastic problem in an infinite circular cylinder. Abd-Alla *et al*. (2011) investigated the effect of rotation and magnetic field on generalized thermo-viscoelastic in an infinite circular cylinder. Chattopadhyay (2002) investigated the reflection of quasi-P and quasi-SV waves at the free and rigid boundaries of a fibre-reinforced medium. Singh (2007) discussed the wave propagation in an incompressible transversely isotropic fibre-reinforced elastic media. Singh (2005) studied the wave propagation in thermally conducting linear fibre-reinforced composite materials. Abd-Alla *et al*. (2000) studied the thermal stresses in a non-homogeneous orthotropic elastic multilayered cylinder.

Recently, Abd-Alla and Abo-Dahab (2012) investigated the rotation and initial stress effects on an infinite generalized magneto-thermoplastic diffusing body with a spherical cavity. Abouelregal and Abo-Dahab (2012) discusses the dual phase lag model on magneto-thermoelasticity infinite non-homogeneous solid having a spherical cavity.

The aim of this paper is to study the propagation of surface waves in fibre-reinforced anisotropic thermoelastic medium subjected to gravity field leading to particular cases such as Rayleigh waves, Love waves and Stoneley waves. The temperature, displacement and stress are obtained in the physical domain by using the harmonic vibrations.The effects of the gravity, relaxation time, anisotropy and parameters for fibre-reinforced of the material medium on surface waves are studied simultaneously. The numerical result displayed by figures and the physical meaning are explained.The results and discussions presented in this study may be helpful to further understand fibre-reinforced anisotropic thermoelastic medium.

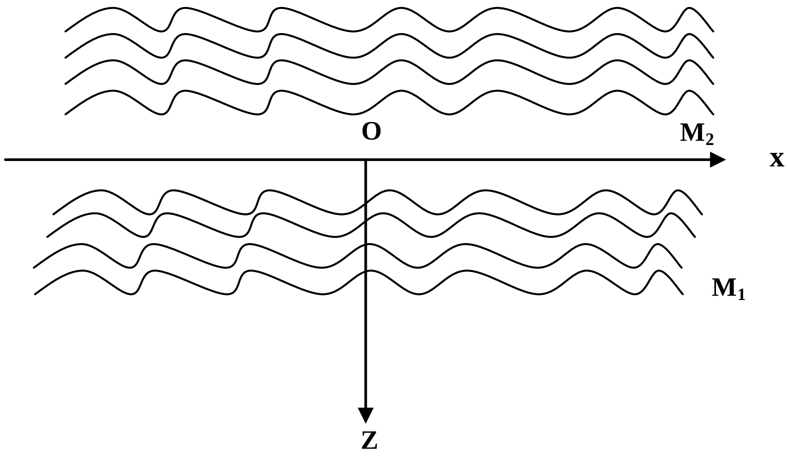


Fig. 1 Schem atic of the problem

2. Formulation of the problem

Let us consider a system of anisotropic Cartesian axes *Oxyz*, M_1 and M_2 be two fibre-reinforced elastic anisotropic semi-infinite solid media. Let *O* be the any point of the plane boundary and *Oz* points vertically downward to the medium M_1. They are perfectly welded in contact to prevent any relative motion or sliding before and after the disturbances and that the continuity of displacement, stress etc. hold good across the common boundary surface. Further the mechanical properties of M_1 being different from those of M_2. These media extend to an infinite great distance from the origin and are separated by a plane horizontal boundary and M_2 is to be taken above M_1. We consider the possibility of a type of wave travelling in the direction *Ox,* in such a manner that the disturbance is largely confined to the neighborhood of the boundary and at any instant, all particles in any line parallel to *y*-axis have equal displacements. These two assumptions conclude that the wave is a surface wave and all partial derivatives with respect to *y* are a zero. Further, let us assume that *u*,*v*,*w* are the components of displacements at any point (*x*,*y*,*z*) at any time *t*. It is also assumed that gravitational field produces a hydrostatic initial stress is produced by a slow process of creep where the shearing stresses tend to become smaller or vanish after a long period of time as shown in Fig. 1.

The equilibrium conditions of the initial stress field (Ohyoshi 2000)

$$\frac{\partial \tau}{\partial x} = 0, \quad \frac{\partial \tau}{\partial z} + \rho g = 0. \tag{1}$$

The stress-temperature equation is given by Rogerson (1992)

$$K\, T_{ij} = \rho\ c_v \frac{\partial}{\partial t}\left(1 + \tau_2 \frac{\partial}{\partial t}\right)\ T + \gamma T_0 \frac{\partial}{\partial t}\left(1 + \tau_2 \delta \frac{\partial}{\partial t}\right) \underline{\nabla}.\ \vec{u} \quad ,i,\, j = 1,2,3 \cdot \tag{2}$$

The dynamical equations of motion for three dimensional elastic solid medium under the influence of initial stress and gravity (Weitsrian 1992)

$$\frac{\partial\tau_{11}}{\partial x}+\frac{\partial\tau_{12}}{\partial y}+\frac{\partial\tau_{13}}{\partial z}+\rho g\frac{\partial w}{\partial x}=\rho\frac{\partial^2 u}{\partial t^2}, \tag{3}$$

$$\frac{\partial\tau_{12}}{\partial x}+\frac{\partial\tau_{22}}{\partial y}+\frac{\partial\tau_{23}}{\partial z}+\rho g\frac{\partial w}{\partial y}=\rho\frac{\partial^2 v}{\partial t^2}, \tag{4}$$

$$\frac{\partial\tau_{13}}{\partial x}+\frac{\partial\tau_{23}}{\partial y}+\frac{\partial\tau_{33}}{\partial z}-\rho g\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=\rho\frac{\partial^2 w}{\partial t^2}. \tag{5}$$

The constitutive equations for a fibre-reinforced linearly thermoelastic anisotropic medium with respect to a preferred direction $\vec{a}$ are Huang *et al.* (1995)

$$\begin{aligned}\tau_{ij} &= \lambda e_{kk}\delta_{ij}+2\mu_T e_{ij}+\alpha(a_k a_m e_{km}\delta_{ij}+e_{kk}a_i a_j)\\ &\quad +2(\mu_L-\mu_T)\ (a_i a_k e_{kj}+a_j a_k e_{ki})+\beta\ (a_k a_m e_{km}a_i a_j)-\gamma(T+\tau_1\dot{T})\ \delta_{ij}\end{aligned} \tag{6}$$

where, τ is a function of depth, ρ is density of the material, K is thermal conductivity, c_v is specific heat of the material per unit mass, τ_1, τ_2 are the thermal relaxation parameter, $\gamma=\alpha_t(3\lambda+2\mu_T)$, α_t is the coefficient of linear thermal expansion, λ, μ_T are elastic parameters, θ is the absolute temperature, T_0 is the reference temperature solid, T is the temperature difference $(\theta-T_0)$, g be the acceleration due to gravity and $\tau_{ij}=\tau_{ji}$, $\forall\ i,j$ are the stress components, $e_{ij}=1/2(u_{i,j}+u_{j,i})$ are components of strain; α, β, $(\mu_L-\mu_T)$ are reinforced anisotropic elastic parameters; $\vec{a}=(a_1, a_2, a_3)$, $a_1^2+a_2^2+a_3^2=1$. If $\vec{a}$ has components that are (1, 0, 0) so that the preferred direction is the x-axis, (6) can be written as follows

$$\begin{aligned}\tau_{11} &= (\lambda+2\alpha+4\mu_L-2\mu_T+\beta)\ e_{11}+(\lambda+\alpha)\ e_{22}+(\lambda+\alpha)\ e_{33}-\gamma(T+\tau_1\dot{T}),\\ \tau_{22} &= (\lambda+\alpha)\ e_{11}+(\lambda+2\mu_T)\ e_{22}+\lambda e_{33}-\gamma(T-\tau_1\dot{T}),\\ \tau_{33} &= (\lambda+\alpha)\ e_{11}+\lambda e_{22}+(\lambda+2\mu_T)\ e_{33}-\gamma(T-\tau_1\dot{T}),\\ \tau_{23} &= 2\mu_T e_{23},\ \ \tau_{13}=2\mu_L e_{13},\ \ \tau_{12}=2\mu_L e_{12}.\end{aligned} \tag{7}$$

Introducing Eq. (7) into Eqs. (3)-(5), we have

$$(\lambda+2\alpha+4\mu_L-2\mu_T+\beta)\frac{\partial^2 u}{\partial x^2}+(\lambda+\alpha+\mu_L)\frac{\partial^2 w}{\partial x\partial z}+\mu_L\frac{\partial^2 u}{\partial z^2}+\rho g\frac{\partial w}{\partial x}-\gamma(1+\tau_1\frac{\partial}{\partial t})\frac{\partial T}{\partial x}=\rho\frac{\partial^2 u}{\partial t^2}, \tag{8}$$

$$(\mu_L-\mu_T)\frac{\partial^2 v}{\partial x^2}+\mu_T\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)v=\rho\frac{\partial^2 v}{\partial t^2}, \tag{9}$$

$$\mu_L\frac{\partial^2 w}{\partial x^2}+(\lambda+\alpha+\mu_L)\frac{\partial^2 u}{\partial x\partial z}+(\lambda+2\mu_T)\frac{\partial^2 w}{\partial z^2}-\rho g\frac{\partial u}{\partial x}-\gamma(1+\tau_1\frac{\partial}{\partial t})\frac{\partial T}{\partial z}=\rho\frac{\partial^2 w}{\partial t^2}. \tag{10}$$

By Helmholtz's theorem (Fu and Zhang 2006), the displacement vector $\vec{u}$ can be written in the displacement potentials ϕ and ψ form, as

$$\vec{u}=grad\,\phi+curl\vec{\psi},\ \ \vec{\psi}=(0,\psi,0) \tag{11}$$

which reduces to

$$u = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial z}, \quad w = \frac{\partial \phi}{\partial z} + \frac{\partial \psi}{\partial x}. \tag{12}$$

The component v is associated with purely distortional movement. We note that ϕ, ψ and v are respectively associated with P waves. SV waves and SH waves. The symbols have their usual significances.

Now using Eq. (12) in Eqs. (8)-(9) we obtain the following wave equation in M_1 satisfied by ϕ and ψ as

$$(\lambda + 2\alpha + 4\mu_L - 2\mu_T + \beta)\frac{\partial^2 \phi}{\partial x^2} + (\lambda + \alpha + 2\mu_L)\frac{\partial^2 \phi}{\partial z^2} + \rho g \frac{\partial \psi}{\partial x} - \gamma(1 + \tau_1 \frac{\partial}{\partial t})T = \rho \frac{\partial^2 \phi}{\partial t^2}, \tag{13}$$

$$(\mu_L - \mu_T)\frac{\partial^2 v}{\partial x^2} + \mu_T\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) v = \rho \frac{\partial^2 v}{\partial t^2}, \tag{14}$$

$$(\alpha + 3\mu_L - 2\mu_T + \beta)\frac{\partial^2 \psi}{\partial x^2} + \mu_L \frac{\partial^2 \psi}{\partial z^2} - \rho g \frac{\partial \phi}{\partial x} = \rho \frac{\partial^2 \psi}{\partial t^2}. \tag{15}$$

Substituting from Eq. (12) into Eq. (2), we obtain

$$K \nabla^2 T = \rho\, c_v \frac{\partial}{\partial t}\left(1 + \tau_2 \frac{\partial}{\partial t}\right) T + \gamma T_0 \frac{\partial}{\partial t}\left(1 + \tau_2 \delta \frac{\partial}{\partial t}\right) \nabla^2 \phi \tag{16}$$

where, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$, and similar relations in M_2 with ρ, λ, α, μ_L, β replaced by ρ', λ', α', μ'_L, β'.

3. Solution of the problem

Since we consider the propagation of surface waves in the direction of x only, we restrict our attention only to Eqs. (13)-(16), which their solutions take the form

$$(\phi, \psi, v, T) = [\Phi(z),\ \Psi(z),\ V(z),\ \Theta(z)]\ e^{i\omega(x-ct)}. \tag{17}$$

Using Eqs. (17) in (13-16), we get a set of differential equations for medium M_1 as follows

$$(D^2 + m_1^2)\Phi + m_2^2\Psi - m_3^2\Theta = 0, \tag{18}$$

$$(D^2 + m_4^2)\Psi - m_5^2\Phi = 0, \tag{19}$$

$$(D^2 + m_6^2)V = 0, \tag{20}$$

$$(D^2 + A_1^2)\Theta + A_2^2\Phi = 0 \tag{21}$$

where

$$\Gamma_1=\frac{\lambda+2\alpha+4\mu_L-2\mu_T+\beta}{\rho},\ \Gamma_2=\frac{\alpha+\lambda+2\mu_L}{\rho},\ \Gamma_3=\frac{\alpha+3\mu_L-2\mu_T+\beta}{\rho},\Gamma_4=\frac{\mu_L}{\rho},\ \Gamma_5=\frac{\mu_T}{\rho},$$
$$m_1^2=\frac{\omega^2(c^2-\Gamma_1)}{\Gamma_2},\ m_2^2=\frac{i\,\omega g}{\Gamma_2},\ m_3^2=\frac{L}{\Gamma_2},\ m_4^2=\frac{\omega^2(c^2-\Gamma_3)}{\Gamma_4},\ m_5^2=\frac{i\,\omega g}{\Gamma_4},m_6^2=\frac{\omega^2(c^2-\Gamma_4)}{\Gamma_5},$$
$$L=\frac{\gamma(1-i\,\omega c\tau_1)}{\rho},\ A_1^2=\frac{\omega^2-i\,\omega c\,\rho c_v\,(1-i\,\omega c\tau_2)}{K},A_2^2=\frac{i\,\omega c\gamma T_0(1-i\,\omega c\delta\tau_2)}{K},\ D=\frac{d}{dz}.$$

and those of the medium M_2 are given by

$$(D^2+m_1'^2)\Phi+m_2'^2\Psi-m_3'^2\Theta=0, \tag{22}$$

$$(D^2+m_4'^2)\Psi-m_5'^2\Phi=0, \tag{23}$$

$$(D^2+m_6'^2)V=0, \tag{24}$$

$$(D^2+A_1'^2)\Theta+A_2'^2\Phi=0 \tag{25}$$

where

$$\Gamma_1'=\frac{\lambda'+2\alpha'+4\mu_L'-2\mu_T'+\beta'}{\rho'},\ \Gamma_2'=\frac{\alpha'+\lambda'+2\mu_L'}{\rho'},\ \Gamma_3'=\frac{\alpha'+3\mu_L'-2\mu_T'+\beta'}{\rho'},\Gamma_4'=\frac{\mu_L'}{\rho'},\Gamma_5'=\frac{\mu_T'}{\rho'},$$
$$m_1'^2=\frac{\omega^2(c^2-\Gamma_1')}{\Gamma_2'},\ m_2'^2=\frac{i\,\omega g}{\Gamma_2'},\ m_3'^2=\frac{L'}{\Gamma_2'},\ m_4'^2=\frac{\omega^2(c^2-\Gamma_4')}{\Gamma_3'},\ m_5'^2=\frac{i\,\omega g}{\Gamma_3'},m_6'^2=\frac{\omega^2(c^2-\Gamma_4')}{\Gamma_5'},$$
$$L'=\frac{\gamma'(1-i\,\omega c\tau_1)}{\rho'},\qquad A_1'^2=\frac{\omega^2-i\,\omega c\,\rho' c_v\,(1-i\,\omega c\tau_2)}{K},\qquad A_2'^2=\frac{i\,\omega c\gamma' T_0(1-i\,\omega c\delta\tau_2)}{K}.$$

Eliminating the temperature Θ from Eqs. (22) and (25), we obtain

$$\left[D^4+(A_1^2+m_1^2)D^2+A_1^2m_1^2+m_3^2A_2^2\right]\Phi+m_2^2(D^2+A_1^2)\ \Psi=0, \tag{26}$$

Using Eq. (19) and Eq. (26), we get

$$\left[D^6+Q_1D^4+Q_2D^2+Q_3\right]\Phi=0 \tag{27}$$

Eqs. (22) - (25) tend to the solutions

$$\Phi=\sum_{j=1}^{3}\left[B_je^{i\omega\lambda_jz}+C_je^{-i\omega\lambda_jz}\right], \tag{28}$$

$$\Psi=\sum_{j=1}^{3}\left[D_je^{i\omega\lambda_jz}+E_je^{-i\omega\lambda_jz}\right], \tag{29}$$

$$V=Fe^{i\omega\ m_7z}+Ge^{-i\omega\ m_6z} \tag{30}$$

where, the constants D_j and E_j are related to the constants B_j and C_j in the form

$$D_j=N_jB_j,\quad E_j=N_jC_j,\qquad j=1,2,3,$$

$$N_j=\frac{m_5^2}{[m_4^2-\omega^2\lambda_j^2]}\ . \tag{31}$$

From Eqs. (28) and (29) in Eq. (18), we get

$$\Theta = \frac{1}{m_3^2} \sum_{j=1}^{3} \ (m_1^2 - \omega^2 \lambda_j^2 + m_2^2 N_j) \ [B_j e^{i\omega\lambda_j z} + C_j e^{-i\omega\lambda_j z}] \cdot \tag{32}$$

where, λ_1, λ_2 and λ_3 are taken to be the complex roots of the equation

$$\lambda^6 + Q_1 \lambda^4 + Q_2 \lambda^2 + Q_3 = 0, \tag{33}$$

$$\begin{aligned} &Q_1 = m_4^2 + m_1^2 + A_1^2, \ \ Q_2 = A_1^2 (m_4^2 + m_1^2) + m_1^2 m_4^2 + m_3^2 A_2^2 + m_5^2 m_2^2, \\ &Q_3 = A_1^2 (m_4^2 m_1^2 + m_2^2 m_5^2) + m_3^2 m_4^2). \end{aligned} \tag{34}$$

Since the solutions in order Φ, *Ψ*, *V* and Θ will describe surface waves, they must become vanishing as $z \to \infty$, hence for medium M_1

$$\Phi = \sum_{j=1}^{3} [C_j e^{-i\omega\lambda_j z}], \tag{35}$$

$$\Psi = \sum_{j=1}^{3} \left[N_j C_j e^{-i\omega\lambda_j z} \right], \tag{36}$$

$$V = G e^{-i\omega \ m_6 z}, \tag{37}$$

$$\Theta = \frac{1}{m_3^2} \sum_{j=1}^{3} \ (m_1^2 - \omega^2 \lambda_j^2 + m_2^2 N_j) \ [C_j e^{-i\omega\lambda_j z}] \tag{38}$$

and for medium M_2

$$\Phi = \sum_{j=1}^{3} [C'_j e^{-i\omega\lambda'_j z}], \tag{39}$$

$$\Psi = \sum_{j=1}^{3} \left[N'_j C'_j e^{-i\omega\lambda'_j z} \right], \tag{40}$$

$$V = G' e^{-i\omega \ m'_6 z}, \tag{41}$$

$$\Theta = \frac{1}{m_3'^2} \sum_{j=1}^{3} \ (m_1'^2 - \omega^2 \lambda_j'^2 + m_2'^2 N'_j) \ [C'_j e^{-i\omega\lambda'_j z}] \cdot \tag{42}$$

Finally from Eq. (17), and substituting from Eqs. (35)-(38), we have

$$\phi = \sum_{j=1}^{3} \ \left[C_j e^{i\omega(x - ct - \lambda_j z)} \right], \tag{43}$$

$$\psi = \sum_{j=1}^{3} \left[N_j C_j e^{i\omega(x - ct - \lambda_j z)} \right], \tag{44}$$

$$T = \frac{1}{m_3^2} \sum_{j=1}^{3} \ (m_1^2 - \omega^2 \lambda_j^2 + m_2^2 N_j) \left[\ \ C_j e^{i\omega(x - ct - \lambda_j z)} \ \ \right], \tag{45}$$

$$u = i\omega \sum_{j=1}^{3} \left[\ 1 + N_j \lambda_j \ \right] C_j e^{i\omega (x - ct - \lambda_j z)}, \tag{46}$$

$$w = i\omega \sum_{j=1}^{3} \left[\ -\lambda_j + N_j \ \right] C_j e^{i\omega (x - ct - \lambda_j z)}, \tag{47}$$

$$v = G e^{i\omega \ (m_6 z + x - ct)} \ . \tag{48}$$

4. Boundary conditions

The boundary conditions for the problem are

(i) The displacement components at the boundary surface between the media M_1 and M_2 must be continued at all times and positions.

i.e., $[u, v, w]_{M_1} = [u, v, w]_{M_2}$ at $z=0$

(ii) The stress components τ_{31}, τ_{32} and τ_{33} must be continuous at the boundary $z=0$.

i.e., $[\tau_{31}, \tau_{32}, \tau_{33}]_{M_1} = [\tau_{31}, \tau_{32}, \tau_{33}]_{M_2}$, at $z=0$

(iii) The thermal boundary conditions must be continuous at the boundary $z=0$.

i.e., $\left(\frac{\partial T}{\partial z} + hT \right)_{M_1} = \left(\frac{\partial T}{\partial z} + hT \right)_{M_2}$, at $z=0$.

$$T_{M_1} = T_{M_2}$$

Applying the boundary conditions (i)-(iii), we have

$$(1 + N_j \lambda_j) C_j = (1 + N'_j \lambda'_j) C'_j, \tag{49}$$

$$G = G', \tag{50}$$

$$(\lambda_j - N_j) C_j = (\lambda'_j - N'_j) C'_j, \tag{51}$$

$$\mu_L \left\{ \ 2\lambda_j + (\lambda_j^2 - 1) N_j \ \right\} C_j = \mu'_L \left\{ \ 2\lambda'_j + (\lambda_j'^2 - 1) N'_j \ \right\} C'_j, \tag{52}$$

$$\mu_T m_6 G = \mu'_T m'_6 G', \tag{53}$$

$$\left[\lambda_j (\lambda + 2\mu_T)[\lambda_j + N_j] - (\lambda + \alpha)[1 + \lambda_j N_j] - \frac{\gamma (1 - i\omega c \tau_1)(m_1^2 - \omega^2 \lambda_j^2 + m_2^2 N_j)}{\omega^2 m_3^2} \right] C_j =$$
$$\left[\lambda'_j (\lambda' + 2\mu'_T)[\lambda'_j + N'_j] - (\lambda' + \alpha')[1 + \lambda'_j N'_j] - \frac{\gamma' (1 - i\omega c \tau_1)(m_1'^2 - \omega^2 \lambda_j'^2 + m_2'^2 N'_j \ )}{\omega^2 m_3'^2} \right] C'_j, \tag{54}$$

$$\frac{1}{m_3^2} (m_1^2 - \omega^2 \lambda_j^2 + m_2^2 N_j)[h - i\,\omega \lambda_j] C_j = \frac{1}{m_3'^2} (m_1'^2 - \omega^2 \lambda_j'^2 + m_2'^2 N'_j)[h - i\,\omega \lambda'_j] C'_j, \tag{55}$$

$$\frac{1}{m_3^2} (m_1^2 - \omega^2 \lambda_j^2 + m_2^2 N_j) \ C_j = \frac{1}{m_3'^2} (m_1'^2 - \omega^2 \lambda_j'^2 + m_2'^2 N'_j) \ C'_j. \tag{56}$$

From Eqs. (50) and (53), we have $G=G'=0$. Thus, there is no propagation of displacement v. Hence SH-waves are decoupled in this case. The constants C_j and C'_j we can determinate where $j=1,2,3$

Finally, eliminating the constants C_j and C'_j from Eqs. (49), (51), (52), (54), (55) and (56), we get

$$\det(a_{ij}) = 0, \quad i, j = 1,2,3,4,5,6 \tag{57}$$

From Eq. (57), we get the velocity of surface waves in common boundary between two fibre-reinforced anisotropic semi-infinite thermoelasticity solid media under the influence of gravity. Since the wave velocity c obtained from (57) depends on the particular value of ω which indicates the dispersion of the general wave form and in the gravity field, imposing a certain changes in the waveform.

We discuss this case and special cases in two models as the following:

(I) Lord-Shulman (1967)-model ($\tau_1=0$, $\tau_2>0$, $\delta=1$)

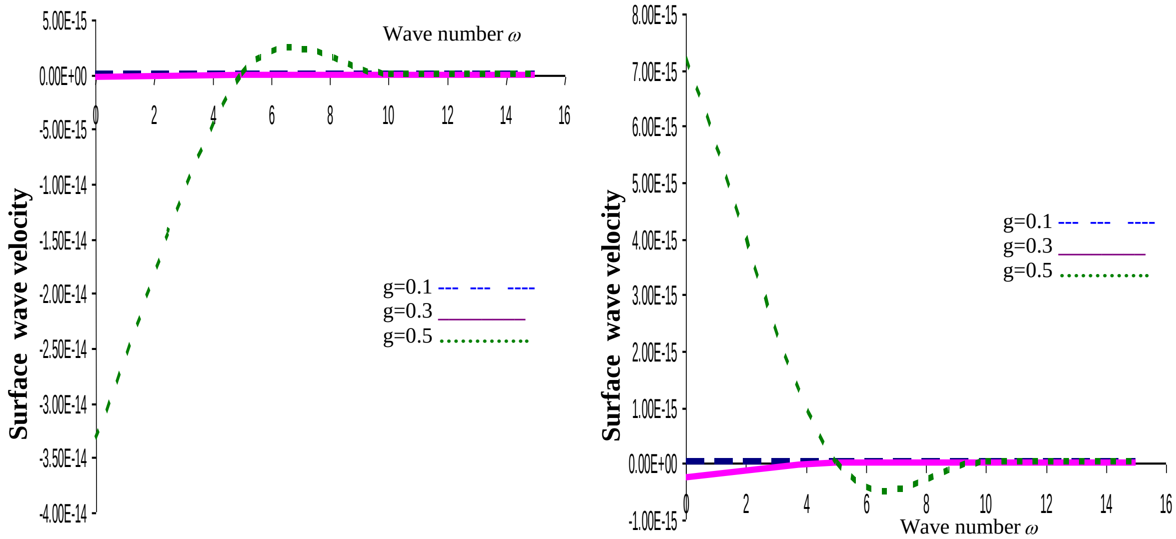
(II) Green-Lindsay (1972)-model ($\tau_1 \geq \tau_2 > 0$, $\delta=0$).

The discussion is clear up from Figs. 1- 4.

5. Special cases of surface waves

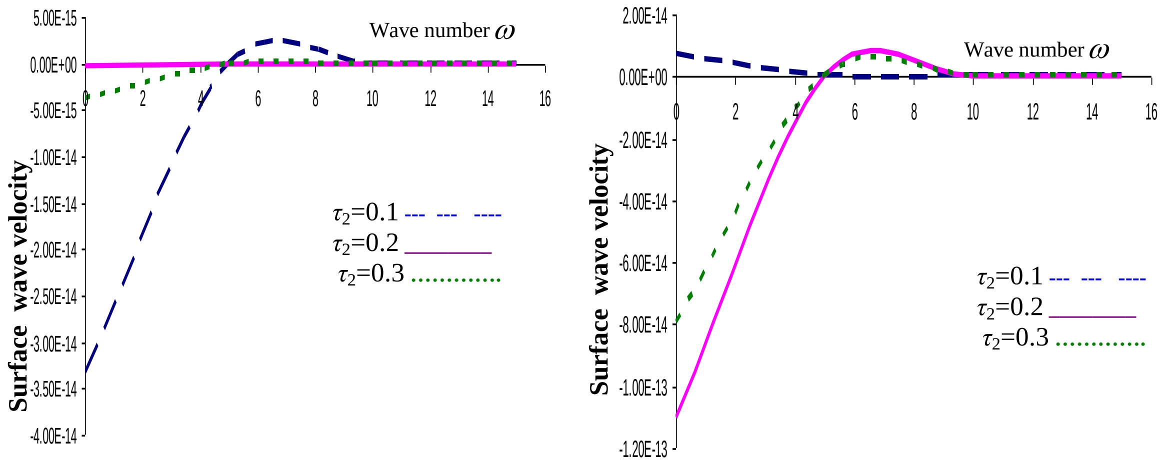
5.1 Anisotropic generalized thermoelastic medium with gravity

Stoneley Waves: It is the generalized form of Rayleigh waves in which we assume that the waves are propagated along the common boundary of two semi-infinite media M_1 and M_2. Therefore, Eq. (57) determines the wave velocity equation for Stoneley waves in anisotropic



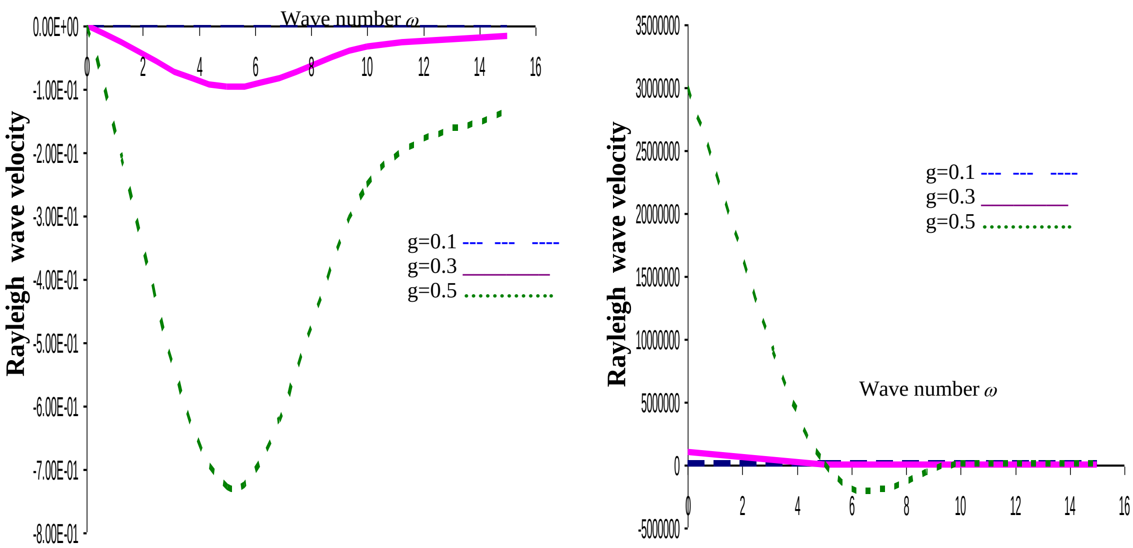
(I) LS-model ($\tau_1=0$, $\tau_2>0$, $\delta=1$) (II) GL-model ($\tau_1 \geq \tau_2 > 0$, $\delta=0$)

Fig. (2) Variation of surface wave velocity with respect to the wave number under effect of g



(I) LS-model (τ_1=0, τ_2>0, δ=1) (II) GL-model ($\tau_1 \geq \tau_2$>0, δ=0)

Fig. 3 Variation of surface wave velocity with respect to the wave number under effect of τ_2



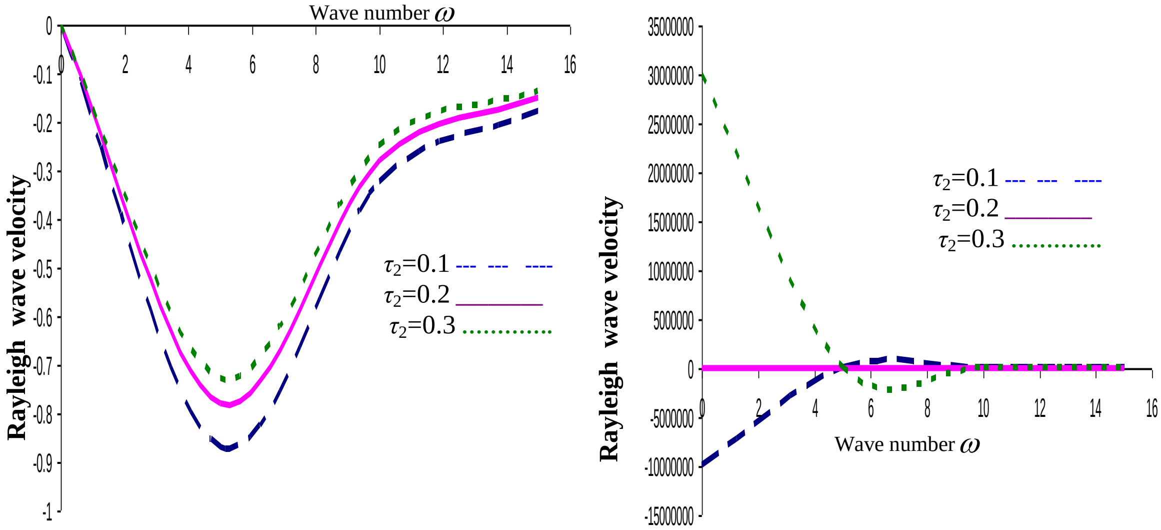
(I) LS-model (τ_1=0, τ_2>0, δ=1) (II) GL-model ($\tau_1 \geq \tau_2$>0, δ=0)

Fig. 4 Variation of Rayleigh waves velocity with respect to the wave number under effect of g

fibre- reinforced solid thermoelastic media under the influence of gravity.

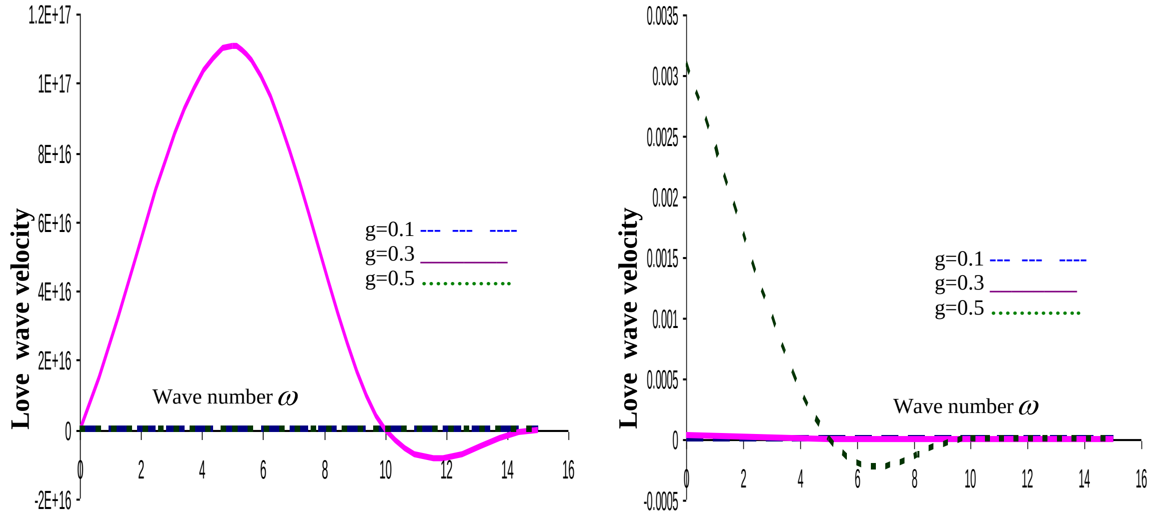
Rayleigh waves: To investigate the possibility of Rayleigh waves in anisotropic fiber-reinforced elastic media, we replace the medium M_2 by vacuum, in the preceding problem. Since the boundary z=0 is adjacent to vacuum, it is free from surface traction. So the stress boundary condition in this case may be expressed as τ_{31}=0, τ_{33}=0 and T=0 on z=0, i.e., the boundary conditions as

$$\mu_L \left\{2\lambda_j + (\lambda_j^2 - 1)N_j\right\} C_j = 0, \tag{58}$$



(I) LS-model (τ_1=0, τ_2>0, δ=1) (II) GL-model ($\tau_1 \geq \tau_2$>0, δ=0)

Fig. 5 Variation of Rayleigh waves velocity with respect to the wave number under effect of τ_2

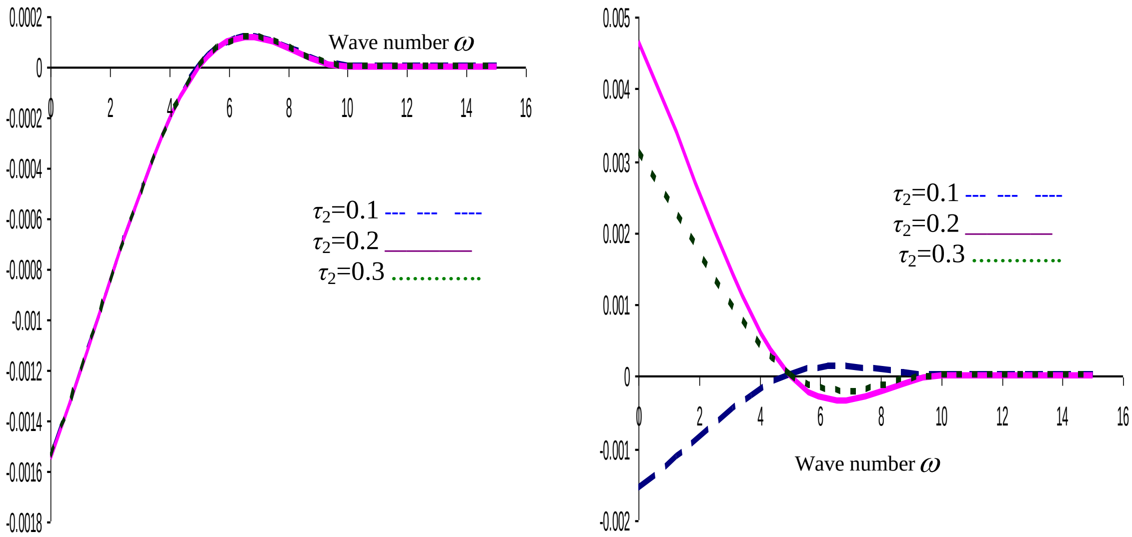


(I) LS-model (τ_1=0, τ_2>0, δ=1) (II) GL-model ($\tau_1 \geq \tau_2$>0, δ=0)

Fig. 6 Variation of Love wave velocity with respect to the wave number under effect of *g*

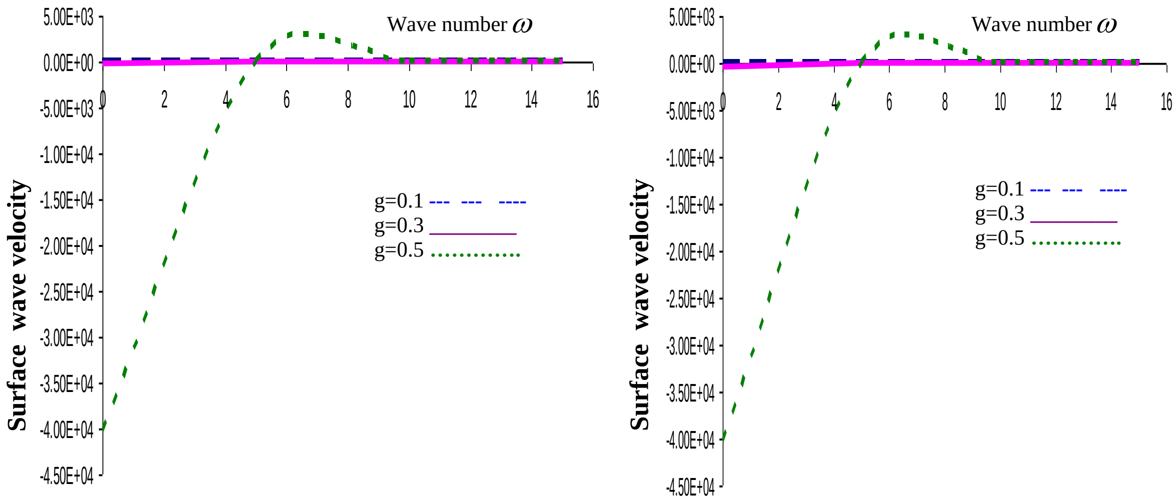
$$\left[\lambda_j(\lambda+2\mu_T)[\lambda_j+N_j]-(\lambda+\alpha)[1+\lambda_j N_j]-\frac{\gamma(1-i\omega c\tau_1)(m_1^2-\omega^2\lambda_j^2+m_2^2 N_j)}{\omega^2 m_3^2}\right]C_j=0 \tag{59}$$

$$\frac{1}{m_3^2}(m_1^2-\omega^2\lambda_j^2+m_2^2 N_j)\ C_j=0\cdot \tag{60}$$



(I) LS-model (τ_1=0, τ_2>0, δ=1) (II) GL-model ($\tau_1 \geq \tau_2$>0, δ=0)

Fig. 7 Variation of Love wave velocity with respect to the wave number under effect of τ_2



(I) LS-model (τ_1=0, τ_2>0, δ=1) (II) GL-model ($\tau_1 \geq \tau_2$>0, δ=0)

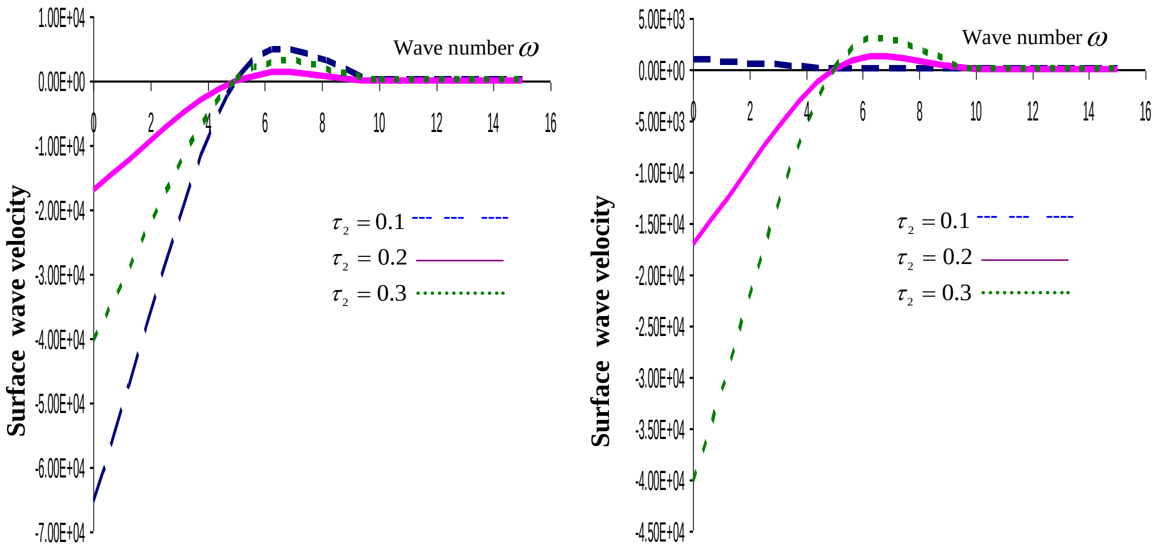
Fig. 8 Variation of surface wave velocity with respect to the wave number under effect of g

In this case , the velocity of Rayleigh waves can be determined from

$$\det(a_{ij}) = 0, \quad i, j = 1,2,3 \tag{61}$$

Love waves:

For the existence of Love waves, we consider a layered semi-infinite medium, in which M_1 is obtained by two horizontal plane surfaces, a finite distance H apart, in this case we can put the stress boundary condition equal zero on $z=H$.



(I) LS-model (τ_1=0, τ_2>0, δ=1) (II) GL-model ($\tau_1 \geq \tau_2$>0, δ=0)

Fig. 9 Variation of surface wave velocity with respect to the wave number under effect of τ_2

5.2 Isotropic generalized thermoelastic medium with gravity

In this case, substituting $\mu_L=\mu_T=\mu$, $\gamma=\alpha_t(3\lambda+2\mu)$ and β=0 in Eq. (57), we obtain the surface waves in fibre-reinforced isotropic solid thermoelastic medium with gravity.

6. Numerical results and discussions

The following values of elastic constants are considered Singh (2005) and Chattopadhyay *et al*. (2002), for mediums M_1 and M_2 respectively.

$$\rho = 2660\,Kg/m^3\ ,\ \ \lambda = 5.65\times10^{10}\,Nm^{-2},\ \ \mu_T = 2.46\times10^{9}\,Nm^{-2}, \mu_L = 5.66\times10^{9}\,Nm^{-2},$$

$$\alpha = -1.28\times10^{9}\,Nm^{-2},\ \ \beta = 220.90\times10^{9}\,Nm^{-2,}$$

$$\rho = 7800\,Kg/m^3\ ,\ \ \lambda = 5.65\times10^{9}\,Nm^{-2},\ \ \mu_T = 2.46\times10^{10}\,Nm^{-2}, \mu_L = 5.66\times10^{10}\,Nm^{-2},$$

$$\alpha = -1.28\times10^{10}\,Nm^{-2},\ \ \beta = 220.90\times10^{10}\,Nm^{-2,}$$

$$c_v = 0.787\times10^3\ \ J/kg\ \ K,\ \ K = 0.0921\times10^3\,Jm^{-1}\deg^{-1} s^{-1}\ \ ,T_0 = 293K, c = 1.2\times10^4\,m^2/s^2K.$$

The numerical technique outlined above was used to obtain surface wave velocity and with respect to wave number under the effects of gravity and thermal relaxation time parameter in two models. For the sake of brevity,some computational results are being presented here.
The variations are shown in Figs. 2-9 respectively.

Fig. (2): shows the variations in the value of velocity of surface waves u with respect to wave number, which it has oscillatory behavior with gravity in the whole range of the wave number. Both figures indicate that the medium along wave number ξ in two models Lord-Shulman and

Green-Lindsay undergoes expansion deformation because of thermal shock while the other compressive deformation. The effect of gravity g on surface wave velocity, which it shifts from the positive into the negative gradually. At a given instant, the velocity of surface waves is finite, which is due to the effect of gravity. It is noticeable that the velocity of surface wave in LS-model increases with increasing the values of g, while it decreases with increasing the values of g, as well it decreases with increasing of the value of wave number ξ until approaching to zero.

Fig. (3): shows the variations in the value of velocity of surface waves u with respect to wave number, which it has oscillatory behavior with relaxation time τ_2 in the whole range of the wave number. Both figures indicate that the medium along wave number ξ in two models Lord-Shulman and Green-Lindsay undergoes expansion deformation because of thermal shock while the other compressive deformation. The effect of relaxation time τ_2 on the velocity of Surface waves becomes large while the effect of the relaxation time on surface wave velocity, which it shifts from the positive into the negative gradually. At a given instant, the velocity of Surface waves is finite, which is due to the effect of relaxation time. It is noticeable that the velocity of surface wave in LS-model increases with increasing the values of τ_2, while it decreases in the GL-model, as well it decreases with increasing of the value of wave number ξ until approaching to zero. A high velocity variation of surface waves was found to exist within the LS-model and GL-model.

Fig. (4): shows that the variation of Rayleigh wave velocity with respect to wave number under the effects of gravity in two models Lord-Shulman model and Green-Lindsay model. The value of the velocity of Rayleigh waves u has oscillatory behavior with gravity in the whole range of the wave number. Both figures indicate that the medium along wave number ξ in two models Lord-Shulman and Green- Lindsay undergoes expansion deformation because of thermal shock while the other compressive deformation. The effect of gravity g on Rayleigh wave velocity, which it shifts from the positive into the negative gradually. At a given instant, the velocity of Rayleigh waves is finite, which is due to the effect of gravity. It is noticeable that the velocity of Rayleigh waves in two models decreases with increasing the values of g, as well it decreases with increasing of the value of wave number ξ until approaching to zero. It is found that in LS- model the values of Rayleigh wave velocity are a semi-stable with increasing values of g.

Fig. (5): shows that the variation of Rayleigh wave velocity with respect to wave number under the effects of relaxation time τ_2 in two models Lord-Shulman model and Green-Lindsay model. The value of the velocity of Rayleigh waves u has oscillatory behavior with relaxation time in the whole range of the wave number. Both figures indicate that the medium along wave number ξ in two models Lord-Shulman and Green-Lindsay undergoes expansion deformation because of thermal shock while the other compressive deformation. The effect of relaxation time on Rayleigh wave velocity, which it shifts from the positive into the negative gradually. At a given instant, the velocity of Rayleigh waves is finite, which is due to the effect of relaxation time. It is noticeable that the velocity of surface wave in LS-model increases with increasing the values of τ_2 , while it decreases in the GL-model, as well it decreases with increasing of the value of wave number ξ until approaching to zero. A high velocity variation of Rayleigh waves was found to exist within the LS-model and GL-model.

Fig. (6): shows that the variation of Love waves velocity with respect to wave number under the effects of gravity in two models Lord-Shulman model and Green-Lindsay model. The values of the velocity of Love waves u have an oscillatory behavior with gravity in the whole range of the wave number. Both figures indicate that the medium along wave number ξ in two models Lord-Shulman and Green-Lindsay undergoes expansion deformation because of thermal shock while the other compressive deformation. The effect of gravity g on Love wave velocity, which it shifts from the positive into the negative gradually. At a given instant, the velocity of Love waves is finite, which is due to the effect of gravity . It is noticeable that the velocity of Love waves in two models increases with increasing the values of g, while it decreases with increasing of the values of the wave number ξ until approaching to zero.

Fig. (7): presents the variation of Love waves velocity with respect to wave number under the effects of relaxation time τ_2 in two models Lord-Shulman model and Green-Lindsay model. The value of the velocity of Love waves u has oscillatory behavior with relaxation time in the whole range of the wave number. Both figures indicate that the medium along wave number ξ in two models Lord-Shulman and Green-Lindsay undergoes expansion deformation because of thermal shock while the other compressive deformation. The effect of relaxation time on Love wave velocity, which it shifts from the positive into the negative gradually. At a given instant, the velocity of Love waves is finite, which is due to the effect of relaxation time. It is noticeable that the velocity of Love waves in LS-models coincides with increasing the values of τ_2, while it increases with increasing of the value of wave number ξ, as well the values of Love wave velocity in GL-model increases with increasing of relaxation time until approaching to zero with the wave number. A high velocity variation of Love waves was found to exist within the GL-model.

Fig. (8): shows that the variation of the value of the velocity of surface waves u with respect to wave number, which it has an oscillatory behavior with gravity in the whole range of the wave number in an isotropic case. Both figures indicate that the medium along wave number ξ in two models Lord-Shulman and Green-Lindsay undergoes expansion deformation because of thermal shock while the other compressive deformation. The effect of gravity g on surface wave velocity, which it shifts from the positive into the negative gradually. At a given instant, the velocity of Surface waves is finite, which is due to the effect of gravity. It is noticeable that the velocity of surface wave in two models increases with increasing the values of g, while it decreases and increases with increasing of the value of wave number ξ until approaching to zero.

Fig. (9): illustrates the value of the variation of surface waves u with respect to wave number, which it has an oscillatory behavior with relaxation time τ_2 in the whole range of the wave number in isotropic case. Both figures indicate that the medium along wave number ξ in two models Lord-Shulman and Green-Lindsay undergoes expansion deformation because of thermal shock while the other compressive deformation. The effect of relaxation time τ_2 on the velocity of Surface waves becomes large while the effect of the relaxation time on surface wave velocity, which it shifts from the negative into the positive gradually. At a given instant, the velocity of Surface waves is finite, which is due to the effect of relaxation time. It is noticeable that the velocity of surface wave in LS-models increases with increasing the values of τ_2, while it decreases with increasing the values of τ_2 in the GL-model, as well the values of surface wave velocity approaching to zero with increasing of the wave number. A high velocity variation of surface waves was found to exist within the LS-model and GL-model.

Comparing with previous studies, we find that our results (shown in Figs. 1-9) without gravity, anisotropy, relaxation times and parameters for fibre-reinforced with the results obtained by Fu and Zhang (2006). Also, these results agree with those of Weitsman and Benveniste (1974), Sengupta (2001) when the gravity, anisotropy, relaxation times and parameters for fibre-reinforced almost equal zero. In case of a gravity g=0, our results are in agreement with that of Singh (2005) The analytical results obtained by Singh (2007) can be considered as a limiting case (by taking g=0 and τ_2=0), which are in agreement with earlier results obtained by Huang and Rokhlin (1995).

7. Conclusions

Due to the complicated nature of the governing equations of the generalized thermo elasticity fiber-reinforced theory, the work done in this field is unfortunately limited in number. The method used in this study provides a quite successful in dealing with such problems. This method gives exact solutions in the elastic medium without any assumed restrictions on the actual physical quantities that appear in the governing equations of the problem considered. Important phenomena are observed in all these computations:

• It was found that for large values of time the coupled and the generalized give close results. The case is quite different when we consider small value of relaxation time. The coupled theory predicts infinite speeds of wave propagation. The solutions obtained in the context of generalized thermoelasticity theory, however, exhibit the behavior of finite speeds of wave propagation.

• By comparing Figs. 2-9 for Lord-Shulman theory with Figures for Green-Lindsay, it was found that wave velocity has the same behavior in both media. But with the passage of relaxation time and gravity, numerical values of wave velocity in the generalized thermoplastic medium are large in comparison with those in thermoplastic medium due to the influences of relaxation time and gravity.

• Special cases are considered as Rayleigh waves, Love wave and surface waves in anisotropic generalized thermoelastic medium, as well in the isotropic case.

• The results presented in this paper should prove useful for researchers in material science, designers of new materials, low-temperature physicists, as well as for those working on the development of a theory of hyperbolic propagation of hyperbolic thermoelastic. Relaxation time and gravity exchange with the environment arising from and inside nuclear reactors influence their and operations. Study of the phenomenon of relaxation time and gravity is also used to improve the conditions of oil extractions.

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Appendix A

Elements of Eq. (57) are

$$a_{11} = 1 + N_1\lambda_1, \quad a_{12} = 1 + N_2\lambda_2, \quad a_{13} = 1 + N_3\lambda_3,$$

$$a_{14} = 1 + N_1'\lambda_1', \quad a_{15} = 1 + N_2'\lambda_2', \quad a_{16} = 1 + N_3'\lambda_3',$$

$$a_{21} = \lambda_1 - N_1, \quad a_{22} = \lambda_2 - N_2, \quad a_{23} = \lambda_3 - N_3,$$

$$a_{24} = \lambda_1' - N_1', \quad a_{25} = \lambda_2' - N_2', \quad a_{26} = \lambda_3' - N_3',$$

$$a_{31} = \mu_L\{\ 2\lambda_1 + (\lambda_1^2 - 1)N_1\ \}, \quad a_{32} = \mu_L\{\ 2\lambda_2 + (\lambda_2^2 - 1)N_2\ \},$$

$$a_{33} = \mu_L\{\ 2\lambda_3 + (\lambda_3^2 - 1)N_3\ \},$$

$$a_{34} = \mu_L'\{\ 2\lambda_1' + (\lambda_1'^2 - 1)N_1'\ \}, \quad a_{35} = \mu_L'\{\ 2\lambda_2' + (\lambda_2'^2 - 1)N_2'\ \},$$

$$a_{36} = \mu_L'\{\ 2\lambda_3' + (\lambda_3'^2 - 1)N_3'\ \},$$

$$a_{41} = \left[\lambda_1(\lambda + 2\mu_T)[\lambda_1 + N_1] - (\lambda + \alpha)[1 + \lambda_1 N_1] - \frac{\gamma(1 - i\omega c\tau_1)(m_1^2 - \omega^2\lambda_1^2 + m_2^2 N_1)}{\omega^2 m_3^2}\right],$$

$$a_{42} = \left[\lambda_2(\lambda + 2\mu_T)[\lambda_2 + N_2] - (\lambda + \alpha)[1 + \lambda_2 N_2] - \frac{\gamma(1 - i\omega c\tau_1)(m_1^2 - \omega^2\lambda_2^2 + m_2^2 N_2)}{\omega^2 m_3^2}\right],$$

$$a_{43} = \left[\lambda_3(\lambda + 2\mu_T)[\lambda_3 + N_3] - (\lambda + \alpha)[1 + \lambda_3 N_3] - \frac{\gamma(1 - i\omega c\tau_1)(m_1^2 - \omega^2\lambda_3^2 + m_2^2 N_3)}{\omega^2 m_3^2}\right],$$

$$a_{44} = \left[\lambda_1'(\lambda' + 2\mu_T')[\lambda_1' + N_1'] - (\lambda' + \alpha')[1 + \lambda_1' N_1'] - \frac{\gamma'(1 - i\omega c\tau_1)(m_1'^2 - \omega^2\lambda_1'^2 + m_2'^2 N_1')}{\omega^2 m_3'^2}\right],$$

$$a_{45} = \left[\lambda_2'(\lambda' + 2\mu_T')[\lambda_2' + N_2'] - (\lambda' + \alpha')[1 + \lambda_2' N_2'] - \frac{\gamma'(1 - i\omega c\tau_1)(m_1'^2 - \omega^2\lambda_2'^2 + m_2'^2 N_2')}{\omega^2 m_3'}\right],$$

$$a_{46} = \left[\lambda_3'(\lambda' + 2\mu_T')[\lambda_3' + N_3'] - (\lambda' + \alpha')[1 + \lambda_3' N_3'] - \frac{\gamma'(1 - i\omega c\tau_1)(m_1'^2 - \omega^2\lambda_3'^2 + m_2'^2 N_3')}{\omega^2 m_3'^2}\right],$$

$$a_{51} = \frac{1}{m_3^2}(m_1^2 - \omega^2\lambda_1^2 + m_2^2 N_1)[h - i\omega\lambda_1], \quad a_{52} = \frac{1}{m_3^2}(m_1^2 - \omega^2\lambda_2^2 + m_2^2 N_2)[h - i\omega\lambda_2],$$

$$a_{53} = \frac{1}{m_3^2}(m_1^2 - \omega^2\lambda_3^2 + m_2^2 N_3)[h - i\omega\lambda_3],$$

$$a_{54} = \frac{1}{m_3'^2}(m_1'^2 - \omega^2\lambda_1'^2 + m_2'^2 N_1')[h - i\omega\lambda_1'], \quad a_{55} = \frac{1}{m_3'^2}(m_1'^2 - \omega^2\lambda_2'^2 + m_2'^2 N_2')[h - i\omega\lambda_2'],$$

$$a_{56} = \frac{1}{m_3'^2}(m_1'^2 - \omega^2 \lambda_3'^2 + m_2'^2 N_3')[h - i\omega\lambda_3'],$$

$$a_{61} = \frac{1}{m_3^2}(m_1^2 - \omega^2 \lambda_1^2 + m_2^2 N_1), \quad a_{62} = \frac{1}{m_3^2}(m_1^2 - \omega^2 \lambda_2^2 + m_2^2 N_2),$$

$$a_{63} = \frac{1}{m_3^2}(m_1^2 - \omega^2 \lambda_3^2 + m_2^2 N_3),$$

$$a_{64} = \frac{1}{m_3'^2}(m_1'^2 - \omega^2 \lambda_1'^2 + m_2'^2 N_1'), \quad a_{65} = \frac{1}{m_3'^2}(m_1'^2 - \omega^2 \lambda_2'^2 + m_2'^2 N_2'),$$

$$a_{66} = \frac{1}{m_3'^2}(m_1'^2 - \omega^2 \lambda_3'^2 + m_2'^2 N_3').$$