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Reliability-based design optimization of structural systems using a hybrid genetic algorithm

Reza Abbasnia*, Mohsenali Shayanfar and Ali Khodam

School of Civil Engineering, Iran University of Science and Technology, P.O. Box 16765-163, Narmak, Tehran, Iran

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Abstract. In this paper, reliability-based design optimization (RBDO) of structures is addressed. For this purpose, the global search and optimization capabilities of genetic algorithm (GA) are combined with the efficiency and reasonable accuracy of an advanced moment-based finite element reliability method. For performing RBDO, three variants of GA including a real-coded, a binary-coded and an improved binary-coded GA are developed. In these methods, GA performs (finite element) reliability analyses to evaluate reliability constraints. For truss structures which include finite element modeling, reliability constraints are evaluated using finite element reliability analysis. Response sensitivity required for finite element reliability analysis is obtained by direct differentiation method (DDM) rather than finite difference method (FDM). The proposed methods are examined within four standard test examples and real-world design problems. The results illustrate the superiority and efficiency of the improved binary-coded GA. Results also illustrate that DDM significantly reduces the computational cost and improves the efficiency of the optimization procedure.

Keywords: reliability-based optimization; finite element reliability analysis; genetic algorithm; optimization of structures; direct differentiation method

1. Introduction

Design of structures accounting for probability concepts necessitates the development of appropriate automatic procedures to solve the RBDO problems. Enevoldsen and Sorensen (1994) have presented several formulations for RBDO problem. They have solved the formulated problems using reliability index approach (RIA). Tu *et al*. (1999) have presented an approach denoted performance measure approach (PMA) in which the probabilistic constraints are defined in terms of performance measure obtained by inverse first-order reliability method (iFORM) approach (Togan *et al*. 2011). Since these approaches are very time-consuming, a number of methods such as single loop methods (Agarwal *et al*. 2007) and decoupling approaches (Der Kiureghian and Polak 1998, Royset *et al*. 2001, Aoues and Chateauneuf 2010) have been developed to improve the efficiency of RBDO. For more comprehensive review on the approaches to the RBDO problem, see (Valdebenito and Schueller 2010).

*Corresponding author, Associate Professor, E-mail: abbasnia@iust.ac.ir

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As seen in the literature, most of the methods presented for solving the RBDO problems are based on traditional optimization methods including gradient-based mathematical programming and hill-climbing algorithms (Chen *et al*. 2013, Cho and Lee 2011, Royset *et al*. 2006, Lee and Lee 2005). In the situations where the objective function is not continuously differentiable or has multiple local optima, these methods are ineffective. In comparison, genetic algorithm (GA) due to its probabilistic nature does not encounter these difficulties. Moreover, high computational effort required for solving the RBDO problem necessitates the use of parallel computing in order to render the application of RBDO feasible for practical problems (Valdebenito and Schueller 2010). This can be accomplished by means of the methods that are amenable to parallelization. GA is amenable to parallel computing due to its exceptional ability to be parallelized. These advantages of GA motivate us to develop three hybrid optimization methods for solving reliability-based structural design problems in which GA is used as an optimization tool.

In the last decades, GA has been successfully used for a wide range of difficult problems in structural engineering due to its simple implementation and robust performance. A GA methodology for reliability-based design optimization of truss structures has been proposed by Dimou and Koumousis (2003). In their research, system reliability estimation is performed using Ditlevsen bounds. Shayanfar *et al*. (2013) recently proposed a GA combined with second-order reliability method (SORM) for mathematical RBDO problems having explicit limit-state functions without finite element modeling.

In this paper, three hybrid methods for reliability-based design optimization of structures that combine GA with reliability and finite element reliability analysis are proposed. For this purpose, three different variants of GA are developed for RBDO problems. The first two variants, namely GA-V1 and GA-V2, have been previously introduced only for deterministic problems and in this paper, we extend them for RBDO problems and study their effectiveness for this particular problem. The third variant (GA-V3) is proposed by the authors as an improved optimization method adjusted for RBDO problems. In this variant of GA, a novel representation method is introduced in order to improve the performance of the hybrid method. In general, because of their unacceptable computational costs, simulation methods may not be efficient tools for reliability analysis within the RBDO problems. Unlike the simulation methods, advanced first-order moment-based methods due to their appealing balance between accuracy and efficiency are appropriate choices for reliability assessment within the optimization procedure (Aoues and Chateauneuf 2010). Therefore, in this work, an advanced first-order moment-based method is used to perform reliability and finite element reliability analysis. Particularly, for truss structures which consist of finite element modeling, reliability assessment is performed by finite element reliability analysis.

GA like other meta-heuristic algorithms requires many function evaluations to search and find the optimum design. Moreover, finite element reliability can be very time-consuming due to repeated structural analyses necessary to compute the gradients by finite difference method. To tackle these problems, in this research, response sensitivity analysis required for finite element reliability is performed by direct differentiation method (DDM) in order to decrease the computational time of optimization procedure. To handle the constraints, the hybrid GAs are integrated with the penalty function method.

The structure of the paper is as follows: In section 2, the definition and formulation of the RBDO problem are presented. Section 3 discusses the issue of reliability and finite element reliability analysis. Response sensitivity analysis and DDM are also discussed in this section. Section 4 addresses the methods for solving the RBDO problem using hybridization of GA and (finite element) reliability analysis. The details of the hybrid methods and implementation issues are discussed in this section. Four numerical examples are given in section 5.

2. Definition and formulation of RBDO problem

2.1 Definition

The formulation of RBDO problems can be presented in different ways (Valdebenito and Schueller 2010). According to (Der Kiureghian and Polak 1998), three different types of reliability-based optimization problems can be defined:

1- Minimization of the cost of the design subject to reliability and deterministic structural constraints

2- Maximization of the reliability of structural design subject to deterministic and cost constraints

3- Minimization of initial cost plus expected cost of failure subject to reliability and deterministic structural constraints.

This paper focuses on the first problem. However, the proposed method could be extended for solving the third problem without additional difficulties.

2.2 Formulation

This paper considers a single-objective optimization problem, where particular constraints of the optimization problem are expressed in terms of the reliability of the design. The basic formulation of RBDO involves minimizing the objective function subject to reliability and deterministic constraints. In this context, the cost of the design (e.g., the total weight of structure) is defined as the objective function. In mathematical terms, typical RBDO problem can be represented as follows

$$
\begin{array}{lll}
\min & C(\mathbf{d}) & \\
\text{subject to} & prob(g_i(\mathbf{d},\mathbf{x}) \le 0) \le \Phi(-\beta_i^t) & i = 1,\ldots, n_{rc} \\
 & h_j(\mathbf{d}) \le 0 & j = 1,\ldots, n_{dc}
\end{array} \tag{1}
$$

where $C(.)$ denotes the cost of the design, $\mathbf{d}$ denotes the vector of design variables, $\mathbf{x}$ is the vector of random variables, $g_i(\mathbf{d},\mathbf{x})$ is the ith performance function, $h_j(.)$ is the jth deterministic constraint, β_i^t is the target reliability index for ith performance function, and n_{rc} and n_{dc} are the number of reliability and deterministic constraints, respectively. Notice that the design variables can be either deterministic variables or parameters of probability distribution such as mean value of random variables.

3. Structural reliability

3.1 Reliability analysis

Formulation of structural reliability problems is usually based on the definition of performance function(s), i.e., $g=g(\mathbf{d},\mathbf{x})$. This function is defined in terms of design variables and uncertain parameters such that $g(.)\leq 0$ represents the failure domain. The probability of failure of a component, p_f, for the design **d** is calculated by the following integral

$$p_f(\mathbf{d}) = \int \underset{g_i(\mathbf{d},\mathbf{x})\leq 0}{\cdots} \int f_{\mathbf{x}}(\mathbf{x})d\mathbf{x} \tag{2}$$

where p_f is the probability of failure, and $f_{\mathbf{x}}(\mathbf{x})$ is the joint PDF of **x**. Since the multi-dimensional integral of Eq. (2) does not have closed-form solution except for a few simple cases, some approximate methods have been developed. These methods consist of first-order reliability method (FORM), second-order reliability method (SORM), advanced simulation methods, and response surface methods (Basaga *et al*. 2012, Jiang *et al*. 2014).

Among the various available methods, FORM due to its appropriate balance between accuracy and computational cost is a suitable choice for reliability assessment within the RBDO problems (Aoues and Chateauneuf 2010). FORM usually needs 5-10 evaluations of performance function in order to get sufficiently accurate probability approximation (Koduru and Haukaas 2010). For this reason, FORM is widely used for reliability assessment within RBDO problems.

The first step in FORM is to transform the random variables of the problem into uncorrelated standard normal random variables. There exist various transformations in the literature which can be used in FORM. Rosenblatt and Nataf are two important transformations. Nataf presented by Der Kiureghian and Liu (1986) is an approximate transformation which requires marginal distributions and correlations of random variables. This transformation used in this research can be applied to almost all such reliability analyses.

After transforming the input random variables, in the space of standard normal random variables, limit-state surface is replaced by a hyper-plane tangent at the closest point to the origin. The point of tangency, denoted by $\mathbf{y}^*$, is called "design point" or the most probable failure point (MPP). The reliability index, β, is defined as the Euclidean distance of design point (MPP) from the origin in the standard normal space. Therefore, structural reliability problem is converted to a constraint optimization problem for finding the design point expressed as

$$\begin{aligned} &\beta = \min && \|\mathbf{y}\| \\ &\text{subject to} && g(\mathbf{d}, \mathbf{x}) = 0 \end{aligned} \tag{3}$$

The reliability index and probability of failure are related by following formula

$$p_f = \Phi(-\beta) \tag{4}$$

where Φ denotes the standard normal cumulative distribution function.

Finding the design point can be performed by standard nonlinear optimization algorithms such as the gradient projection method, the augmented Lagrangian method, the sequential quadratic programming method, etc. However, one of the most popular algorithms for solving the optimization problem of Eq. (3) is the well-known HL-RF algorithm. The HL-RF algorithm is an iterative scheme based on Newton-Raphson root finding approach. This algorithm can be expressed in mathematical terms as

$$\beta^{(j)} = \frac{-\nabla_{\mathbf{y}} g}{\|\nabla_{\mathbf{y}} g\|}\mathbf{y}^j + \frac{g(\mathbf{y}^j)}{\|\nabla_{\mathbf{y}} g\|} \tag{5}$$

$$\mathbf{y}^{(j+1)} = [\frac{-\nabla_{\mathbf{y}} g}{\|\nabla_{\mathbf{y}} g\|}\mathbf{y}^{j} + \frac{g(\mathbf{y}^{j})}{\|\nabla_{\mathbf{y}} g\|}]\frac{-(\nabla_{\mathbf{y}} g)^{T}}{\|\nabla_{\mathbf{y}} g\|} \tag{6}$$

where $\nabla_{\mathbf{y}} g$ is the gradient vector of performance function $g(.)$ with respect to the transformed random variables.

The HL-RF algorithm has been improved by adding a line search scheme. It is proved that the improved algorithm, denoted by iHL-RF, is unconditionally convergent. In particular, in this paper, iHL-RF is used to find the design point in structural reliability and finite element reliability analysis.

3.2 Finite element reliability analysis

Finite element reliability merges advanced reliability methods with finite element analysis to estimate the probability of failure for predefined performance functions (Haukaas and Der Kiureghian 2007). Finite element method (FEM) enters the reliability analysis through the definition of performance function. A typical performance function for truss structures has the following form

$$g(\mathbf{x}) = u(\mathbf{x}) - u_0 \tag{7}$$

where $u(\mathbf{x})$ denotes a response quantity obtained by FEM and u_0 is a response threshold. Notice that $u(\mathbf{x})$ is a function of random variables. According to Eqs. (5) and (6), to perform reliability analysis for performance function of Eq. (7) by iHL-RF, it is necessary to compute the gradients of performance function with respect to random variables. The required gradient has the following form

$$\frac{\partial g}{\partial \mathbf{y}} = \frac{\partial g}{\partial u}\frac{\partial u}{\partial \mathbf{x}}\frac{\partial \mathbf{x}}{\partial \mathbf{y}} \tag{8}$$

In the above equation, $\partial g/\partial u$ is easily computed and $\partial \mathbf{x}/\partial \mathbf{y}$ is the Jacobian of Nataf transformation. The remaining gradient, $\partial u/\partial \mathbf{x}$, needs to be obtained by finite element method and represents the response sensitivity.

However, a significant drawback is the high computational time required for repeated structural analyses to obtain this gradient using finite difference method (FDM). Moreover, FDM might lead to inaccurate results depending on the size of perturbation.

Direct differentiation method (DDM) is an attractive alternative to FDM for sensitivity analysis within the reliability assessment. DDM includes deriving the analytical differentiation of the governing equations of structural response. The sensitivity equations are then implemented along with finite element response equations and calculated with the same precision. Therefore, unlike FDM, DDM provides the sensitivity of structural response without repeated analyses of structures for perturbed value of each random parameter. This leads to the efficiency of sensitivity algorithm which motivates us to employ DDM for sensitivity calculations of reliability assessment within the RBDO procedure proposed for truss structures.

The structural response is obtained by solving the equations of static equilibrium. These equations have the general form

$$P_{\text{int}}(\mathbf{u}(\mathbf{X}), \mathbf{X}) = P_{ext}(\mathbf{X}) \tag{9}$$

where P_{ext} and P_{int} are the external and internal force vector, and **X** denotes the vector of random parameters of structures including material, geometric and load parameters and **u**(**X**) denotes the nodal displacement vector. The derivative of Eq. (9) with respect to a single parameter **x** in **X** is

$$\mathbf{K}_T \frac{\partial \mathbf{u}}{\partial \mathbf{x}} + \left.\frac{\partial \mathbf{P}_{\text{int}}}{\partial \mathbf{x}}\right|_{\mathbf{u}\,Fixed} = \frac{\partial \mathbf{P}_{ext}}{\partial \mathbf{x}} \tag{10}$$

where

$$\mathbf{K}_T = \frac{\partial \mathbf{P}_{\text{int}}}{\partial \mathbf{u}} \tag{11}$$

where $\mathbf{K}_T$ is the tangent stiffness matrix equal to the partial derivative of internal force vector with respect to nodal displacements, $\left.\frac{\partial \mathbf{P}_{\text{int}}}{\partial \mathbf{x}}\right|_{\mathbf{u}\,Fixed}$ is the conditional derivative of internal force vector with respect to parameter **x** while the nodal displacements are held fixed, and $\frac{\partial \mathbf{P}_{ext}}{\partial \mathbf{x}}$ is equal to zero except in the situation where **x** presents a nodal force.

Therefore, the nodal response sensitivity is obtained by solving the following system of linear equations

$$\frac{\partial \mathbf{u}}{\partial \mathbf{x}} = \mathbf{K}^{-1}{}_T \left(\frac{\partial \mathbf{P}_{ext}}{\partial \mathbf{x}} - \left.\frac{\partial \mathbf{P}_{\text{int}}}{\partial \mathbf{x}}\right|_{\mathbf{u}\,Fixed}\right) \tag{12}$$

More details about DDM equations and the solution procedures can be found in (Kleiber *et al*. 1997).

4. Proposed hybrid genetic algorithms for RBDO

This section describes three hybrid approaches in which, GA is combined with finite element reliability analysis to perform reliability-based design optimization. More precisely in the proposed approaches GA is used as an optimization tool to minimize the cost of structural design while it performs finite element reliability analyses in order to evaluate the reliability constraints.

In the following subsections three variants of GA, denoted by GA-V1, GA-V2 and GA-V3, are developed for RBDO problems with real-valued design variables. The general features of these variants are borrowed from the standard GA and include the following steps: 1) Encoding and decoding. 2) Initialization 3) Selection. 4) Crossover. 5) Mutation. 6) Fitness evaluation and constraint handling (see Fig. 1). On the other hand each variant utilizes different versions of the encoding, crossover and mutation steps which are proposed based on the nature of the current problem of reliability-based design. The constraints will be handled by penalty function method which transforms the constrained optimization problem into an unconstrained problem.

The first step of a GA is generating an initial population. Before starting the iterative loop of the GA, the initial population of individuals is randomly generated. The population size should be selected in a way to provide a balance between diversification of the population and efficiency of the algorithm. The fitness value of each individual in the population is evaluated in all generations. A roulette wheel approach is then used to select the individuals for reproduction. As it is not

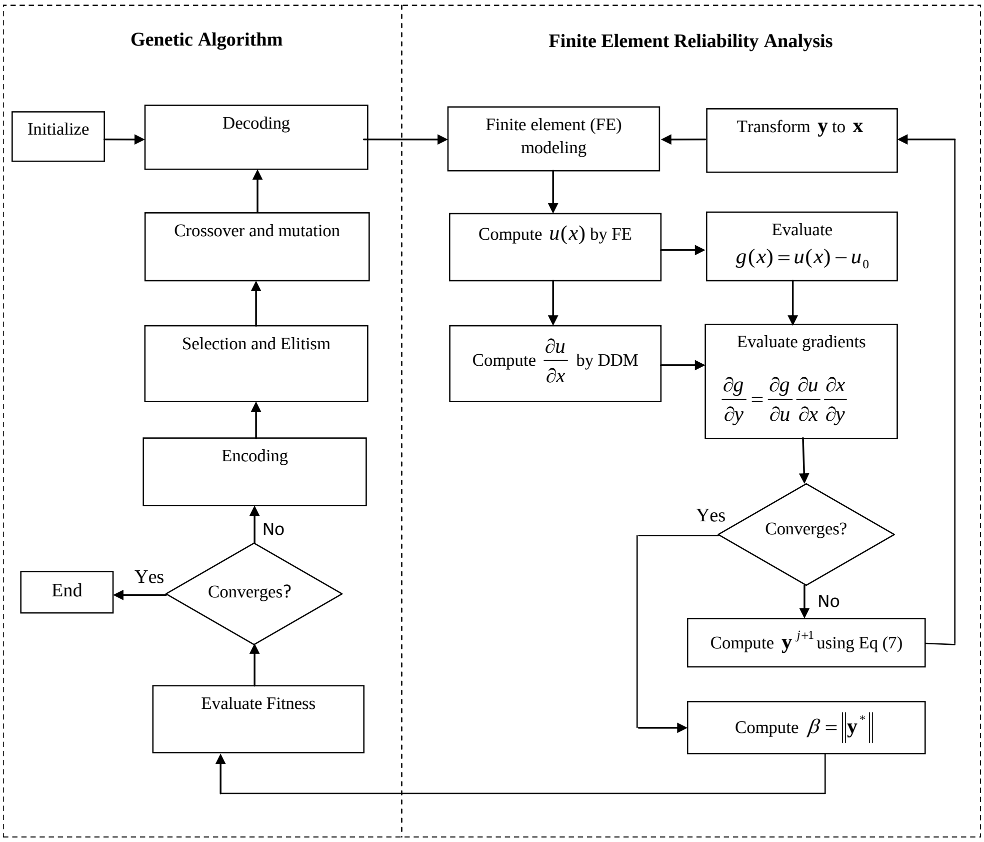

Fig. 1 Schematic representation of the hybrid optimization methodology

guaranteed to select the fittest individual by using the roulette wheel approach, an elitist strategy is also employed in the proposed algorithm. Using this strategy, the best five individuals of the current population are selected and directly carried over to the next generation.

After that, Crossover and mutation operators are applied to the selected individuals to construct the next population. Mutation operator is used to prevent the algorithm to be trapped into a local minimum. In addition, mutation prevents permanent loss of genetic data and also produces new data during the optimization process.

4.1 GA-V1: a real-coded GA

In the real-coded GA, design variables are applied as real-valued variables. Therefore, unlike the other variants there is no need to encode and decode the design variables.

In this variant of GA, an arithmetic crossover is employed to create two new offspring. The basic idea of this method is taken from convex set theory. In this method, the resulting offspring, c_1 and c_2, are produced using linear combination of two parents, P_1 and P_2, as given bellow

$$\begin{aligned} c_1 &= \alpha P_1 + (1-\alpha) P_2 \\ c_2 &= (1-\alpha) P_1 + \alpha P_2 \end{aligned} \tag{13}$$

where α is a random number uniformly distributed between 0 and 1.

GA-V1 utilizes a nonlinear mutation operator. For this purpose, Consider an individual $x=(x_1,x_2,...,x_k,...,x_n)$, a component of **x**, *xk*, is randomly selected and replaced by x'_k obtained as follows

$$x'_k=\begin{cases} x_k+\Delta\left(t,x_k^u-x_k\right) & \text{if random binary digit is 0} \\ x_k-\Delta\left(t,x_k-x_k^l\right) & \text{if random binary digit is 1} \end{cases} \tag{14}$$

where x_k^u and x_k^l are upper and lower bounds of x_k. The function $\Delta(t, dx)$ is defined as

$$\Delta(t,dx)=dx.r.\left(1-\frac{t}{T}\right)^b \tag{15}$$

In the above equation, t is the generation number, T denotes the maximum number of generation and b is the parameter that determines the nonlinearity assumed as b=3, and r is a real random number between 0 and 1.

4.2 GA-V2: a binary-coded GA

In GA-V2, a binary encoding method is utilized. In this method, each design variable is transformed into a set of binary digits and a total combination of these sets forms an individual. Since the design variables of the problem are real-valued, for encoding and decoding a binary string, the following equation is used (Shayanfar *et al*. 2013)

$$C=C_{\min}+\frac{B}{2^L}(C_{\max}-C_{\min}) \tag{16}$$

where C is the real value the string represents, $C_{\max}$ and $C_{\min}$ are respectively upper and lower bounds of the design variable, B denotes the decimal value of the binary string, and L is the length of binary string. The required accuracy specifies the length of the binary string. In this paper, the length of binary strings used for encoding each design variable is taken to 10.

The RBDO problems usually have multiple design variables. Encoding the multiple design variables is performed by concatenating each of encoded design variables next to each other. For example, consider a problem with two design variables *w* and *t* where each of the design variables is encoded by a 10-digit string. In this case, an individual is formed by concatenating two encoded design variables and contains 20 digits. This process is shown schematically in Fig. 2.

In the GA-V2, a multi-point crossover method with three cutting points is applied to the parents. In multi-point crossover, first, 3 points are randomly chosen along the binary bits of

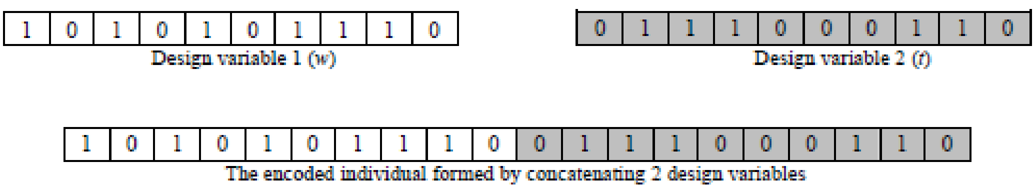

Fig. 2 Schematic representation of encoding process for multiple design variables in GA-V2

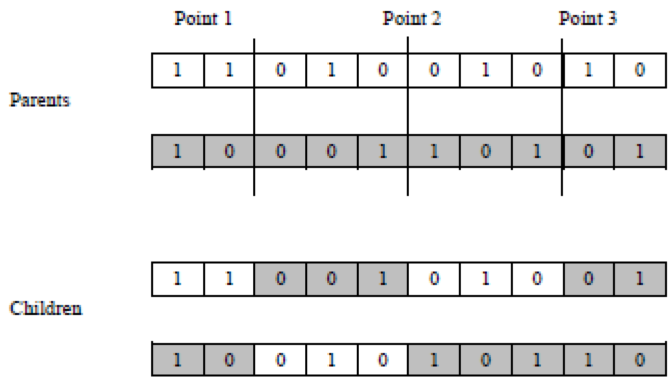

Fig. 3 Schematic representation of multi-point crossover

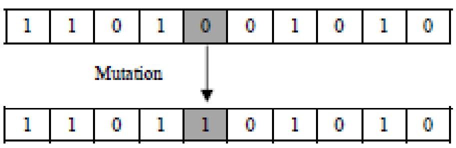

Fig. 4 Schematic representation of bit flip mutation

encoded individuals. Then, the contents between these points are swapped to produce two new children. This process is shown schematically in Fig. 3.

In this variant of GA, with a mutation probability of P_m, a bit flip mutation is performed, in which a binary bit along the encoded individual is randomly selected and flipped as shown in Fig. 4.

4.3 GA-V3: an improved binary-coded GA

The main difference of GA-V3 compared with GA-V2 is in the representation method used for individuals. In GA-V3, a novel representation is introduced in order to improve the performance of the hybrid RBDO method. In this variant of GA, each design variable is represented using two binary strings. The first string, str_1, represents the integer part of the design variable. The length of this string is set to the length of the maximum allowable value of the design variable in base-2 numeral system. The second string, str_2, represents the fractional part of design variable. The length of the second string is specified by the required accuracy. Coding and decoding of a binary string are based on the following formulas

$$\begin{gathered} \mathrm{int}(x_1) = \mathrm{binary2decimal}(\mathrm{str}_1) \\ \mathrm{frac}(x_1) = \frac{B}{2^L} \\ x_1 = \mathrm{int}(x_1) + \mathrm{frac}(x_1) \end{gathered} \tag{17}$$

where x_1 is the real valued design variable, binary2decimal(.) is an operator that converts the binary string to its decimal value which equals integer part of the design variable x_1. B is decimal

Section 1 (integer parts)		Section 2 (fractional parts)	
Design Variable 1	Design variable 2	Design Variable 1	Design variable 2
1 0 0 1 0 1	0 0 1 1 0 1	1 1 1 0 1 10 1	0 1 1 0 0 1 1 0

Fig. 5 Schematic representation of encoding for problems with 2 design variables in GA-V3

value of the second binary string, and L is the length of str_2 set to 8.

In this method, each design solution (individual) including multiple parameters consists of two sections. The first section is formed by concatenating integer parts of all design variables and the second section is constructed by concatenating fractional parts of all design variables. This process for a problem with 2 design variables is illustrated in Fig. 5.

In the GA-V3, like GA-V2 the multipoint crossover with 3 cutting points chosen randomly along the binary string and bit flip mutation are applied to the selected individuals for reproduction.

4.4 Fitness evaluation and constraint handling

To evaluate the fitness value of individuals, first, each one is decoded to a set of real-valued design variables. However, as mentioned above, in the case of GA-V1 there is no need to encode the design variables. Reliability analysis is then performed by iHL-RF method for each individual. In each generation of iHL-RF method, it is necessary to evaluate the limit state function and its derivatives with respect to random variables. As another novelty in RBDO problem, for truss structures, response sensitivity analysis required for finite element reliability is performed by DDM. Since DDM does not need to repeated structural analysis for computing the gradients, the computational cost of RBDO significantly decreases.

The hybrid methods use the penalty function approach to handle the constraints. Therefore, poor fitness values will be assigned to the infeasible solutions by adding a penalty term to the objective function. Using a quadratic penalty function, the constrained optimization problem of Eq. (1) is converted into the following unconstrained optimization problem

$$\min \qquad \text{fit}(D) = C(D) + \alpha . \sum_{1}^{N} P_i^{\,2} \tag{18}$$

where P_i is defined in Eq. (19), D is an individual corresponding to the vector of design variables x, fit(D) is the fitness of the individual D, $C(D)$ is the cost of the design, N is the number of reliability constraints, β_i is the reliability index of ith performance function, $\beta_{allowale,i}$ is the minimum allowable reliability index corresponding to the ith performance function, and α is a penalty coefficient.

$$P_i = \left(1 - \frac{\beta_i}{\beta_{allowable,i}}\right)^{+} = \max\left(1 - \frac{\beta_i}{\beta_{allowable,i}}, 0\right) \tag{19}$$

For minimization problems, it is required to rescale the fitness values using the following equation

$$\text{fit}_{\text{rescaled}}(D) = 2F_{\max} - \text{fit}(D) \tag{20}$$

Table 1 Pseudo-code of the hybrid optimization method

Notation

N_{itr} : Number of allowed generation
Size : Size of population
P_c : probability of crossover
P_m : probability of mutation
P : Current population of individuals
SelectBest (**P**) : An operator that returns the best five individuals in the population **P**
$\mathbf{P}_{\mathbf{Mate}}$: A set of offspring produced by mating operators
RouletteWheel (**P**,k) : An operator that returns *k* individuals of population **P** based on roulette wheel approach
$\varnothing$: An empty set
$\leftarrow$: Assignment operator

Pseudo-code

```
//Initialization
Generate random initial population, P
Decode individuals in P to design parameters
Perform reliability analysis for each individual in P using (finite element) reliability analysis
Evaluate fitness values of individuals in P using Eq. (18)
//Main Loop
FOR i=1 to N_itr
        P_Mate ← ∅
        best ← SelectBest(P)   // elitist strategy
        // Mating (Crossover and Mutation)
        FOR j=1 to (P_C×Size)
                (X1,X2) ← RouletteWheel (P,2)
                (Y1,Y2) ← MultiPointCrossover (X1,X2)
                        // Mutation
                        IF  rand (0,1) < P_m
                                Y1 ← FlipMutation (Y1)
                        END IF
                        IF  rand (0,1) < P_m
                                Y2 ← FlipMutation (Y2)
                        END IF
                        P_Mate ← (Y1, Y2) ∪ P_Mate
        END FOR
Decode individuals in P_Mate to design parameters
Perform reliability analysis for each individual in P_Mate using (finite element) reliability analysis
Evaluate fitness values of individuals in P_Mate using Eq. (18)
Pool ← P_Mate ∪ P
P ← ∅
P ← RouletteWheel(Pool,Size-5) ∪ best
END FOR
Return P            \\ Final result
```

where the parameter $F_{\max}$ represents the maximum fitness of the individuals in the current population. Notice that the coefficient of 2 in Eq. (20) is added to improve the diversity of the population and prevent the algorithm from premature convergence. This coefficient has been obtained by a trial and error process.

4.5 Implementation of hybrid RBDO method

In this research, first, GA generates randomly an initial population of N individuals. The reliability (finite element reliability) analysis for each individual, i, is then performed by iHL-RF method. In the case of truss structure, the gradients are obtained by DDM within finite element reliability analysis. The cost of the design and the penalty terms obtained using the reliability indices are utilized to calculate the fitness value for each individual according to Eq. (18). As seen in pseudo-code in Table 1, this procedure is repeated for each generation until the given number of generations is met.

5. Numerical examples

In GAs the parameter values such as population size, crossover and mutation probability, and the number of elite individuals are usually tuned by trial and error. The parameters tuned for the current problem are as follows: the size of population is 200, the probability of crossover is 0.7, the probability of mutation is 0.1, and the number of elite individuals is 5. The penalty coefficient, α, in Eq. (18) is set to 10^4.

The hybrid optimization procedures have been implemented in MATLAB™ 7.8. Optimization runs have been performed on a PC with 2.2 GHz Intel Core 2 Duo processor and 2GB of RAM memory. The solutions have been compared with the results recently published in the literature. To evaluate the accuracy of reliability constraints at optimum design, Monte Carlo simulation (Schueller 2009) with 10^7 sample size is utilized.

5.1 Example 1: mathematical problem with multiple nonlinear limit-state functions

The first example is a mathematical problem (Chen *et al*. 2013, Cho and Lee 2011) with two random design variables and three nonlinear reliability constraints. The random variables are statistically independent and normal. The formulation of the problem is

$$
\begin{aligned}
&\text{find} \;\; \mathbf{d} = [d_1\, d_2]^T \\
&\min \;\; f(\mathbf{d}) = d_1 + d_2 \\
&\text{subject to} \quad prob\left(g_1 = \frac{X_1^{\,2} X_2}{20} - 1 \le 0\right) \le \Phi\left(-\beta_1^{\,t}\right) \\
&\qquad\qquad\quad prob\left(g_2 = \frac{\left(X_1 + X_2 - 5\right)^2}{30} - \frac{\left(X_1 - X_2 - 12\right)^2}{120} - 1 \le 0\right) \le \Phi\left(-\beta_2^{\,t}\right) \\
&\qquad\qquad\quad prob\left(g_3 = \frac{80}{\left(X_1^{\,2} + 8X_2 + 5\right)} - 1 \le 0\right) \le \Phi\left(-\beta_3^{\,t}\right) \\
&0 \le d_i \le 10.0 \;\; , i = 1,2 \\
&X_i \sim N(d_i, 0.3^2), \;\; i = 1,2. \\
&\beta_i^{\,t} = 3.0 \qquad i = 1,2,3
\end{aligned} \tag{21}
$$

The optimization results are listed in Table 2 where the first row of the table is taken from (Chen *et al*. 2013) for comparison. Table 2 shows that the proposed method (GA-V3) leads to the best solution in comparison to other variants of GA implemented in this research and other methods in the literature. In this table, β_i^{MCS} stands for reliability index of *i*th performance function calculated using Monte Carlo simulation (MCS) with 10^7 sample size to confirm whether the reliability constraint is satisfied. Since the 3th reliability constraint is inactive in optimum, the corresponding reliability index is infinite. The results also show that GA-V3 leads to the better results compared with other variants of GA implemented in this research.

5.2 Example 2: welded beam design

This example (Chen *et al*. 2013) investigates the reliability-based optimization of a welded beam shown in Fig. 6. The problem has four random design variables and five reliability constraints. In this example the objective function is welding cost and reliability constraints are imposed on quantities such as maximum allowable stress, buckling and tip displacement. The random variables are normal and statistically independent. The deterministic parameters of the system are given in Table 3. The RBDO problem is formulated as

$$\text{find } \mathbf{d} = [d_1\, d_2\, d_3\, d_4]^T$$

$$\min\ f(\mathbf{d},\mathbf{z}) = c_1 d_1^{\,2} d_2 + c_2 d_3 d_4 (z_2 + d_2)$$

$$\text{subject to} \qquad prob(g_i \le 0) \le \Phi\left(-\beta_i^{\,t}\right) \qquad i = 1,2,3,4,5$$

where

$$g_1(\mathbf{X},\mathbf{z}) = \frac{\tau(\mathbf{X},\mathbf{z})}{z_6} - 1 \quad , \; g_2(\mathbf{X},\mathbf{z}) = \frac{\sigma(\mathbf{X},\mathbf{z})}{z_7} - 1 \quad , \; g_3(\mathbf{X},\mathbf{z}) = \frac{X_1}{X_4} - 1$$

$$g_4(\mathbf{X},\mathbf{z}) = \frac{\delta(\mathbf{X},\mathbf{z})}{z_5} - 1 \quad , g_5(\mathbf{X},\mathbf{z}) = 1 - \frac{P_c(\mathbf{X},\mathbf{z})}{z_1}$$

$$3.175 \le d_1 \le 50.8 \;\; , 0.0 \le d_2 \le 254.0 \;\; , 0.0 \le d_3 \le 254.0 \;\; , 0.0 \le d_4 \le 50.8$$

$$\tau(\mathbf{X},\mathbf{z}) = \left\{ \frac{t(\mathbf{X},\mathbf{z})^2 + 2t(\mathbf{X},\mathbf{z})^2\, tt(\mathbf{X},\mathbf{z}) X_2}{2R(\mathbf{X})} + tt(\mathbf{X},\mathbf{z})^2 \right\}^{\frac{1}{2}}$$

$$t(\mathbf{X},\mathbf{z}) = \frac{z_1}{\sqrt{2} X_1 X_2}, \qquad tt(\mathbf{X},\mathbf{z}) = \frac{M(\mathbf{X},\mathbf{z}) R(\mathbf{X})}{J(\mathbf{X})}$$

$$M(\mathbf{X},\mathbf{z}) = z_1\left(z_2 + \frac{X_2}{2}\right) \quad , R(\mathbf{X}) = \left(\frac{\sqrt{X_2^2 + (X_1 + X_3)^2}}{2}\right)$$

$$J(\mathbf{X}) = \sqrt{2} X_1 X_2 \left\{ \frac{X_2^{\,2}}{12} + \frac{(X_1 + X_3)^2}{4} \right\}$$

$$\sigma(\mathbf{X},\mathbf{z}) = \frac{6 z_1 z_2}{X_3^{\,2} X_4} \;,\; \delta(\mathbf{X},\mathbf{z}) = \frac{4 z_1 z_2^{\,3}}{z_3 X_3^{\,3} X_4}$$

$$P_c(\mathbf{X},\mathbf{z}) = \frac{4.013 X_3 X_4^{\,3} \sqrt{z_3 z_4}}{6 z_2^2} \left(1 - \frac{X_3}{4 z_2}\sqrt{\frac{z_3}{z_4}}\right)$$

$$X_i \sim N(d_i, 0.1693^2), \qquad i = 1,2$$

$$X_i \sim N(d_i, 0.0107^2), \qquad i = 3,4$$
$$\beta_i^t = 3.0 \qquad i = 1,2,3,4,5 \tag{22}$$

Table 4 presents the results of the optimization problem. Reliability constraints are also evaluated using MCS with a sample size of 10^7 and listed in Table 4. The results show that in comparison with other variants, GA-V3 leads to better results. It can be seen that the improved

Table 2 Summary of results for example 1

RBDO Methods	Design variables		Objective	β_1^{MCS}	β_2^{MCS}	β_3^{MCS}
	d_1	d_2				
Chen *et al*. (2013)	3.4391	3.2866	6.7260	2.97	3.05	infinite
GA-V1	3.4287	3.3007	6.7294	2.95	3.11	infinite
GA-V2	3.4306	3.3013	6.7319	2.96	3.10	infinite
GA-V3	3.4351	3.2815	6.7166	2.97	3.04	infinite

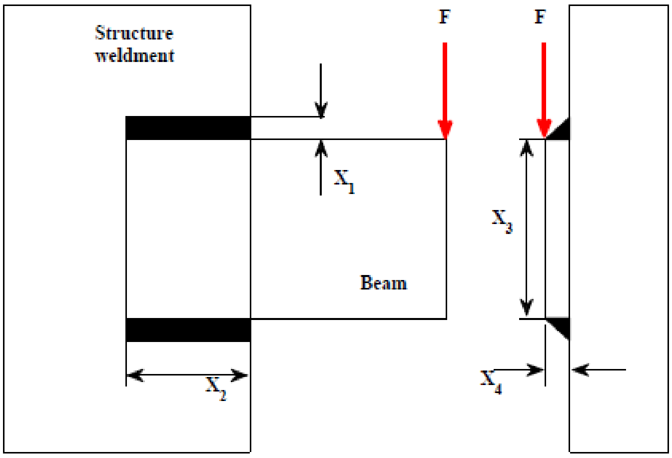

Fig. 6 Welded beam structure

Table 3 Deterministic parameters of the system for welded beam problem

z_1	2.6688×10^4 (N)
z_2	3.556×10^2 (mm)
z_3	2.0685×10^5 (MPa)
z_4	8.274×10^4 (MPa)
z_5	6.35 (mm)
z_6	9.377×10 (MPa)
z_7	2.0685×10^2 (MPa)
c_1	6.74135×10^{-5} (\$/mm^3)
c_2	2.93585×10^{-6} (\$/mm^3)

Table 4 Summary of results for example 2

RBDO Methods	Objective	β_1^{MCS}	β_2^{MCS}	β_3^{MCS}	β_4^{MCS}	β_5^{MCS}
Chen *et al*. (2013)	2.5914	2.99	3.01	2.99	infinite	3.00
Lee and Lee (2005)	3.09	-	-	-	-	-
GA-V1	3.1893	4.11	Infinite	3.29	Infinite	Infinite
GA-V2	2.6904	2.96	Infinite	3.01	Infinite	infinite
GA-V3	2.5928	2.98	3.10	3.14	Infinite	3.84

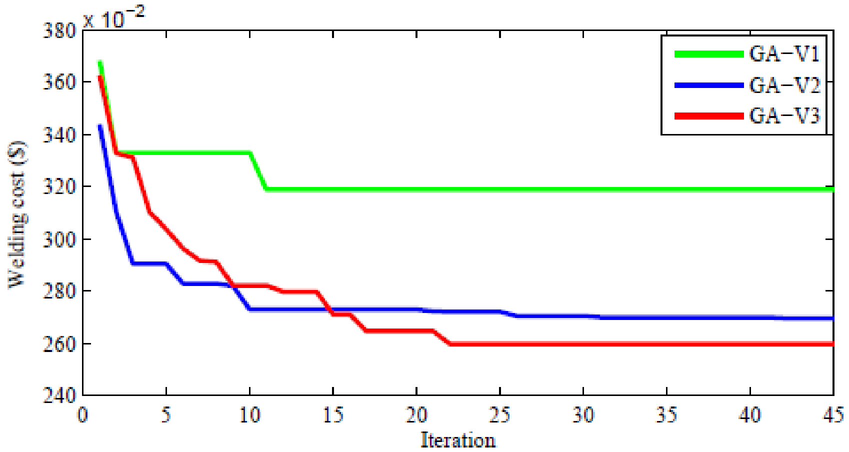

Fig. 7 Convergence history for example 2

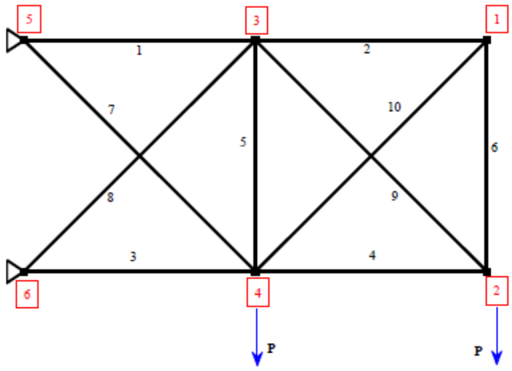

Fig. 8 A ten-bar truss structure

proposed GA (GA-V3) outperforms the GA-V1 and GA-V2 by 22.86% and 3.64%, respectively. Convergence history is illustrated in Fig. 7. This figure shows that GA-V3 achieves the best answer in near 20 generations.

5.3 Example 3: ten-bar truss

The 10-bar truss shown in Fig. 8 includes six nodes and ten truss elements constructed from aluminum with the Young's modules of E=10^7 psi. This example and its modified versions are

Table 5 Statistic parameters of ten-bar truss

Random variable	Dist	Mean	S.D
A_1	Normal	d_1	$0.05d_1$
A_2	Normal	d_2	$0.05d_2$
A_3	Normal	d_3	$0.05d_3$
A_4	Normal	d_4	$0.05d_4$
A_5	Normal	d_5	$0.05d_5$
A_6	Normal	d_6	$0.05d_6$
A_7	Normal	d_7	$0.05d_7$
A_8	Normal	d_8	$0.05d_8$
A_9	Normal	d_9	$0.05d_9$
A_{10}	Normal	d_{10}	$0.05d_{10}$

investigated by a number of researchers (Rahman and Wei 2008, Luo and Grandhi 1997).

The truss is under two concentrated load of $P=10^5$ *lb* at the nodes 2 and 4. The truss design is performed by minimizing the weight such that the transverse displacement at node 2 is less than or equal to 2 *in*. The problem has ten random design variables and one reliability constraint. The random variables are statistically independent with the characteristics listed in Table 5.

The RBDO problem can be formulated as follows:

$$
\begin{aligned}
&\text{find} \quad \mathbf{d} = [A_1\ A_2\ A_3\ A_4\ A_5\ A_6 A_7\ A_8\ A_9\ A_{10}]^T \\
&\min \quad W = \rho \sum_{i=1}^{10} A_i L_i \qquad i = 1,2,...,10 \\
&\text{subject to} \qquad prob(g(X) \le 0) \le \Phi\left(-\beta^{\,t}\right) \\
&\qquad\qquad\qquad 0.10 \le \mathrm{A_i} \le 35.0 \\
&\text{where} \\
&\mathrm{g(X)} = u_2 - 2 \\
&\beta^{\,t} = 3.09
\end{aligned}
\tag{23}
$$

where A_i is the cross-sectional area of *i*th element of truss, L_i is the length of *i*th element, ρ is the material density equal to 0.1 *lb/in*^{3}, W denotes the total weight of the truss, and u_2 is the vertical displacement at the node 2.

The results of the optimization problem are listed in Table 6. The reliability index for the performance function at the optimum design is calculated using MCS with the sample size of 10^7 to verify the validity of solutions and presented in Table 6. The results show that the proposed method outperforms the other approaches in finding the minimum design without violating the reliability constraint. GA-V3 achieves the results that are 25%, 6.2% and 1.8% better than the GA-V1, GA-V2, and the method used by Luo and Grandhi (1997), respectively. Convergence history is illustrated in Fig. 9. This figure shows that the GA-V3 achieves the best answer in near 100 generations. In addition, the figure indicated that the GA-V3 has the higher convergence rate in comparison to GA-V1 and GA-V2.

In this example, the limit-state function for reliability assessment is an implicit function of random variables. Here, the reliability constraint of the optimization problem is evaluated using finite element reliability analysis. Gradients of structural responses required for finite element

Table 6 Summary of results for example 3

Design Variables	RBDO Methods			
	Luo and Grandhi 1997	GA-V1	GA-V2	GA-V3
A_1	33.955	24.6390	32.5461	35.00
A_2	0.7092	11.4374	1.2929	0.116
A_3	23.436	29.0491	20.7878	23.516
A_4	16.066	14.9618	19.9357	17.921
A_5	0.10	2.5213	0.3045	0.1
A_6	0.6626	8.9616	1.7359	0.108
A_7	4.9484	20.0429	6.7801	1.835
A_8	224.101	19.0886	24.0938	23.57
A_9	23.748	17.3400	25.5252	24.611
A_{10}	0.5333	9.2821	0.3045	0.108
Objective function	5412.8 lb	6644.15 lb	5644.6 lb	5315.2 lb
β^{MCS}	2.972	3.217	3.323	3.144

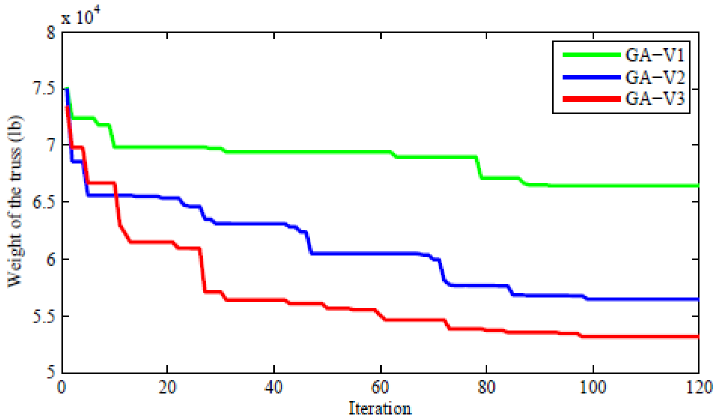

Fig. 9 Convergence history for example 3

reliability analysis are obtained by DDM. In order to compare the efficiency of DDM with FDM, GA-V3 is also implemented in a way that FDM is used for computing the gradients. The numbers of function evaluations necessary for a single reliability assessment by DDM and FDM are 4 and 44, respectively. The comparison shows that the number of function evaluations for a single reliability analysis is reduced by 40 (91 percent) using DDM.

5.4 Example 4: seventy two-bar spatial truss

Reliability-based optimization of 72-bar spatial truss shown in Fig. 10 is considered in this example. The material density is equal to 0.1 lb/in^3 (2767.9907 kg/m^3). The truss structure is subjected to three nodal loads in the *x*, *y*, and *z* directions at node 1. A deterministic version of this example is solved by a number of researchers to evaluate different optimization algorithms (Kaveh *et al*. 2014). The objective is minimization of the weight of the truss under eight reliability constraints. The reliability constraints are imposed on lateral displacements of uppermost nodes of

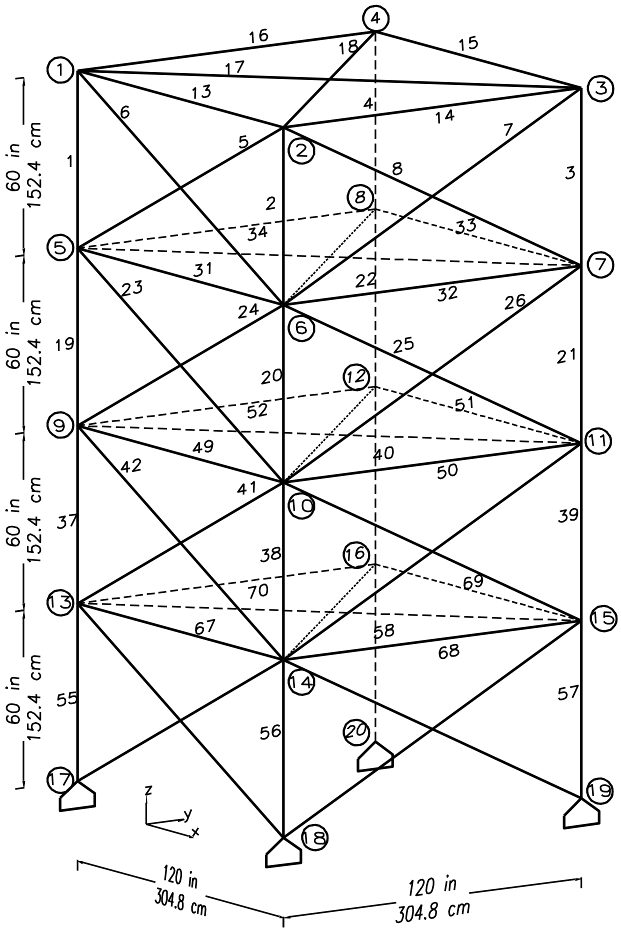

Fig. 10 Seventy two-bar spatial truss

Table 7 Uncertain parameters of 72-bar spatial truss

Parameter	Distribution	Mean	COV
P_X	Lognarmal	5 kip	0.1
P_Y	Lognarmal	5 kip	0.1
P_Z	Lognarmal	-5 kip	0.1
E	Lognarmal	10^4 ksi	0.05
A_i	Normal	Assigned by RBDO method	0.05

the truss in both x and y directions. The uncertain parameters of this problem are given in Table 7. The elements of truss structure are collected in 16 design groups given in table 8. Thus, the RBDO problem has 16 random design variables. The formulation of RBDO problem is as follows:

$$\text{find } \mathbf{d} = [A_1\ A_2\ A_3\ A_4\ A_5\ A_6 A_7\ A_8\ A_9\ A_{10}\ A_{11}\ A_{12}\ A_{13}\ A_{14}\ A_{15}\ A_{16}]^T$$

$$\min\ W = \rho \sum_{i=1}^{16} A_i L_i$$

$$\begin{aligned}
&\text{subject to} \quad prob(g_j(X) \le 0) \le \Phi\left(-\beta^t\right) \qquad j = 1,2,...,8 \\
&\qquad 0.10 \le \mathrm{A_j} \le 8.0 \qquad j = 1,2,...,16 \\
&\text{where} \\
&\mathrm{g_1(X)} = u_{1x} - 0.3;\ \mathrm{g_2(X)} = u_{1y} - 0.3;\ \mathrm{g_3(X)} = u_{2x} - 0.3;\ \mathrm{g_4(X)} = u_{2y} - 0.3; \\
&\mathrm{g_5(X)} = u_{3x} - 0.3;\ \mathrm{g_6(X)} = u_{3y} - 0.3;\ \mathrm{g_7(X)} = u_{4x} - 0.3;\ \mathrm{g_8(X)} = u_{4y} - 0.3; \\
&\beta_{\mathrm{i}}^{t} = 3.0 \qquad i = 1,2,...,8
\end{aligned} \tag{24}$$

where u_{ix} and u_{iy} are the lateral displacement of node *i* in the direction of *x* and *y*, respectively.

The results of optimization problem are listed in Table 8. Reliability indices for all performance functions computed by MCS are shown in Table 9. The results indicate that the GA-V3 outperforms other methods in finding optimum design. However, Table 9 shows that all the methods have some violations due to highly nonlinear behavior of limit state function. Convergence history for all three methods is illustrated in Fig. 11.

Table 10 indicates the reduction in the number of function evaluations obtained by using DDM rather than FDM for examples 3 and 4. As shown in Table 10, for 72-bar truss the numbers of function evaluations required for a single reliability analysis for the first limit state function using

Table 8 Optimization results for 72-bar spatial truss

Elements group	Elements number	GA-V1	GA-V2	GA-V3
1	1-4	0.787	0.153	0.141
2	5-12	1.346	0.939	0.756
3	13- 16	0.250	0.810	0.557
4	17,18	0.353	1.618	0.953
5	19- 22	0.735	0.830	0.672
6	23-30	0.667	1.123	0.838
7	31- 34	0.168	0.426	0.184
8	35, 36	0.168	0.142	0.363
9	37-40	2.364	0.751	1.177
10	41-48	1.234	0.719	0.599
11	49-52	0.104	0.278	0.112
12	53, 54	1.337	0.167	0.906
13	55-58	1.376	1.344	1.691
14	59-66	0.942	0.844	0.872
15	67-70	0.186	0.165	0.378
16	71,72	0.1	0.239	0.146
	Weight	676.6	617.22	556.9

Table 9 Reliability constraints at optimal design evaluated by MCS for spatial truss

RBO methods	β_1^{MCS}	β_2^{MCS}	β_3^{MCS}	β_4^{MCS}	β_5^{MCS}	β_6^{MCS}	β_7^{MCS}	β_8^{MCS}
Proposed Method (GA-V3)	2.95	2.95	3.78	infinite	4.33	4.2618	infinite	3.80
GA-V2	2.85	2.81	3.37	3.64	infinite	3.64	infinite	3.42
GA-V1	2.98	2.97	4.3	infinite	infinite	infinite	infinite	4.35

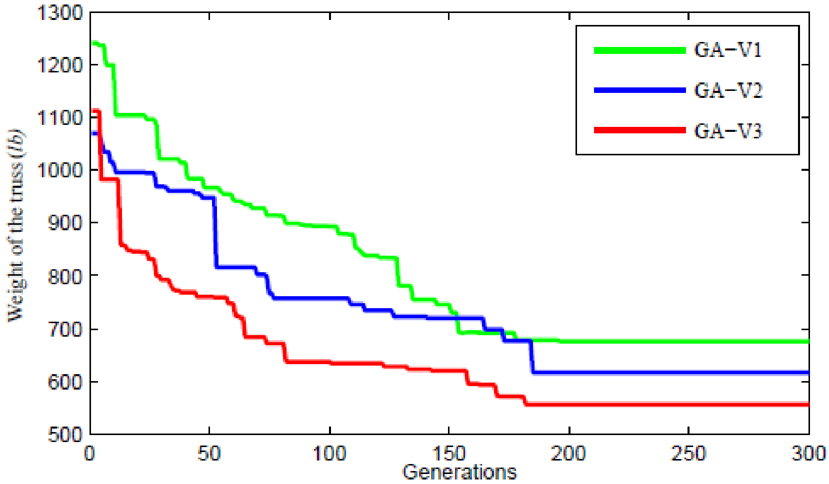

Fig. 11 Convergence history for spatial truss

Table 10 The number of function evaluations (NFE) for a single reliability analysis

truss	Number of random variables	NFE for reliability analysis using Sensitivity method		Reduction in the NFE
		FDM	DDM	
10-bar truss	10	44	4	40 (91 percent)
72-bar truss	20	116	36	80 (69 percent)

DDM and FDM are 36 and 116, respectively. This reveals that the number of function evaluations for a single reliability analysis is reduced by 80 (69 percent) using DDM.

6. Discussion

GA like other meta-heuristic algorithms requires many finite element reliability analyses to deal with RBDO problems. Moreover, finite element reliability analysis needs to perform repeated structural analyses to compute the gradients of structural responses. As it is shown in this research, using DDM to compute the gradients decreases the number of limit-state function evaluations significantly, and consequently there is no need to repeated structural analysis for perturbed values of random parameters. This leads to a significant decrease in computational time.

Since the proposed method employs the penalty function method to handle the constraints, the penalty coefficient has a major effect on the convergence of the solution. The coefficient has been selected by numerical experiments and a trial and error process. According to the numerical experiments performed, selecting a small value for this coefficient leads to the infeasible solution. This is because, in this case, the penalty term is negligible with respect to objective function and eventually has a small contribution in optimizing the fitness function. Moreover, small values of the penalty coefficient cause the proposed GAs to spend a large amount of the search time exploring infeasible regions in the search space. On the other hand, if the penalty coefficient is too high, the solution obtained will be feasible but non-optimal. It is due to the fact that the GA will be pushed into a feasible region quickly and will converge prematurely to a feasible solution which is not near-global optimal solution. In other words, an overly high penalty coefficient discourages the GA from being explorative in the search space.

A significant drawback of any GA-based approach is the computational cost. However, the exceptional ability of the proposed method (GA-V3) to be implemented together with parallel computing techniques opens the possibility of solving time-consuming RBDO problems in significantly reduced time. This is due to independence of each individual in the population.

Since the reliability assessment within the proposed method is performed in a black-box fashion, any available method can be used for reliability assessment. However, the method of reliability analysis within the optimization framework should be selected according to its efficiency. For example, if Monte Carlo method is used for reliability assessment, the computational time for performing RBDO will be unacceptable. On the other hand, reliability assessment within the optimization procedure can be performed by other efficient reliability methods such as SORM and importance sampling.

7. Conclusions

This paper successfully addresses a particular type of reliability-based design optimization problem involving minimization of the cost of structural design subject to reliability and deterministic constraints. To perform RBDO, this research has proposed three hybrid variants of GA (namely GA-V1, GA-V2 and GA-V3) as optimization tools for minimizing the cost of the design. In these methods, GA performs (finite element) reliability analysis using iHL-RF for evaluation of reliability constraints. Response sensitivity required for finite element reliability analysis is performed using DDM rather than FDM. To handle the constraint, the proposed GAs are integrated with the penalty function method.

The proposed GA variants are examined within four numerical examples. Results show the superiority and efficiency of GA-V3 in comparison with other methods. Comparison also shows that utilizing DDM within the finite element reliability analysis leads to a significant decrease in computational time and improves the efficiency of optimization procedure.

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