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Symplectic analysis of functionally graded beams subjected to arbitrary lateral loads

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Abstract. The rational analytical solutions are presented for functionally graded beams subjected to arbitrary tractions on the upper and lower surfaces. The Young's modulus is assumed to vary exponentially along the thickness direction while the Poisson's ratio keeps unaltered. Within the framework of symplectic elasticity, zero eigensolutions along with general eigensolutions are investigated to derive the homogeneous solutions of functionally graded beams with no body force and traction-free lateral surfaces. Zero eigensolutions are proved to compose the basic solutions of the Saint-Venant problem, while general eigensolutions which vary exponentially with the axial coordinate have a significant influence on the local behavior. The complete elasticity solutions presented here include homogeneous solutions and particular solutions which satisfy the loading conditions on the lateral surfaces. Numerical examples are considered and compared with established results, illustrating the effects of material inhomogeneity on the localized stress distributions.

Keywords: functionally graded material; symplectic framework; exact solution; eigensolution

1. Introduction

In recent decades, functionally graded materials (FGMs) as a new generation of inhomogeneous composite materials have been used in many different applications, such as aircraft, armor plating, rocking motor casing, fusion energy devices, biomedical sectors and other engineering structures. Due to the smooth variation of material properties, FGM structures have received continuous and even enormous scientific attention. The pure mechanical behavior of FGM beams and plates have also been studied by many researchers (Chan *et al*. 2004, Ying *et al*. 2008, Lü *et al*. 2008, Vel 2010, Mantari *et al*. 2012). Sankar (2001) presented the analytical solutions for FGM beams with material properties varying exponentially along the thickness. Zhong and Yu (2007) obtained a general solution for functionally graded beams with arbitrary variations of material properties by means of Airy stress function approach. Ding *et al*. (2007) proposed the proper generalized stress function method to obtain the analytical elasticity solutions for functionally graded anisotropic beams with arbitrarily graded material properties. Huang *et al*. (2009) extended their method to plane analysis of functionally graded beams subjected to arbitrary loads. Nie *et al*. (2013) presented the analytical solutions for functionally graded beams with

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arbitrary material inhomogeneity along the thickness by the displacement function approach.

Although there is a huge volume of published literature on analytical solutions of functionally graded beams, it is very difficult to obtain complete stress distribution solutions based on two-dimensional elasticity theory. For example, the stress function method usually requires ample skills and experiences in seeking the potential function. Besides, the elastic field is inaccurate near the two ends which is covered up by Saint-Venant principle. Recently, Zhong (1995), Yao and Zhong (2002) developed a rational analytical method based on Hamiltonian system for elasticity problems of homogeneous materials. Leung and Zheng (2007) extended the work to derive the whole stress distributions for cantilever beams based on rigorous two-dimensional elasticity. The symplectic approach has been further extended to many various branches of applied mechanics (Yao and Xu 2001, Yao and Li 2006, Lim *et al*. 2007, Xu *et al*. 2008, Tarn *et al*. 2009, Tarn *et al*. 2010, Zhong and Li 2009). Recently, Zhao *et al*. systematically developed the symplectic framework for isotropic elastic FGMs (Zhao *et al*. 2012a, b), transversely isotropic piezoelectric FGMs (Zhao and Chen 2009) and magneto-electro-elastic FGMs (Zhao and Chen 2010). The material constants of the plane beams are assumed to vary in the length direction. Zhao *et al*. (2012c) presented exact solutions for bi-directional functionally graded beams with elastic modulus varying exponentially both along the axial and transverse coordinates.

It should be mentioned that, our previous analyses of axial-directional and bi-directional FGMs based on Hamiltonian system were carried out for plane beams with traction-free boundary conditions at the lateral surfaces. This paper attempts to obtain the complete stress distributions of functionally graded beams with material properties varying exponentially in the thickness direction. A particular solution is presented for generally supported beams subjected to arbitrary form tractions on the upper and lower surfaces. For homogeneous problem, a matrix state equation is derived with an operator matrix whose eigenvalues are classified into zero and general eigenvalues. The eigensolutions corresponding to zero eigenvalues compose the basic solutions of the Saint-Venant problem. Meanwhile, the general eigensolutions decay exponentially with the axial coordinate which are usually covered up by Saint-Venant principle. Two numerical examples are given to compare the accurate stress field with those for the homogeneous materials and established functionally graded loaded beam, respectively.

2. Symplectic framework

2.1 Analysis model

As shown in Fig. 1, we consider an isotropic functionally graded beam occupying the rectangular domain V: $0\le z\le l$ and $-h\le x\le h$, and subjected to normal and shear tractions, $\tilde{\sigma}_{xi}$ and $\tilde{\tau}_{xzi}$ ($i=1,2$) at the upper and lower surfaces

For the plane beam, the Young's modulus E is assumed to vary exponentially with the thickness in the form of $E=E_0e^{\beta x}$, while Poisson's ratio ν keeps unaltered. Here, E_0 is a constant and β is the gradient index of the material.

In absence of body force, the governing equations of equilibrium are

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xz}}{\partial z}=0,\quad \frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\sigma_z}{\partial z}=0 \tag{1}$$

The constitutive relations for the two-dimensional elasticity are

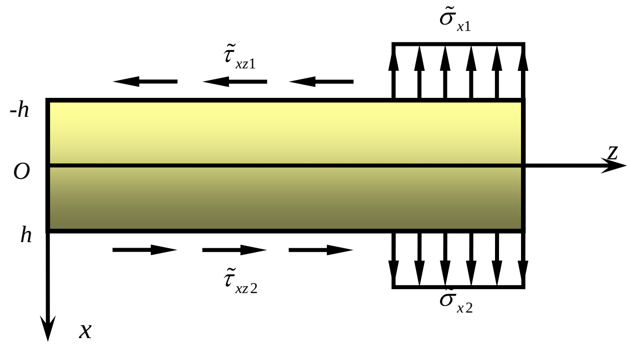


Fig. 1 The plane problem of FGM beam

$$\sigma_x = \frac{E}{1-\nu^2}\left(\frac{\partial u_x}{\partial x} + \nu\frac{\partial u_z}{\partial z}\right),\quad \sigma_z = \frac{E}{1-\nu^2}\left(\nu\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z}\right),\quad \tau_{xz} = \frac{E}{2(1+\nu)}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \tag{2}$$

By introducing new form of stress variables $\hat{\sigma}_x = \sigma_x \mathrm{e}^{-\beta x}$, $\hat{\sigma}_z = \sigma_z \mathrm{e}^{-\beta x}$ and $\hat{\tau}_{xz} = \tau_{xz} \mathrm{e}^{-\beta x}$, the matrix state equation can be deduced from the governing equations in Eqs. (1) and (2) as

$$\dot{\boldsymbol{v}} = \boldsymbol{H}\boldsymbol{v} \tag{3}$$

in which $\boldsymbol{v} = [u_z, u_x, \hat{\sigma}_z, \hat{\tau}_{xz}]^{\mathrm{T}}$ is the state vector, and the operator matrix $\boldsymbol{H}$ is defined as

$$\boldsymbol{H} = \begin{bmatrix} 0 & -\nu\frac{\partial}{\partial x} & \frac{1-\nu^2}{E_0} & 0 \\ -\frac{\partial}{\partial x} & 0 & 0 & \frac{2(1+\nu)}{E_0} \\ 0 & 0 & 0 & -\beta-\frac{\partial}{\partial x} \\ 0 & -E_0\frac{\partial^2}{\partial x^2} - E_0\beta\frac{\partial}{\partial x} & -\nu\frac{\partial}{\partial x} - \nu\beta & 0 \end{bmatrix} \tag{4}$$

Eq. (3) is usually referred to as the state equation. If β equals zero, the matrix $\boldsymbol{H}$ will degenerate to the conventional Hamiltonian matrix for homogeneous materials (Yao and Zhong 2002).

2.2 Formulation of the eigen-problem

In this section a brief review of symplectic formulations for FGM plane problems (Zhao *et al*. 2012b) is first presented. Considering the state equation of Eq. (3) along with the free boundary conditions on the lateral surfaces

$$x = \pm h:\quad \hat{\sigma}_x = E_0\frac{\partial u_x}{\partial x} + \nu\hat{\sigma}_z = 0,\quad \hat{\tau}_{xz} = 0 \tag{5}$$

we can assume the solution as follows using the method of separation of variables

$$\boldsymbol{v}(x,z)=\xi(z)\boldsymbol{\Phi}(x)=\mathrm{e}^{\mu z}[w(x),u(x),\sigma(x),\tau(x)]^{\mathrm{T}} \tag{6}$$

where μ is the eigenvalue of the operator matrix $\boldsymbol{H}$, and $\Phi(x)$ is the corresponding eigenvector. Eq. (3) and the homogeneous boundary conditions in Eq. (5) constitute a well-defined eigen-problem.

Because of the inhomogeneity parameter β, the operator matrix $\boldsymbol{H}$ exhibits different properties compared with the Hamiltonian matrix (Yao and Zhong 2002) and the shift-Hamiltonian matrix (Zhao *et al*. 2012b). Using the same analytical method as that for bi-directional functionally graded materials (Zhao *et al*. 2012c), we investigate the eigenfunction of the eigenvalue μ and divide them into particular eigensolutions and general ones.

To obtain the solution of the eigen-problem, the following eigenequation is derived from Eqs. (6) and (3)

$$\boldsymbol{H\Phi}(x)=\mu\boldsymbol{\Phi}(x) \tag{7}$$

We assume the solution in the form of

$$\boldsymbol{\Phi}(x)=\mathrm{e}^{\eta x}\boldsymbol{V} \tag{8}$$

where $\boldsymbol{V}$ is an undetermined constant vector, and η is the eigen-root of the following characteristic polynomial

$$\eta^4+2\beta\eta^3+(\beta^2+2\mu^2)\eta^2+2\beta\mu^2\eta-\beta^2\nu\mu^2+\mu^4=0 \tag{9}$$

Its four roots can be obtained as

$$\begin{aligned}\eta_1&=-\frac{1}{2}\beta+\frac{1}{2}\sqrt{\beta^2-4\mu^2-4\mu\beta\sqrt{\nu}}, &\quad \eta_2&=-\frac{1}{2}\beta-\frac{1}{2}\sqrt{\beta^2-4\mu^2-4\mu\beta\sqrt{\nu}}\\ \eta_3&=-\frac{1}{2}\beta+\frac{1}{2}\sqrt{\beta^2-4\mu^2+4\mu\beta\sqrt{\nu}}, &\quad \eta_4&=-\frac{1}{2}\beta-\frac{1}{2}\sqrt{\beta^2-4\mu^2+4\mu\beta\sqrt{\nu}}\end{aligned} \tag{10}$$

To obtain the eigen-solutions explicitly, we investigate all possible situations of repeated eigen-roots (Zhao *et al*. 2012c). Following the similar procedure as that for bi-directional FGMs, it is easy to prove that Eq. (9) has a pair of repeated roots only when μ=0. The zero eigenvalue is referred to as particular eigenvalue later on. Otherwise, the four roots of Eq. (9) are distinct from each other for the general eigenvalues $\mu\neq0$. Thus, the general solution of Eq. (7) can be obtained

$$w=\sum_{i=1}^{4}A_i\mathrm{e}^{\eta_i x},\quad u=\sum_{i=1}^{4}B_i\mathrm{e}^{\eta_i x},\quad \sigma=\sum_{i=1}^{4}C_i\mathrm{e}^{\eta_i x},\quad \tau=\sum_{i=1}^{4}D_i\mathrm{e}^{\eta_i x} \tag{11}$$

where A_i, B_i, C_i and D_i are constants to be determined; they are not independent as to be shown below.

2.3 Zero eigensolutions

Since zero eigenvalue is multiple, its fundamental eigensolutions and Jordan form eigensolutions need to be considered.

Following the similar procedure as that for homogeneous materials (Yao and Zhong 2002), the fundamental and first-order Jordan normal form eigenvectors corresponding to the first Jordan chain are

$$\boldsymbol{\Phi}_{0,1}^{(0)} = [1,0,0,0]^{\mathrm{T}} , \quad \boldsymbol{\Phi}_{0,1}^{(1)} = [0, -\nu x, E_0, 0]^{\mathrm{T}} \tag{12}$$

The eigensolutions of the problem can be constructed as

$$\boldsymbol{v}_{0,1}^{(0)} = \boldsymbol{\Phi}_{0,1}^{(0)} , \quad \boldsymbol{v}_{0,1}^{(1)} = \boldsymbol{M}_t \left(\boldsymbol{\Phi}_{0,1}^{(1)} + z\boldsymbol{\Phi}_{0,1}^{(0)} \right) = [z, -\nu x, E_0 \mathrm{e}^{\beta x}, 0]^{\mathrm{T}} \tag{13}$$

which represent rigid translation along the z-axis and the solution of extension.

The eigenvectors of the second Jordan chain can be obtained as

$$\boldsymbol{\Phi}_{0,2}^{(0)} = [0,1,0,0]^{\mathrm{T}} , \quad \boldsymbol{\Phi}_{0,2}^{(1)} = [-x,0,0,0]^{\mathrm{T}} , \quad \boldsymbol{\Phi}_{0,2}^{(2)} = [0, \frac{1}{2}\nu x^2, -E_0 x, 0]^{\mathrm{T}} \tag{14}$$

Then, the original solutions associated with the above eigenvectors are expressed as

$$\begin{gathered} \boldsymbol{v}_{0,2}^{(0)} = \boldsymbol{\Phi}_{0,2}^{(0)} , \quad \boldsymbol{v}_{0,2}^{(1)} = \boldsymbol{\Phi}_{0,2}^{(1)} + z\boldsymbol{\Phi}_{0,2}^{(0)} = [-x, z, 0, 0]^{\mathrm{T}} \\ \boldsymbol{v}_{0,2}^{(2)} = \boldsymbol{M}_t \left(\boldsymbol{\Phi}_{0,2}^{(2)} + z\boldsymbol{\Phi}_{0,2}^{(1)} + \frac{1}{2} z^2 \boldsymbol{\Phi}_{0,2}^{(0)} \right) = [-xz, \frac{1}{2}(\nu x^2 + z^2), -E_0 \mathrm{e}^{\beta x} x, 0]^{\mathrm{T}} \end{gathered} \tag{15}$$

where $\boldsymbol{M}_t = \mathrm{diag}[1,1,\mathrm{e}^{\beta x},\mathrm{e}^{\beta x}]$ is the transform matrix between the eigenvectors and original solutions of the problem. The physical interpretations of Eq. (15) represent rigid translation along the x-axis, rigid rotation about the y axis and the solution of pure bending, respectively.

It should be noted that the eigenvectors $\boldsymbol{\Phi}_{0,1}^{(0)}$, $\boldsymbol{\Phi}_{0,2}^{(0)}$, $\boldsymbol{\Phi}_{0,2}^{(1)}$ and their corresponding original solutions are either symmetric or antisymmetric deformations. But for the eigenvector $\boldsymbol{\Phi}_{0,1}^{(1)}$ and $\boldsymbol{\Phi}_{0,2}^{(2)}$, the original solutions of their eigenvectors are asymmetric deformations.

It should be emphasized that the first Jordan chain terminates for $\boldsymbol{\Phi}_{0,1}^{(1)}$ because there is no next grade eigenvector that satisfies the lateral boundary conditions. But the second Jordan chain does not terminate. To obtain the shear bending solution as that for the homogeneous material (Yao and Zhong 2002), a new eigenvector is introduced by combining two fundamental eigenvectors as

$$\boldsymbol{\Phi}_{0}^{(2)} = \boldsymbol{\Phi}_{0,2}^{(2)} + A_0 \boldsymbol{\Phi}_{0,1}^{(1)} \tag{16}$$

Then Jordan form eigenvector $\boldsymbol{\Phi}_{0}^{(3)}$ of the reconstructed eigenvector is solved from the equation $\boldsymbol{H}\boldsymbol{\Phi}_{0}^{(3)} = \boldsymbol{\Phi}_{0}^{(2)}$. The eigenvector $\boldsymbol{\Phi}_{0}^{(3)}$ can be obtained

$$\boldsymbol{\Phi}_{0}^{(3)} = \left\{ \begin{array}{c} -\frac{1}{6}\nu x^3 + 2(1+\nu)\left[\frac{1}{2\beta^2} x^2 - \frac{1}{\beta^2} x - \frac{c_1}{E_0 \beta} e^{-\beta x} - \frac{A_0}{\beta} x \right] + \frac{1}{2}\nu A_0 x^2 + c_2 \\ 0 \\ 0 \\ \frac{E_0}{\beta} x - \frac{E_0}{\beta^2} + c_1 e^{-\beta x} - \frac{A_0 E_0}{\beta} \end{array} \right\} \tag{17}$$

in which $A_0 = \left[\beta h \cosh(\beta h) / \sinh(\beta h) - 1 \right] / \beta$, $c_1 = E_0 h / \beta \sinh(\beta h)$ and $c_2 = 2(1+\nu) h / \beta^2 \sinh(\beta h)$.

The solution of the original problem for eigenvector $\boldsymbol{\Phi}_{0}^{(3)}$ can be written as

$$\boldsymbol{v}_0^{(3)} = \boldsymbol{M}_t\left[\boldsymbol{\Phi}_0^{(3)} + \left(z\boldsymbol{\Phi}_{0,2}^{(2)} + \frac{1}{2}z^2\boldsymbol{\Phi}_{0,2}^{(1)} + \frac{1}{6}z^3\boldsymbol{\Phi}_{0,2}^{(0)}\right) + A_0\left(z\boldsymbol{\Phi}_{0,1}^{(1)} + \frac{1}{2}z^2\boldsymbol{\Phi}_{0,1}^{(0)}\right)\right]$$
$$= \begin{Bmatrix} -\frac{1}{6}\nu x^3 + 2(1+\nu)\left[\frac{1}{2\beta^2}x^2 - \frac{1}{\beta^2}x - \frac{c_1}{E_0\beta}e^{-\beta x} - \frac{A_0}{\beta}x\right] - \frac{1}{2}xz^2 + A_0\frac{1}{2}(\nu x^2 + z^2) + c_2 \\ \frac{1}{2}\nu x^2 z + \frac{1}{6}z^3 - A_0\nu xz \\ E_0 \mathrm{e}^{\beta x}(-xz + A_0 z) \\ E_0\mathrm{e}^{\beta x}\left(\frac{x}{\beta} - \frac{1}{\beta^2} + \frac{c_1}{E_0}e^{-\beta x} - \frac{A_0}{\beta}\right) \end{Bmatrix} \tag{18}$$

For detailed comparison with the case of homogeneous materials, we may perform limit analysis by setting $\beta \to 0$ for Eq. (18), which leads to

$$\boldsymbol{v}_0^{(3)} \to \begin{Bmatrix} -(1+\nu)h^2 x + \frac{1}{6}(2+\nu)x^3 - \frac{1}{2}xz^2 \\ \frac{1}{2}\nu x^2 z + \frac{1}{6}z^3 \\ -E_0 xz \\ \frac{1}{2}E_0(x^2 - h^2) \end{Bmatrix} \tag{19}$$

The above degenerated result accords with the one corresponding to zero eigenvalue for homogeneous materials (Yao and Zhong 2002), for which the physical interpretation is the shear-bending in the *x*-*z* plane. Therefore, a new group of particular eigenvectors for eigenvalues zero are consist of $\boldsymbol{\Phi}_{0,1}^{(0)}$, $\boldsymbol{\Phi}_{0,1}^{(1)}$, $\boldsymbol{\Phi}_{0,2}^{(0)}, \boldsymbol{\Phi}_{0,2}^{(1)}, \boldsymbol{\Phi}_{0,2}^{(2)}$ and $\boldsymbol{\Phi}_0^{(3)}$. Moreover, their eigensolutions actually correspond to the classical solutions of the Saint-Venant problem.

2.4 Particular solutions

The particular solutions for FGM beams subjected to external loads on the lateral surfaces can

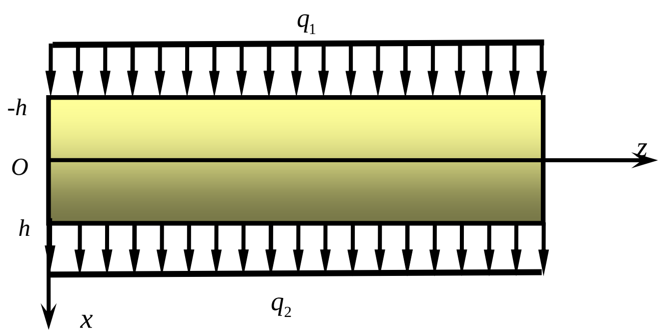


Fig. 2 The rectangular domain with transverse loads at lateral surfaces

be determined from the Jordan normal form solution. Consider the plane rectangular domain as below

The conditions at the upper and lower surfaces are

$$\begin{aligned} x=-h: \ & E_0\frac{\partial u}{\partial x}+\nu\sigma=q_1\mathrm{e}^{\beta h}, \quad \tau=0 \\ x=h: \ & E_0\frac{\partial u}{\partial x}+\nu\sigma=q_2\mathrm{e}^{-\beta h}, \quad \tau=0 \end{aligned} \tag{20}$$

A particular solution of $\boldsymbol{H}\tilde{\boldsymbol{\Phi}}=\boldsymbol{k}\boldsymbol{\Phi}_0^{(3)}$ can be solved as follows

$$\tilde{\boldsymbol{\Phi}}=\begin{Bmatrix} \tilde{w}=0 \\ \tilde{u}=k\left\{\frac{1}{24}\nu^2x^4-\frac{1}{6}\nu^2A_0x^3+(1-\nu^2)\left[\left(c_1x+\frac{c_1}{\beta}-c_3\right)\frac{\mathrm{e}^{-\beta x}}{E_0\beta}-\frac{1}{2\beta^2}x^2+(\frac{2}{\beta^3}+\frac{A_0}{\beta^2})x\right]\right. \\ \left.-\frac{2\nu(1+\nu)}{\beta}\left[\frac{1}{6}x^3-\frac{1}{2\beta}x^2-\frac{A_0}{2}x^2+\frac{c_1}{E_0\beta}\mathrm{e}^{-\beta x}\right]-c_2\nu\right\} \\ \tilde{\sigma}=kE_0\left\{(c_3-c_1x)\frac{\nu}{E_0}\mathrm{e}^{-\beta x}+c_2+\frac{1}{2}\nu A_0x^2-\frac{1}{6}\nu x^3+\frac{2\nu}{\beta^3}+\frac{A_0\nu}{\beta^2}-\frac{\nu}{\beta^2}x\right. \\ \left.+\frac{2(1+\nu)}{\beta}\left[\frac{1}{2}x^2-\frac{1}{\beta}x-A_0x-\frac{c_1}{E_0}\mathrm{e}^{-\beta x}\right]\right\} \\ \tilde{\tau}=0 \end{Bmatrix} \tag{21}$$

in which the coefficients can be expressed as $k=\dfrac{q_1-q_2}{2\left[c_1h+\dfrac{E_0}{\beta^2}h\cosh(\beta h)-\left(\dfrac{2E_0}{\beta^3}+\dfrac{A_0E_0}{\beta^2}\right)\sinh(\beta h)\right]}$

and $c_3=\dfrac{1}{k\sinh(\beta h)}\left[\dfrac{q_1\mathrm{e}^{\beta h}-q_2\mathrm{e}^{-\beta h}}{2}-kh\left(\dfrac{E_0}{\beta^2}+c_1\cosh(\beta h)\right)\right]$.

The particular solution of the plane problem can be constructed as

$$\tilde{\boldsymbol{v}}=\boldsymbol{M}_t\left\{\tilde{\boldsymbol{\Phi}}+k\left[z\boldsymbol{\Phi}_0^{(3)}+\left(\frac{1}{2}z^2\boldsymbol{\Phi}_{0,2}^{(2)}+\frac{1}{6}z^3\boldsymbol{\Phi}_{0,2}^{(1)}+\frac{1}{24}z^4\boldsymbol{\Phi}_{0,2}^{(0)}\right)+A_0\left(\frac{1}{2}z^2\boldsymbol{\Phi}_{0,1}^{(1)}+\frac{1}{6}z^3\boldsymbol{\Phi}_{0,1}^{(0)}\right)\right]\right\} \tag{22}$$

It should be emphasized that particular solutions of the FGM beams can be solved for arbitrary form of external normal forces along *z*-axis, such as uniform loads, linear distribution loads and cosinusoidal loads applying on the upper and lower surfaces. In the numerical examples below we present the complete analytical solutions for FGM beams subjected to uniform and cosinusoidal normal tractions at the top and bottom surfaces.

Following the similar procedure, particular solutions can be deduced from $\boldsymbol{H}\tilde{\boldsymbol{\Phi}}=\boldsymbol{k}\boldsymbol{\Phi}_{0,1}^{(1)}$ for external shear tractions along *z*-axis.

2.5 General eigensolutions

Similar to the analysis procedure for bi-directional FGM (Zhao *et al*. 2012c), we assume that there is no Jordan form solution for the general eigenvalue $\mu\neq 0$. Thus, the general solution (11) is constructed according to the distinct roots $\eta_1\neq\eta_2\neq\eta_3\neq\eta_4$. From Eq. (7) and Eq. (11), we can obtain the relations between coefficients A_i, B_i, C_i and D_i as follows

$$\begin{cases} A_i = \dfrac{1}{E_0\mu(\eta_i+\beta)}\left[\nu\mu - \dfrac{(\eta_i+\beta)^2}{\mu}\right]D_i \\ B_i = -\dfrac{1}{E_0\eta_i(\eta_i+\beta)}\left[\mu - \dfrac{\nu(\eta_i+\beta)^2}{\mu}\right]D_i, \qquad (i=1,2,3,4) \\ C_i = -\dfrac{\eta_i+\beta}{\mu}D_i \end{cases} \tag{23}$$

Thus, we have only four independent constants D_i (i=1,2,3,4) to be determined.

The free boundary conditions on the lateral surfaces lead to the following equations satisfied by the constants D_i (i=1,2,3,4)

$$\begin{bmatrix} e^{\eta_1 h} & e^{\eta_2 h} & e^{\eta_3 h} & e^{\eta_4 h} \\ e^{-\eta_1 h} & e^{-\eta_2 h} & e^{-\eta_3 h} & e^{-\eta_4 h} \\ \dfrac{e^{\eta_1 h}}{(\eta_1+\beta)} & \dfrac{e^{\eta_2 h}}{(\eta_2+\beta)} & \dfrac{e^{\eta_3 h}}{(\eta_3+\beta)} & \dfrac{e^{\eta_4 h}}{(\eta_4+\beta)} \\ \dfrac{e^{-\eta_1 h}}{(\eta_1+\beta)} & \dfrac{e^{-\eta_2 h}}{(\eta_2+\beta)} & \dfrac{e^{-\eta_3 h}}{(\eta_3+\beta)} & \dfrac{e^{-\eta_4 h}}{(\eta_4+\beta)} \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{Bmatrix} = 0 \tag{24}$$

To make sure the nontrivial solutions of Eq. (24) exist, the coefficients determinant must vanish, which yields a transcendental equation in term of the general eigenvalues μ

$$\begin{aligned} &(\eta_1-\eta_2)(\eta_3-\eta_4)+(\eta_1-\eta_4)(\eta_2-\eta_3)\cosh[(\eta_1-\eta_2-\eta_3+\eta_4)h] \\ &+(\eta_1-\eta_3)(\eta_4-\eta_2)\cosh[(\eta_1-\eta_2+\eta_3-\eta_4)h]=0 \end{aligned} \tag{25}$$

Along with Eq. (10), the general eigenvalue μ_n can be obtained from Eq. (25) using appropriate numerical method. Thus, a nontrivial solution D_i (i=1,2,3,4) can be deduced from Eq. (24).

$$\left.\begin{aligned} D_{1n} &= E_0\mu_n^2 \\ D_{2n} &= \frac{(\eta_{2n}+\beta)(\eta_{3n}-\eta_{1n})}{(\eta_{1n}+\beta)(\eta_{2n}-\eta_{3n})}\frac{e^{(\eta_{1n}-\eta_{4n})h}-e^{(\eta_{4n}-\eta_{1n})h}}{e^{(\eta_{2n}-\eta_{4n})h}-e^{(\eta_{4n}-\eta_{2n})h}}D_{1n} \\ D_{3n} &= \frac{(\eta_{3n}+\beta)(\eta_{2n}-\eta_{1n})}{(\eta_{1n}+\beta)(\eta_{3n}-\eta_{2n})}\frac{e^{(\eta_{1n}-\eta_{4n})h}-e^{(\eta_{4n}-\eta_{1n})h}}{e^{(\eta_{3n}-\eta_{4n})h}-e^{(\eta_{4n}-\eta_{3n})h}}D_{1n} \\ D_{4n} &= \frac{(\eta_{4n}+\beta)(\eta_{2n}-\eta_{1n})}{(\eta_{1n}+\beta)(\eta_{2n}-\eta_{4n})}\frac{e^{(\eta_{1n}-\eta_{3n})h}-e^{(\eta_{3n}-\eta_{1n})h}}{e^{(\eta_{3n}-\eta_{4n})h}-e^{(\eta_{4n}-\eta_{3n})h}}D_{1n} \end{aligned}\right\} \tag{26}$$

where the subscript n indicates the *n-th* general eigenvalue.

The eigenvector $\boldsymbol{\Phi}_n$ for each μ_n is then determined and its eigensolution can be obtained as

$$\boldsymbol{v}_n = \boldsymbol{M}_t e^{\mu_n z}\boldsymbol{\Phi}_n \tag{27}$$

where $\boldsymbol{\Phi}_n$ is expressed in Eq. (11).

2.6 Complete analytical solution

The complete analytical solution of a loaded beam is formed by zero eigensolutions, general eigensolutions and particular solution corresponding to nonhomogeneous boundary conditions on the lateral surfaces. It should be emphasized that each general eigenvalue is a complex number whose eigensolution decaying with distance from the end of the beam. The effects of these eigensolutions are usually covered by the well-known Saint-Venant principle. All the general eigenvalues can be divided into two groups as follows

$$\begin{aligned} &(\mathrm{A}) \quad \mu_i, \quad \mathrm{Re}(\mu_i) < 0 \quad (i = 1, 2, \cdots, n) \\ &(\mathrm{B}) \quad \mu_{-i}, \quad \mathrm{Re}(\mu_i) > 0 \quad (i = 1, 2, \cdots, n) \end{aligned} \tag{28}$$

In the above classification, μ_{-i} is the symplectic conjugate adjoint of μ_i. The eigenvalues with negative real part correspond to the eigensolutions decaying along the positive z-direction. Meanwhile, the eigensolutions in group B decay along the negative z-direction.

The complete solution can be expressed as

$$\boldsymbol{v} = \tilde{\boldsymbol{v}} + m_1\boldsymbol{v}_{0,1}^{(0)} + m_2\boldsymbol{v}_{0,1}^{(1)} + m_3\boldsymbol{v}_{0,2}^{(0)} + m_4\boldsymbol{v}_{0,2}^{(1)} + m_5\boldsymbol{v}_{0,2}^{(2)} + m_6\boldsymbol{v}_{0}^{(3)} + \sum_{i=1}^{N}(a_i\boldsymbol{v}_i + b_i\boldsymbol{v}_{-i}) \tag{29}$$

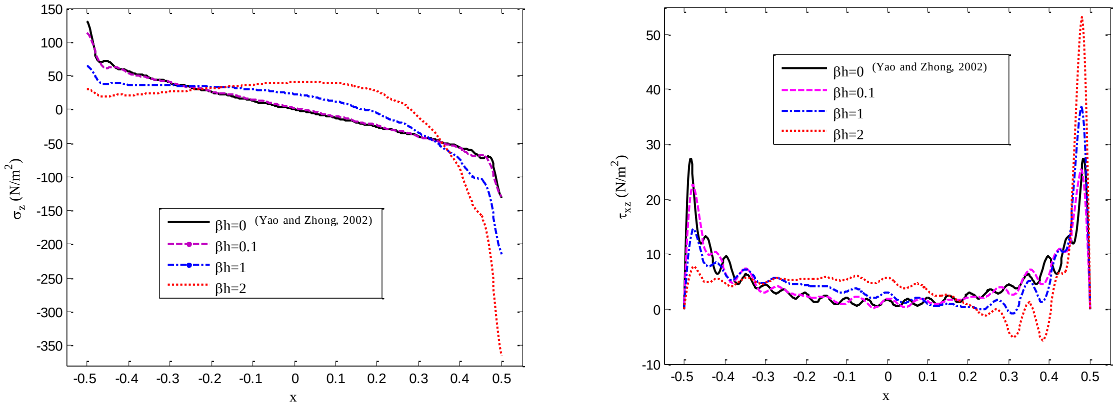
In the expansion series above, $\tilde{\boldsymbol{v}}$ is the particular solution for certain lateral loads applying on the upper and lower surfaces, $\boldsymbol{v}_i$, $\boldsymbol{v}_{-i}$ represent A-set and B-set eigensolutions, respectively. N is a truncated number that should be large enough to ensure the accuracy of the symplectic expansion. The constants m_i (i=1,2,…,6), a_i and b_i can be determined by a linear system of equations resulted from the Hamiltonian variational principle (Yao and Zhong 2002). Similar as that of Zhao and Chen (2008), B-set eigensolutions in the symplectic expansion series are rewritten to avoid the overflow problem.

3. Numerical examples

3.1 Example 1

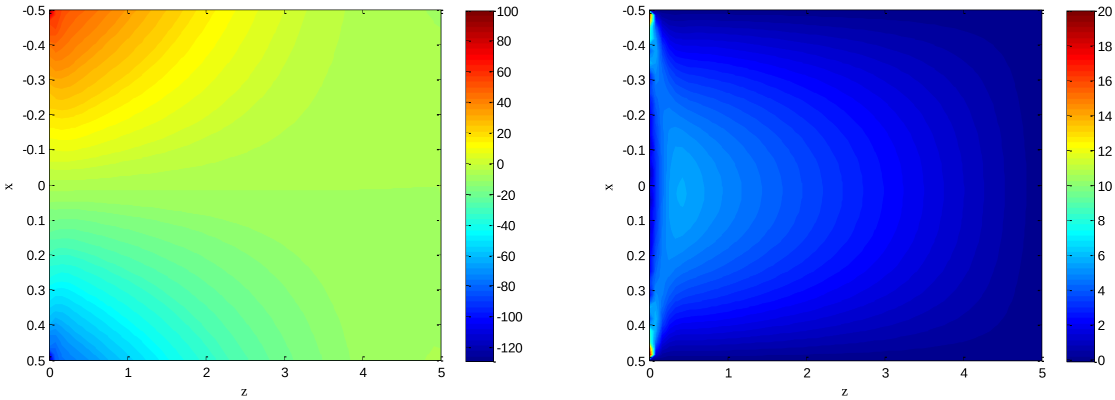
The clamped-free beam of thickness $2h$=1 m and length-to-thickness ratio $l/(2h)$=5 is subjected to prescribed normal tractions $q_1=q_2=0.5$ N/m^2 at its upper and lower surfaces (Fig. 2). The Young's modulus varies exponentially along x with its value at x=0 being $E_0=2.0\times10^{11}$ N/m^2, and the Poisson's ratio v=0.29 keeps constant. The material gradient index takes three values as βh=0.1, βh=1 and βh=2.

Fig. 3 shows the normal and shear stress distributions through the thickness of the beam at the clamped end. The stress distributions corresponding to the homogeneous material are given for comparison (Yao and Zhong, 2002). Seen from Figs. 3(a)-(b), with the material inhomogeneity parameter βh decreasing (βh=2, βh=1, βh=0.1), the stress distributions gradually tend to those of homogeneous material (βh=0). It is verified that the symplectic framework is suitable for obtaining highly accurate local stress distributions, which are completely covered up the Saint-Venant principle employed in the conventional analytical methods.



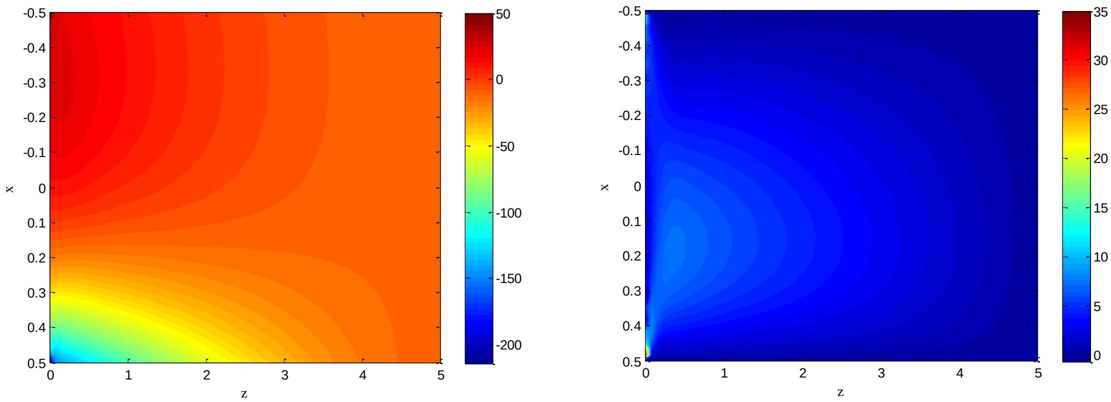
(a) σ_z (b) τ_{xz}

Fig. 3 Stress distributions at the clamped end



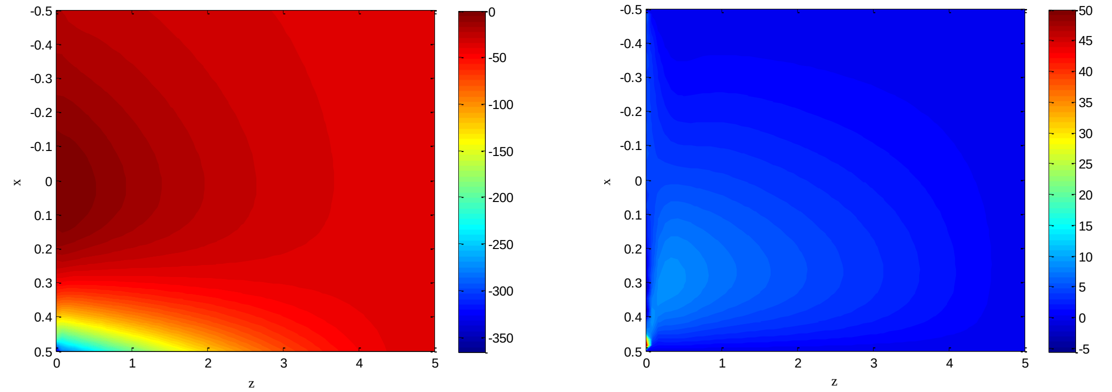
(a) σ_z (b) τ_{xz}

Fig. 4 Contours of normal and shear stresses (βh=0.1)



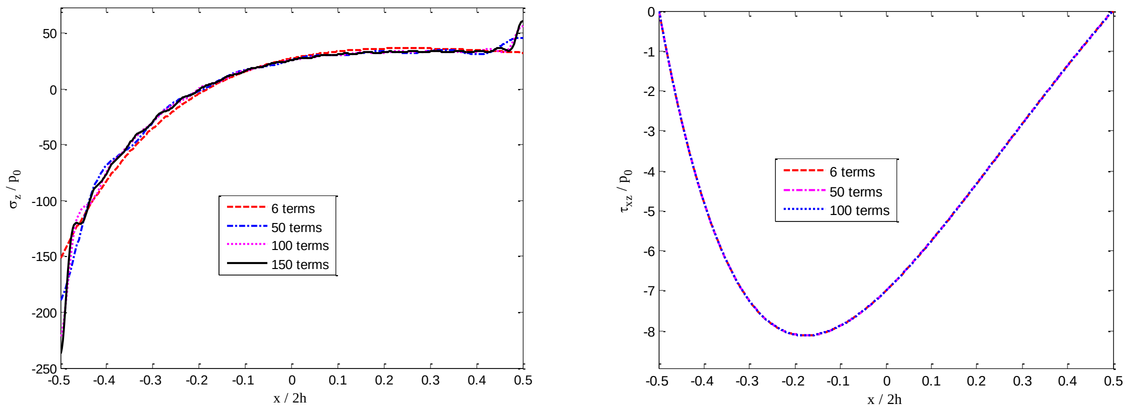
(a) σ_z (b) τ_{xz}

Fig. 5 Contours of normal and shear stresses (βh=1)



(a) σ_z (b) τ_{xz}

Fig. 6 Contours of normal and shear stresses (βh=2)



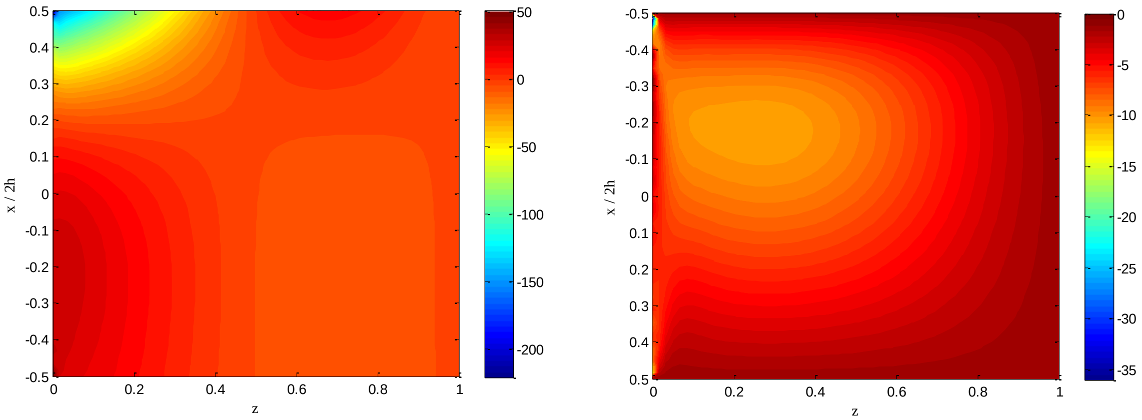
(a) The normal stress σ_z/p_0 at z=0 (b) The shear stress τ_{xz}/p_0 at $z=l/2$

Fig. 7 The normal and shear stress distributions along the thickness direction

Figs. 4-6 depict the contours of the analytical normal and shear stress distributions for the cantilever FGM beams with βh=0.1, 1 and 2, respectively. For both normal and shear stresses, the local accurate behaviors near the clamped end can be displayed evidently. It can also be seen that the gradient index βh has evident effects on the normal and shear stresses at the vicinity of clamped end. The contours of normal stress σ_z are dissymmetrical about x=0 for functionally graded materials which are different from that of homogeneous materials. Furthermore, with the material gradient increasing (βh=0.1, 1 and 2), the dissymmetry tends to be more evident. There are similar properties for the contours of shear stress τ_{xz}. With the high accurate local stress distributions and stress contours, the beam can be optimally designed according to certain loaded conditions.

3.2 Example 2

Consider the cantilever FGM beam with the length l=1 m and the thickness $2h$=0.2 m,



(a) σ_z/p_0 (b) τ_{xz}/p_0

Fig. 8 Contours of normal and shear stresses

subjected to a cosinusoidal normal traction $p(z)=-p_0\cos(\pi z/l)$ (p_0=0.1 MPa) at its upper surface. We assume the Young's modulus at the upper and lower surfaces being $E(h)=-10$ MPa, $E(h)=1$ MPa and $E_0=\sqrt{10}\text{MPa}$, meanwhile, the Poisson's ratio v=0.3 keeps constant. Then, the material gradient index is taken to be $\beta=-5\ \ln 10$. (Nie *et al*. 2003)

Figs. 7(a)-(b) depict the distributions of the normal stress σ_z at the clamped end and the shear stress τ_{xz} at $z=l/2$ with certain truncated terms in the symplectic expansion, respectively. It is obvious that the general eigensolutions play a significant role in the accurate local stress distributions, especially at the corner of the clamped end. However, it has little effect on the shear stress far away from the clamped end which is not covered up by Saint-Venant principle. The stress distributions accord with the results presented by Nie *et al*. (2013) which employed the displacement function approach with one term remained in the Fourier Cosine series.

To further illustrate the complete stress distributions of the beam, Fig. 8(a) depicts the contour of the normal stress while Fig. 8(b) depicts the shear stress contour. To consider local effects, one hundred expansion terms in Eq. (29) are concerned in the calculation. Within the symplectic framework, the whole normal and shear stress field of the beam can be obtained accurately. Also the local behaviors are displayed around the clamped end which is usual covered up by Saint-Venant principle.

4. Conclusions

The complete stress distributions are presented for functionally graded beams subjected to arbitrary lateral loads (either normal or shear load) in the framework of symplectic analysis. The Young's modulus of the beam is assumed to vary exponentially along the thickness direction and the Poisson's ratio keeps constant. In the symplectic framework, a particular solution of the FGM beam is obtained which satisfies the lateral boundary conditions. The complete solution is obtained by superposing the particular solution and the eigen-solutions. Numerical results show that the symplectic approach is effective in predicting accurate local stress distributions and exhibiting obvious singularity behavior at the corner of the clamped end. Also, complete stress distribution contours of the FGM beams are displayed for the first time.

Along with the plane analysis of axially directional and bi-directional functionally graded beams, this paper further completes analytical solutions of functionally graded plane beams with exponential FGM model within the framework of symplectic elasticity. It should be pointed that, the similar analytical procedure is valid for the complete stress field of axial-directional and bi-directional FGMs beams subjected to arbitrary tractions on the lateral surfaces.

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