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Buckling analysis of semi-rigid connected and partially embedded pile in elastic soil using differential transform method

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Abstract. The parts of semi-rigid connected and partially embedded piles in elastic soil, above the soil and embedded in the soil are called the first region and second region, respectively. The upper end of the pile in the first region is supported by linear-elastic rotational spring. The forth order differential equations of both region for critical buckling load of partially embedded and semi-rigid connected pile with shear deformation are established using small-displacement theory and Winkler hypothesis. These differential equations are solved by differential transform method (DTM) and analytical method and critical buckling loads of semi-rigid connected and partially embedded pile are obtained, results are given in tables and graphs are presented for investigating the effects of relative stiffness of the pile and flexibility of rotational spring.

Keywords: differential transform method, semi-rigid connected, partially embedded pile, non-trivial solution, buckling

1. Introduction

The piles partially embedded in the soil are widely used marine, harbor, bridge structures. Due to some manufacturing errors the structural behavior of the connection between beams or plates of these structures and the upper ends of the piles are neither rigid nor flexible. These types of connections are called semi-rigid connections and these connections are modeled mostly by linear-elastic rotational spring. The soil is idealized mostly by Winkler hypothesize in the mathematical models of the piles partially embedded in the soil (Chen 1997). Elastic soil is idealized by Winkler foundation modulus in also this study and effect friction through the pile length is neglected. The analysis of the beams on elastic foundation and elastic buckling of columns, beams, plates and shells have been studied by many researchers in the past. Hetenyi (1995) has studied beams on Winkler foundations. Reddy and Valsangkar (1970) have obtained buckling loads for fully and partially embedded piles using vibration functions and Rayleight-Ritz method. Smith (1979) has obtained discrete element matrices for stability analysis of selender piles. West *et al*. (1997) have neglected shear effect and assumed the coefficient of horizontal subgrade reaction varies linearly with depth and investigated stability of end-bearing piles in elastic foundation. Heelis *et al*. (2004)

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have calculated bucking load of Euler-Bernoulli pile embedded in Winkler foundation. Heelis *et al*. (1999) have investigated the stability of uniform-friction piles in homogeneous and non-homogeneous elastic foundation using a power-series solution and neglecting shear effect. West and Mafi (1984) have determined buckling loads, natural frequencies of Euler-Bernoulli beam rested on elastic supports by using an initial-value numerical method. Chen (1998) has studied Euler-Bernoulli beam resting on elastic foundation using differential quadrature element method. Kumar *et al*. (2007) have calculated buckling capacity of an eccentrically loaded partially embedded reinforced concrete pile in sand using the conventional Davisson and Robinson method. Valsangkar and Pradhanang (1987) have studied the variations of natural circular frequency values of the piles partially embedded in the elastic soil according to relative stiffness, length and buckling load of the piles ignoring the shear effect. Sapountzakis and Kampitsis (2010) have developed a boundary element method for the nonlinear dynamic analysis of Timoshenko beam-columns partially supported on tensionless Winkler foundation. Zhu *et al*. (2011) have studied analysis of nonlinear stability and post-buckling for Euler type beam-column structure located on a nonlinear elastic foundation using the Hamilton variational principle. Yan and Chen (2012) have studied dynamic analysis of semi-rigid connected and partially embedded piles in a two-parameter elastic foundation using reverberation-ray matrix. Vogt *et al*. (2009) have obtained buckling loads of selender piles in soft soil using analytical methods and experimental works. Catal and Alku (1996a) have obtained the second order stiffness matrix of Euler-Bernoulli beam on elastic foundation using analytical method. Catal and Alku (1996b) have calculated vertical displacements of Timoshenko beam on elastic foundation using finite difference equations and matrix-displacement method. Aydogan (1995) has obtained a stiffness matrix for a Timoshenko beam on elastic foundation using differential equation. Li (2001a) has obtained critical buckling load of multi-step cracked columns with shear deformation by using transfer matrix. Li (2001b) has governed differential equation for buckling of a multi-step non-uniform beam. Banarjee and Williams (1994) have investigated the effects of shear deformation on the critical buckling of columns. Yang and Ye (2002) have studied a dynamic elastic load buckling analysis for a pile subjected to an axial impact load using a perturbation technique. Wang *et al*. (2002) have investigated exact stability criteria and buckling loads of Timoshenko columns under intermediate using analytical method. Catal (2002) has obtained fourth order differential equations for free vibration of partially embedded pile in soil. Catal (2006) has studied the variations of natural circular frequency values of the semi-rigid connected piles partially embedded in the elastic soil according to relative stiffness, rigidity factor, length of the piles. Yesilce and Catal (2008) have studied natural circular frequency values of the semi-rigid connected Reddy-Bickford piles embedded in elastic soil using analytical method. Yesilce and Catal (2006) have calculated natural frequency of the piles embedded in the soil having different modulus of subgrade reaction.

The differential transform method (DTM) which was introduced by Zhou in 1986 for the solution of initial value problems in electric circuit analysis is based on Taylor series expansions (Zhou 1986). In recent works, DTM is applied to buckling problems and vibration analysis of continuous systems as beams, columns, piles and plates. DTM is applied to solve a second-order non-linear differential equation that describes the under damped and over damped motion of a system subject to external excitations by Jang and Chen (1997). Malik and Dang (1995) have obtained frequency equations and fundamental frequencies of a prismatic Bernoulli-Euler beam using DTM. Chen and Ho (1996), using differential transform technique proposed a method to solve eigenvalue problems for the free and transverse vibration problems of a rotating twisted Timoshenko beam. Ozdemir and Kaya (2006), flapwise bending vibration of a rotating tapared

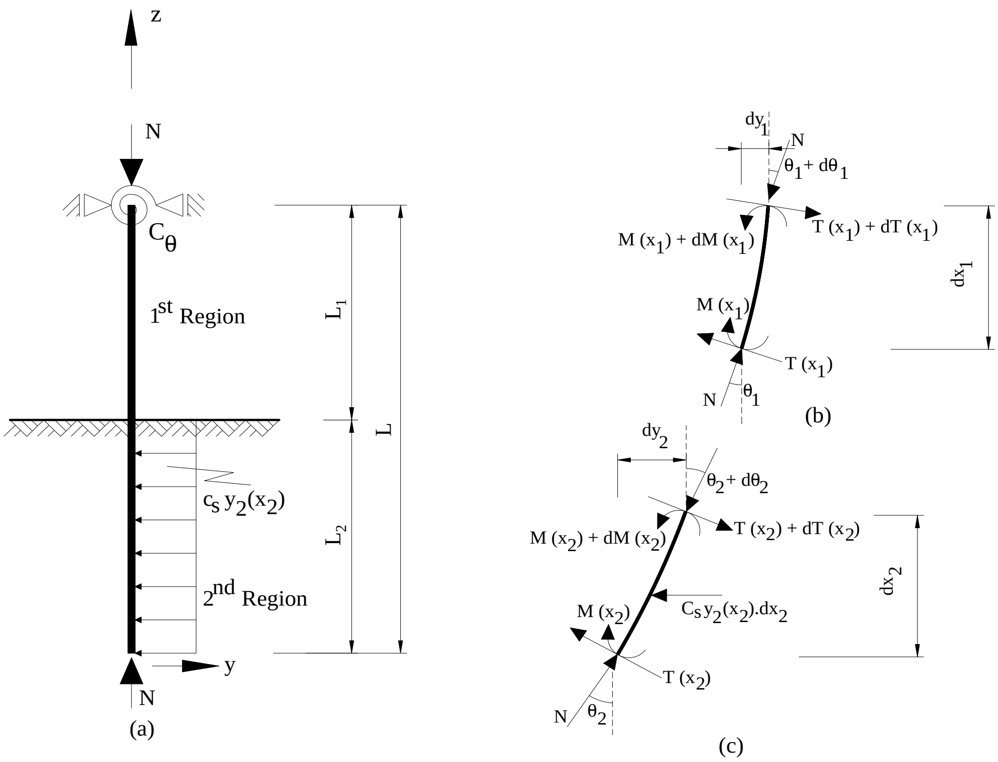


Fig. 1 (a) The semi-rigid connected and partially embedded pile, (b) Internal forces of segment in the first region, (c) Internal forces of segment in the second region

cantilever Bernoulli-Euler beam is considered using DTM. Zou *et al*. (2010) have developed DTM for solving solitary waves by Camassa-Holm equation. Pusjuso and Thongmoon (2010) have presented the definition and operation of the one-dimensional differential transform. Catal and Alku (2006) have calculated buckling load of partially embedded pile in elastic soil using differential transform method (DTM). Yesilce and Catal (2011) have calculated natural frequency values of the semi-rigid connected Reddy-Bickford beams resting on elastic soil using the differential transform method (DTM). Catal (2006) has studied free vibration of the beam on elastic soil using the differential transform method (DTM). DTM is one of the solution methods of ordinary and partial differential equations. DTM has advantage of reducing the ordinary differential equation to the algebraic equation and reducing the partial differential equation to the algebraic equation system. In DTM, the orthogonal polynomials as Taylor series are used for solution of the differential equations and to apply mathematical operations to these polynomials are easier. (Catal and Catal 2006).

In this study, forth-order differential equations of elastic curves for critical buckling load of partially embedded and semi-rigid connected pile are developed, these equations are solved using differential transform method (DTM) and analytical method. Critical buckling loads for the first three modes of the pile are obtained according to relative stiffness, lengths of pile in the first region and second region, the fixity factor. Numerical results are presented and the differential transform solutions are compared with the analytical solutions.

2. Governing equations for buckling of the pile

A pile partially embedded in the soil and semi-rigid connected in Fig. 1(a). The pile parts above the soil and embedded in the soil are called the first region and second region, respectively. The upper end of the pile part in the first region is semi-rigid connected and supported using simple support. The semi-rigid connected is modeled by linear-elastic rotational spring. The internal forces and deformations of the pile having the length of dx_1 and dx_2 at the first and second regions are presented in Fig. 1(b) and Fig. 1(c), respectively.

In this paper the following assumptions are valid: soil behavior is acting in according with the Winkler hypothesis; effect of friction along the pile length is neglected; material behavior of the pile and the spring at the upper end of the pile part in the first region are linear-elastic.

The bending moment functions and fourth order differential equations of the elastic curve functions of the pile in the first and the second region are given in Eqs. (1)-(4) using the equilibrium equations of the lateral load and bending moment acting to segments of the pile (Catal and Catal 2006).

$$M_1(x_1) = -EI\left[1-\frac{N}{\bar{k}AG}\right]\frac{d^2y_1(x_1)}{dx_1^2} \qquad (0 \le x_1 \le L_1) \tag{1}$$

$$M_2(x_2) = -EI\left[1-\frac{N}{\bar{k}AG}\right]\frac{d^2y_2(x_2)}{dx_2^2} + \frac{EIC_s}{\bar{k}AG}y_2(x_2) \qquad (0 \le x_2 \le L_2) \tag{2}$$

$$\frac{d^4y_1(x_1)}{dx_1^4} + \left[\frac{\bar{k}AGN}{(\bar{k}AG-N)EI}\right]\frac{d^2y_1(x_1)}{dx_1^2} = 0 \qquad (0 \le x_1 \le L_1) \tag{3}$$

$$\frac{d^4y_2(x_2)}{dx_2^4} + \left[\frac{\bar{k}AGN-EIC_s}{(\bar{k}AG-N)EI}\right]\frac{d^2y_2(x_2)}{dx_2^2} - \left[\frac{\bar{k}AG}{(N-\bar{k}AG)EI}\right]y_2(x_2) = 0 \quad (0 \le x_2 \le L_2) \tag{4}$$

Where, C_s=C_0. b in which C_0 is the modulus of subgrade reaction and b is width of the pile; $M_1(x_1)$, $M_2(x_2)$, $y_1(x_1)$, $y_2(x_2)$ are bending moment and elastic curve functions for the first and second region, respectively; $\bar{k}$ is the shape factor due to cross-section geometry of the pile; I, A, E, G, N are moment of inertia, cross-section area, modulus of elasticity, shear modulus of the pile and the constant axial compressive force, respectively.

The shape factor of the pile is defined as below (Pöschl 1930)

$$\bar{k} = \frac{A}{I}\int_A \frac{S_x^2}{b^2}dA \tag{5}$$

The shape factor of the pile shown in Fig. 2 is given in Eq. (6) using polar coordinates.

$$\bar{k} = \frac{A}{I^2}\int_0^{2\pi} \frac{S_x^2}{b^2}(tR_0d\theta \tag{6}$$

Where S_x is the first moment of cross-section of the pile, b is thick of pile, R_0 is the average

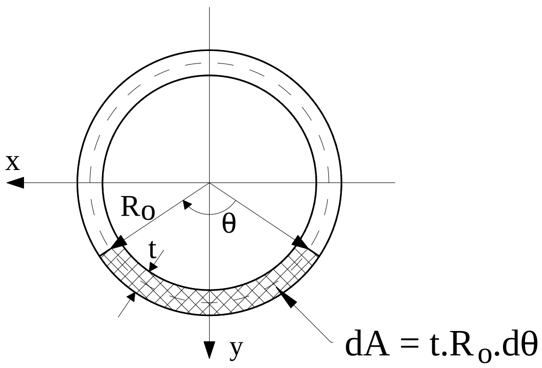


Fig. 2 Cross-section of the pile

radius of the pile.

The cross-section area, the first moment of cross-section, the moment of inertia, and the thick of the pile in Fig. 2 are given Eq. (7) respectively.

$$\left.\begin{aligned} &A = 2\pi R_0 t \\ &S_x = 2\pi R_0^3 t \sin(0{,}5\theta) \\ &I = \pi R_0^3 t \\ &b = 2t \end{aligned}\right\} \tag{7}$$

Substituting Eq. (7) into Eq. (6) respectively, gives

$$\bar{k} = \frac{2\pi R_0 t}{(\pi R_0^3 t)^2}\int_0^{2\pi}\frac{(2R_0^2 t)^2}{(2t)^2}\sin^2(0{,}5\theta)d\theta = 2 \tag{8}$$

Writing the dimensionless parameters z_1, z_2 instead of the position parameters x_1, x_2 in Eqs. (3) and (4) gives the elastic curve differential equations of the pile at the first and the second region as

$$\frac{d^4 y_1(z_1)}{dz_1^4} + D_1 \frac{d^2 y_1(z_1)}{dz_1^2} = 0 \qquad (0 \le z_1 \le \frac{L_1}{L}) \tag{9}$$

$$\frac{d^4 y_2(z_2)}{dz_2^4} + \beta_1 \frac{d^2 y_2(z_2)}{dz_2^2} + \beta_2 y_2(z_2) = 0 \quad (0 \le z_2 \le \frac{L_2}{L}) \tag{10}$$

where $\beta_1 = \frac{(\bar{k}AGN - EIC_s)L^2}{(\bar{k}AG - N)EI}$; $\beta_2 = \frac{L^4\bar{k}AGC_s}{(N - \bar{k}AG)EI}$; $D_1 = \frac{\bar{k}AGNL^2}{(\bar{k}AG - N)EI}$; $\alpha = \frac{C_s L^4}{EI}$ is the relative stiffness of the pile; L_1 is the length of the pile above the soil; L_2 is the length of the pile embedded in the soil; L is the total length of the pile; $z_1 = x_1/L$; $z_2 = x_2/L$.

The rotational spring at the upper end of the pile in the first region are related with fixity factor that is defined as below (Monforton and Wu 1963)

$$f = \frac{1}{1 + \frac{3EI}{L.C_\theta}} \tag{11}$$

Where, C_θ is stiffness of the rotational spring at the upper end of the pile in the first region.

Bending moment at the semi-rigid connected end is written as a linear function of rotational spring stiffness and rotation as follow

$$M_1(z_1 = L_1 / L) = \frac{C_\theta}{L} \frac{dy_1(z_1)}{dz_1}\Big|_{z_1 = L_1 / L} \tag{12}$$

3. Differential transformation

The differential transformation technique, which was first proposed by Zhou in 1986, is one of the numerical methods for ordinary and partial differential equations that use the form of polynomials as the approximation to the exact solutions that are sufficiently differentiable. The function that will be solved and the calculation of following derivatives necessary in the solution become more difficult when the order increases. This is in contrast with the traditional high-order Taylor series method. Instead, the differential transform technique provides an iterative procedure to obtain higher-order series; therefore, it can be applied to the case high order (Catal and Catal 2006).

The differential transformation of the function $y(z)$ is defined as follows

$$Y(k) = \frac{1}{k!}\left[\frac{d^k y(z)}{dz^k}\right]_{z=z_0} \tag{13}$$

Where $y(z)$ is the original function and $Y(k)$ is transformed function which is called the *T*-function. The differential inverse transformation of $Y(k)$ is defined as

$$y(z) = \sum_{k=0}^{\infty} (z - z_0)^k Y(k) \tag{14}$$

from Eq. (13) and Eq. (14) we get

$$y(k) = \sum_{k=0}^{\infty} \frac{(z - z_0)^k}{k!}\left[\frac{d^k y(z)}{dz^k}\right]_{z=z_0} \tag{15}$$

Eq. (14) implies that the concept of the differential transformation is derived from Taylor's series expansion, but the method does not evaluate the derivatives symbolically. However, relative derivative are calculated by iterative procedure that are described by the transformed equations of the original functions.

The basic operations of transformed functions which are given Table 1 can easily be proved using Eqs. (13) and (14).

Table 1 Some basic mathematical operations of DTM

Original function $y(z)$	Transformed function $Y(k)$
$Ay(z)$	$AY(k)$
$y_1(z) \pm y_2(z)$	$Y_1(k) \pm Y_2(k)$
$dy(z)/dz$	$(k+1)\,Y(k+1)$
$d^2y(z)/dz^2$	$(k+1)(k+2)\,Y(k+2)$
$d^3y(z)/dz^3$	$(k+1)(k+2)(k+3)\,Y(k+3)$
$d^4y(z)/dz^4$	$(k+1)(k+2)(k+3)(k+4)\,Y(k+4)$

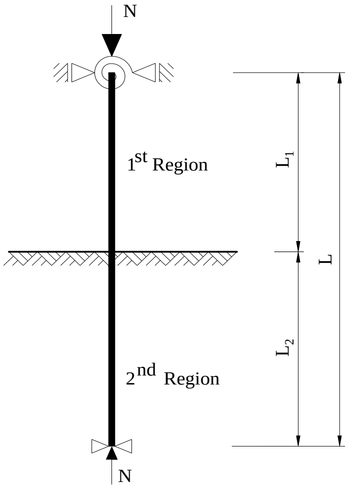


Fig. 3 Semi-rigid connected pile

The function is expressed by finite series and Eq. (14) can be written as $y(z)=\sum_{k=0}^{n}(z-z_0)^k Y(k)$. Eq. (10) implies that $y(z)=\sum_{k=n+1}^{\infty}(z-z_0)^k Y(k)$ is negligibly small. In fact, n is decided by the convergence of buckling load in this paper.

4. Solutions of the equations by differential transformation

The boundary conditions of the pile whose bottom end simply supported, upper end simply supported and semi-rigid connected shown in Fig. 3 are given in Eqs. (16)-(23).

$$y_1(z_1 = L_1 / L) = 0 \tag{16}$$

$$y_2(z_2 = 0) = 0 \tag{17}$$

$$\frac{d^2 y_2(z_2)}{dz_2^2}\Big|_{z_2=0} + \beta_1 y_2(z_2 = 0) = 0 \tag{18}$$

$$\left.\frac{d^2y_1(z_1)}{dz_1^2}\right|_{z_1=\frac{L_1}{L}} + D_1 y_1(z_1 = L_1/L) = -\frac{D_1 C_\theta}{NL}\left.\frac{dy_1(z_1)}{dz_1}\right|_{z_1=\frac{L_1}{L}} \tag{19}$$

$$y_1(z_1 = 0) = y_2(z_2 = L_2/L) \tag{20}$$

$$\left.\frac{dy_1(z_1)}{dz_1}\right|_{z_1=0} = \left.\frac{dy_2(z_2)}{dz_2}\right|_{z_2=\frac{L_2}{L}} \tag{21}$$

$$\left.\frac{d^3y_2(z_2)}{dz_2^3}\right|_{z_2=\frac{L_2}{L}} + \beta_1\left.\frac{dy_2(z_2)}{dz_2}\right|_{z_2=\frac{L_2}{L}} = \left.\frac{d^3y_1(z_1)}{dz_1^{\,3}}\right|_{z_1=0} + D_1\left.\frac{dy_1(z_1)}{dz_1}\right|_{z_1=0} \tag{22}$$

$$\left.\frac{d^2y_2(z_2)}{dz_2^2}\right|_{z_2=\frac{L_2}{L}} + \beta_1 y_2(z_2 = L_2/L) = \left.\frac{d^2y_1(z_1)}{dz_1^2}\right|_{z_1=0} + D_1 y_1(z_1 = 0) \tag{23}$$

By applying the DTM to Eqs. (3), (4), (16), (17) and using the relationship Table 1 following equations are obtained.

$$Y_2(k+4) = -\beta_1\frac{Y_2(k+2)}{(k+3)(k+4)} - \beta_2\frac{Y_2(k)}{(k+1)(k+2)(k+3)(k+4)} \tag{24}$$

$$Y_1(k+4) = -D_1\frac{Y_1(k+2)}{(k+3)(k+4)} \tag{25}$$

$$Y_2(0) = 0 \tag{26}$$

$$Y_2(2) = 0 \tag{27}$$

The recurrence relations of the first region for k=0(1)n are obtained from Eq. (24) using Eqs. (26) and (27) as follows

$$\left.\begin{aligned}
&Y_2(2k) = 0\\
&Y_2(5) = \frac{1}{5!}\{-\beta_1 3!\,Y_2(3) - \beta_2 Y_2(1)\}\\
&Y_2(7) = \frac{1}{7!}\{(\beta_1^2 - \beta_2)3!\,Y_2(3) + \beta_1\beta_2 Y_2(1)\}\\
&Y_2(9) = \frac{1}{9!}\{(-\beta_1^3 + 2\beta_1\beta_2)3!\,Y_2(3) + (-\beta_1^2\beta_2 + \beta_2^2)Y_2(1)\}\\
&Y_2(11) = \frac{1}{11!}\{(\beta_1^4 - 3\beta_1^2\beta_2 + \beta_2^2)3!\,Y_2(3) + (\beta_1^3\beta_2 - 2\beta_1\beta_2^2)Y_2(1)\}\\
&Y_2(13) = \frac{1}{13!}\{(-\beta_1^5 + 4\beta_1^3\beta_2 - 3\beta_1\beta_2^2)3!\,Y_2(3) + (-\beta_1^4\beta_2 + 3\beta_1^2\beta_2^2 - \beta_2^3)Y_2(1)\}\\
&\vdots
\end{aligned}\right\} \tag{28}$$

The recurrence relations of the second region for k=0(1)n are obtained from Eq. (25) as

$$\left.\begin{array}{ll} Y_1(4)=\frac{1}{4!}\{-D_1 2!Y_1(2)\} & Y_1(9)=\frac{1}{9!}\{(-D_1^3)3!Y_1(3)\} \\ Y_1(5)=\frac{1}{5!}\{-D_1 3!Y_1(3)\} & Y_1(10)=\frac{1}{10!}\{(D_1^4)2!Y_1(2)\} \\ Y_1(6)=\frac{1}{6!}\{(D_1^2)2!Y_1(2)\} & Y_1(11)=\frac{1}{11!}\{(D_1^4)3!Y_1(3)\} \\ Y_1(7)=\frac{1}{7!}\{(D_1^2)3!Y_1(3)\} & Y_1(12)=\frac{1}{12!}\{(-D_1^5)2!Y_1(2)\} \\ Y_1(8)=\frac{1}{8!}\{(-D_1^3)2!Y_1(2)\} & Y_1(13)=\frac{1}{13!}\{(-D_1^5)3!Y_1(3)\}\cdots \end{array}\right\} \tag{29}$$

By applying the DTM to Eqs. (18), (19), (20), (21), (22), (23) and using the recurrence relations (28), (29) following equations are obtained

$$b_{11}Y_1(0)+b_{12}Y_1(1)+b_{13}2!Y_1(2)+b_{14}3!Y_1(3)=0 \tag{30}$$

$$b_{21}Y_1(0)+b_{22}Y_1(1)+b_{23}2!Y_1(2)+b_{24}3!Y_1(3)=0 \tag{31}$$

$$b_{35}Y_2(1)+b_{36}3!Y_2(3)=Y_1(0) \tag{32}$$

$$b_{45}Y_2(1)+b_{46}3!Y_2(3)=Y_1(1) \tag{33}$$

$$b_{55}Y_2(1)+b_{56}3!Y_2(3)=3!Y_1(3)+D_1Y_1(1) \tag{34}$$

$$b_{65}Y_2(1)+b_{66}3!Y_2(3)=2!Y_1(2)+D_1Y_1(0) \tag{35}$$

where

$$b_{11}=1;\ b_{12}=\frac{L_1}{L};\ b_{13}=\sum_{k=0}^{n}\frac{D_1^k}{(2k+2)!}\left(\frac{L_1}{L}\right)^{2k+2}(-1)^k;\ b_{14}=\sum_{k=0}^{n}\frac{D_1^k}{(2k+3)!}\left(\frac{L_1}{L}\right)^{2k+3}(-1)^k$$

$$b_{21}=D_1;\ b_{22}=\left(\frac{L1}{L}\right)D_1;\ b_{23}=1+C_\theta\left(\frac{L_1}{L}\right)\left[\sum_{k=0}^{n}\frac{D_1^k(-1)^k}{(2k+1)!}\left(\frac{L_1}{L}\right)^{2k}\right];$$

$$b_{24}=\frac{L_1}{L}+C_\theta\left[\sum_{k=0}^{n}\frac{D_1^k(-1)^k}{(2k+2)!}\left(\frac{L_1}{L}\right)^{2k+2}\right]$$

$$b_{35}=\frac{L_2}{L}-\left(\frac{L_2}{L}\right)^5\frac{\beta_2}{5!}+\sum_{k=3}^{n}\left(\frac{L_2}{L}\right)^{2k+1}\frac{(-1)^k}{(2k+1)!}\left\{\sum_{m=1}^{k\geq 2m}\binom{k-m-1}{m-1}\beta_1^{k-2m}\beta_2^m(-1)^m\right\}$$

$$b_{36}=\left(\frac{L_2}{L}\right)^3\frac{1}{3!}+\sum_{k=2}^{n}\left(\frac{L_2}{L}\right)^{2k+1}\frac{(-1)^k}{(2k+1)!}\left\{\sum_{m=1}^{k\geq 2m-1}\binom{k-m}{m-1}\beta_1^{k-2m+1}\beta_2^{m-1}(-1)^m\right\}$$

$$b_{45}=1+\sum_{k=2}^{n}\left(\frac{L_2}{L}\right)^{2k}\frac{(-1)^k}{(2k)!}\left\{\sum_{m=1}^{k\geq 2m}\binom{k-m-1}{m-1}\beta_1^{k-2m}\beta_2^m(-1)^m\right\}$$

$$b_{46} = \sum_{k=1}^{n}\left(\frac{L_2}{L}\right)^{2k}\frac{(-1)^k}{(2k)!}\left\{\sum_{m=1}^{k\geq 2m-1}\binom{k-m}{m-1}\beta_1^{k-2m+1}\beta_2^{m-1}(-1)^m\right\}$$

$$b_{55} = \beta_1 - \left(\frac{L_2}{L}\right)^2\frac{\beta 2}{2!} + \sum_{k=3}^{n}\left(\frac{L_2}{L}\right)^{2k}\frac{(-1)^k}{(2k)!}\left\{\sum_{m=1}^{k\geq 2m+1}\binom{k-m-2}{m-1}\beta_1^{k-2m-1}\beta_2^{m+1}(-1)^m\right\}$$

$$b_{56} = 1 + \sum_{k=2}^{n}\left(\frac{L_2}{L}\right)^{2k}\frac{(-1)^k}{(2k)!}\left\{\sum_{m=1}^{k\geq 2m}\binom{k-m-1}{m-1}\beta_1^{k-2m}\beta_2^{m}(-1)^m\right\}$$

$$b_{65} = \left(\frac{L_2}{L}\right)\beta_1 - \left(\frac{L_2}{L}\right)^3\frac{\beta_2}{3!} + \sum_{k=3}^{n}\left(\frac{L_2}{L}\right)^{2k+1}\frac{(-1)^k}{(2k+1)!}\left\{\sum_{m=1}^{k\geq 2m+1}\binom{k-m-2}{m-1}\beta_1^{k-2m-1}\beta_2^{m+1}(-1)^m\right\}$$

$$b_{66} = \left(\frac{L_2}{L}\right) + \sum_{k=2}^{n}\left(\frac{L_2}{L}\right)^{2k+1}\frac{(-1)^k}{(2k+1)!}\left\{\sum_{m=1}^{k\geq 2m}\binom{k-m-1}{m-1}\beta_1^{k-2m}\beta_2^{m}(-1)^m\right\}$$

Substituting Eqs. (32) and (33) into Eqs. (34) and (35), respectively, gives

$$3!Y_1(3) = (b_{55} - D_1 b_{45})Y_2(1) + (b_{56} - D_1 b_{46})3!Y_2(3) \tag{36}$$

$$2!Y_1(2) = (b_{65} - D_1 b_{35})Y_2(1) + (b_{66} - D_1 b_{36})3!Y_2(3) \tag{37}$$

Substituting Eqs. (32), (33), (36) and (37) into Eqs. (30) and (31), respectively, gives

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}\begin{Bmatrix} Y_2(1) \\ 3!Y_2(3) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{38}$$

where

$$B_{11} = b_{11}b_{35} + b_{12}b_{45} + b_{13}(b_{65} - D_1 b_{35}) + b_{14}(b_{55} - D_1 b_{45})$$

$$B_{12} = b_{11}b_{36} + b_{12}b_{46} + b_{13}(b_{66} - D_1 b_{36}) + b_{14}(b_{56} - D_1 b_{46})$$

$$B_{21} = b_{21}b_{35} + b_{22}b_{45} + b_{23}(b_{65} - D_1 b_{35}) + b_{24}(b_{55} - D_1 b_{45})$$

$$B_{22} = b_{21}b_{36} + b_{22}b_{46} + b_{23}(b_{66} - D_1 b_{36}) + b_{24}(b_{56} - D_1 b_{46})$$

Thus, the buckling equation of the semi-rigid connected pile in elastic soil is obtained using Eq. (26) as

$$f^{(n)} = B_{11}B_{22} - B_{12}B_{21} = 0 \tag{39}$$

Solving (39) we get $N=N_i^{(n)}$, i=1,2,3,… where $N_i^{(n)}$ is the nth estimated N axial compressive load circular frequency corresponding to n, and n is indicated by

$$\left|N_i^{(n)} - N_i^{(n-1)}\right| \leq \varepsilon$$

where $N_i^{(n-1)}$ is the ith estimated axial compressive load corresponding to n-1 and ε is a positive and small value.

5. Analytical solution of differential equations

The solution of differential equation of the elastic curve for the first region of the pile, Eq. (9), is obtained as (Ross 1984)

$$y_1(z_1) = C_1 + C_2 z_1 + \cos(D_2 z_1)C_3 + \sin(D_2 z_1)C_4 \qquad (0 \le z_1 \le \frac{L_1}{L}) \tag{40}$$

Where $D_2 = \left[\frac{NL^2}{EI}\left[\frac{\bar{k}AG}{\bar{k}AG - N}\right]\right]^{0.5}$.

The solution of Eq. (6) is obtained due to the sign of γ; four possible conditions exist due to the signs of Δ_1 and Δ_2 when γ is positive (Catal and Catal 2006).

Where $\Delta_1 = -\frac{\beta_1}{2} - (\beta_2)^{0.5}$; $\Delta_2 = -\frac{\beta_1}{2} + (\beta_2)^{0.5}$; $D_3 = (\Delta_1)^{0.5}$; $D_4 = (\Delta_2)^{0.5}$; $\gamma = \left(\frac{\beta_1}{2}\right)^2 + \beta_2$

I. $\gamma > 0$, $\Delta_1 > 0$ and $\Delta_2 > 0$

$$y_2(z_2) = \left[C_5 \cosh(D_3 z_2) + C_6 \sinh(D_3 z_2) + C_7 \cosh(D_4 z_2) + C_8 \sinh(D_4 z_2)\right] \qquad (0 \le z_2 \le \frac{L_2}{L}) \tag{41}$$

II. $\gamma > 0$, $\Delta_1 > 0$ and $\Delta_2 < 0$

$$y_2(z_2) = \left[C_5 \cosh(D_3 z_2) + C_6 \sinh(D_3 z_2) + C_7 \cos(D_4 z_2) + C_8 \sin(D_4 z_2)\right] \qquad (0 \le z_2 \le \frac{L_2}{L}) \tag{42}$$

III. $\gamma > 0$, $\Delta_1 < 0$ and $\Delta_2 > 0$

$$y_2(z_2) = \left[C_5 \cos(D_3 z_2) + C_6 \sin(D_3 z_2) + C_7 \cosh(D_4 z_2) + C_8 \sinh(D_4 z_2)\right] \qquad (0 \le z_2 \le \frac{L_2}{L}) \tag{43}$$

VI. $\gamma > 0$, $\Delta_1 < 0$ and $\Delta_2 < 0$

$$y_2(z_2) = \left[C_5 \cos(D_3 z_2) + C_6 \sin(D_3 z_2) + C_7 \cos(D_4 z_2) + C_8 \sin(D_4 z_2)\right] \qquad (0 \le z_2 \le \frac{L_2}{L}) \tag{44}$$

V. $\gamma < 0$

$$y_2(z_2) = \{C_5\left[\cosh(r\alpha_1 z_2)\cos(r\alpha_2 z_2)\right] + C_6\left[\sinh(r\alpha_1 z_1)\cos(r\alpha_2 z_2)\right] + C_7\left[\cosh(r\alpha_1 z_2)\sin(r\alpha_2 z_2)\right] + C_8\left[\sinh(r\alpha_1 z_2)\sin(r\alpha_2 z_2)\right]\} \qquad (0 \le z_2 \le \frac{L_2}{L}) \tag{45}$$

Where $\lambda = Arctg\left[\frac{1}{\beta_1}(-2\sqrt{-\left(\frac{\beta_1}{2}\right)^2 - \beta_2})\right]$; $\alpha_1 = \sin(\lambda/2)$; $\alpha_2 = \cos(\lambda/2)$; $r = \sqrt[4]{-\beta_2}$

C_1, C_2,…, C_8 = constant of integration. Bending moment functions with respect to z for the first and the second regions of pile are obtained from Eqs. (9) and (10), respectively, as

$$M_1(z_1) = -NC_1 - NC_2 z_1 \qquad (0 \le z_1 \le \frac{L_1}{L}) \tag{46}$$

$$M_2(z_2) = -\frac{EI}{L^2}\left[\frac{\bar{k}AG - N}{\bar{k}AG}\right]\frac{d^2 y_2(z_2)}{dz_2^2} + \left[\frac{EIC_s}{\bar{k}AG} - N\right] y_2(z_2) \qquad (0 \le z_2 \le \frac{L_2}{L}) \qquad (47)$$

The summation of horizontal components $V_1(z_1)$ and $V_2(z_2)$ of axial (N) and shear forces ($T_1(z_1)$, $T_2(z_2)$) at initial ends of differential parts at the first and the second regions of pile are written, respectively, as

$$V_1(z_1) = T_1(z_1) - \frac{N}{L}\frac{dy_1(z_1)}{dz_1} = \frac{1}{L}\left[\frac{dM_1(z_1)}{dz_1} - N\frac{dy_1(z_1)}{dz_1}\right] \quad (0 \le z_1 \le \frac{L_1}{L}) \qquad (48)$$

$$V_2(z_2) = T_2(z_2) - \frac{N}{L}\frac{dy_2(z_2)}{dz_2} = \frac{1}{L}\left[\frac{dM_2(z_2)}{dz_2} - N\frac{dy_2(z_2)}{dz_2}\right] \quad (0 \le z_2 \le \frac{L_2}{L}) \qquad (49)$$

substituting Eqs. (42) and (43) into Eqs. (44) and (45), respectively, gives (Catal and Catal 2006).

$$V_1(z_1) = -\frac{EI}{L^3}\left[\frac{\bar{k}AG - N}{\bar{k}AG}\right]\frac{d^3 y_1(z_1)}{dz_1^3} - \frac{N}{L}\frac{dy_1(z_1)}{dz_1} \quad (0 \le z_1 \le \frac{L_1}{L}) \qquad (50)$$

$$V_2(z_2) = -\frac{EI}{L^3}\left[\frac{\bar{k}AG - N}{\bar{k}AG}\right]\frac{d^3 y_2(z_2)}{dz_2^3} + \frac{1}{L}\left[\frac{EIC_s}{\bar{k}AG} - N\right]\frac{dy_2(z_2)}{dz_2} \quad (0 \le z_2 \le \frac{L_2}{L}) \qquad (51)$$

Constants of integration (C_1, ..., C_8) must be obtained by using boundary conditions due to the support type of top and bottom ends in order to the calculate the buckling load of the pile.

Boundary conditions of the pile whose top end simply supported and semi-rigid connected, bottom end simply supported (Fig. 2) are given in relations (52).

$$\left.\begin{array}{ll} y_1(z_1 = \frac{L_1}{L}) = 0 & M_1(z_1 = 0) = M_2(z_2 = \frac{L_2}{L}) \\ M_1(z_1 = \frac{L_1}{L}) = \frac{C_\theta}{L}\left.\frac{dy_1(z_1)}{dz_1}\right|_{z_1 = \frac{L_1}{L}} & V_1(z_1 = 0) = V_2(z_2 = \frac{L_2}{L}) \\ y_1(z_1 = 0) = y_2(z_2 = \frac{L_2}{L}) & y_2(z_2 = 0) = 0 \\ \left.\frac{dy_1(z_1)}{dz_1}\right|_{z_1 = 0} = \left.\frac{dy_2(z_2)}{dz_2}\right|_{z_2 = \frac{L_2}{L}} & M_2(z_2 = 0) = 0 \end{array}\right\} \qquad (52)$$

A set of eight linear homogeneous equations is obtained from Eq.(48) due to boundary conditions of the pile. This equation set is written in matrix form as:

$$[S]\{C\} = \{0\} \qquad (53)$$

Where [S] and {C} indicate coefficient matrix and the unknown coefficients vector, respectively. Hence, the non-trivial solution of this problem is given by

$$|S| = \begin{vmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} & S_{17} & S_{18} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} & S_{27} & S_{28} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} & S_{37} & S_{38} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} & S_{47} & S_{48} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} & S_{57} & S_{58} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} & S_{67} & S_{68} \\ S_{71} & S_{72} & S_{73} & S_{74} & S_{75} & S_{76} & S_{77} & S_{78} \\ S_{81} & S_{82} & S_{83} & S_{84} & S_{85} & S_{86} & S_{87} & S_{88} \end{vmatrix} \tag{54}$$

Where,

$$S_{11} = 1, S_{12} = \frac{L_1}{L}, S_{13} = \cos(D_2 \frac{L_1}{L}), S_{14} = \sin(D_2 \frac{L_1}{L}), S_{15} = 0, S_{16} = 0, S_{17} = 0, S_{18} = 0$$

$$S_{21} = -N, S_{22} = -\frac{1}{L}\left[NL_1 + C_\theta\right], S_{23} = \frac{D_2 C_\theta}{L}\sin(D_2 \frac{L_1}{L}), S_{24} = -\frac{D_2 C_\theta}{L}\cos(D_2 \frac{L_1}{L}),$$

$$S_{25} = 0, S_{26} = 0, S_{27} = 0, S_{28} = 0, S_{31} = 1, S_{32} = 0; S_{33} = 1, S_{34} = 0, S_{41} = 0, S_{42} = \frac{1}{L},$$

$$S_{43} = 0, S_{44} = D_2, S_{51} = -N, S_{52} = 0, S_{53} = 0, S_{54} = 0; S_{61} = 0, S_{62} = -\frac{N}{L}, S_{63} = 0,$$

$$S_{64} = 0;$$

$$U_1 = \frac{EIC_s}{\bar{k}AG},\ U_2 = \frac{EI}{L^2}(\frac{\bar{k}AG - N}{\bar{k}AG}),\quad U_3 = \frac{EI}{L^3}(\frac{\bar{k}AG - N}{\bar{k}AG}), A_1 = U_1 - D_3^2 U_2$$

$$A_2 = U_1 - D_4^2 U_2,\ A_3 = U_1 + D_4^2 U_2, A_4 = U_1 + D_3^2 U_2, A_5 = U_1 - (D_3^2 - D_4^2)U_2 \text{-} N$$

$$A_6 = 2D_3 D_4 U_2,\ B_1 = D_3[(U_1 - \frac{N}{L}) - D_3^2 U_3], B_2 = D_4[(U_1 - \frac{N}{L}) - D_4^2 U_3]$$

$$B_3 = D_4[(-U_1 + \frac{N}{L}) - D_4^2 U_3], B_4 = D_3[(-U_1 + \frac{N}{L}) - D_3^2 U_3], B_5 = U_1 - \frac{N}{L} - (D_3^2 - D_4^2)U_3$$

for $\gamma > 0$, $\Delta_1 > 0$ and $\Delta_2 > 0$

$$S_{35} = -\cosh(D_3 \frac{L_2}{L}), S_{36} = -\sinh(D_3 \frac{L_2}{L}), S_{37} = -\cosh(D_4 \frac{L_2}{L}), S_{38} = -\sinh(D4 \frac{L_2}{L})$$

$$S_{45} = -\frac{D_3}{L}\sinh(D_3 \frac{L_2}{L}), S_{46} = -\frac{D_3}{L}\cosh(D_3 \frac{L_2}{L}), S_{47} = -\frac{D_4}{L}\sinh(D_4 \frac{L_2}{L})$$

$$S_{48} = -\frac{D_4}{L}\cosh(D_4 \frac{L_2}{L}), S_{55} = -A_1\cosh(D_3 \frac{L_2}{L}), S_{56} = -A_1\sinh(D_3 \frac{L_2}{L})$$

$$S_{57} = -A_2\cosh(D_4 \frac{L_2}{L}), S_{58} = -A_2\sinh(D_4 \frac{L_2}{L}), S_{65} = -B_1\sinh(D_3 \frac{L_2}{L})$$

$$S_{66} = -B_1\cosh(D_3 \frac{L_2}{L}), S_{67} = -B_2\sinh(D_4 \frac{L_2}{L}), S_{68} = -B_2\cosh(D_4 \frac{L_2}{L}), S_{75} = -1, S_{76} = 0$$

$$S_{77}=-1, S_{78}=0, S_{85}=A_1, S_{86}=0, S_{87}=A_2, S_{88}=0$$

for $\gamma>0$, $\Delta_1>0$ and $\Delta_2<0$

$$S_{35}=-\cosh(D_3\frac{L_2}{L}), S_{36}=-\sinh(D_3\frac{L_2}{L}), S_{37}=-\cos(D_4\frac{L_2}{L}), S_{38}=-\sin(D4\frac{L_2}{L})$$

$$S_{45}=-\frac{D_3}{L}\sinh(D_3\frac{L_2}{L}), S_{46}=-\frac{D_3}{L}\cosh(D_3\frac{L_2}{L}), S_{47}=\frac{D_4}{L}\sin(D_4\frac{L_2}{L})$$

$$S_{48}=-\frac{D_4}{L}\cos(D_4\frac{L_2}{L}), S_{55}=-A_1\cosh(D_3\frac{L_2}{L}), S_{56}=-A_1\sinh(D_3\frac{L_2}{L})$$

$$S_{57}=-A_3\cos(D_4\frac{L_2}{L}), S_{58}=-A_3\sin(D_4\frac{L_2}{L}), S_{65}=-B_1\sinh(D_3\frac{L_2}{L}),$$

$$S_{66}=-B_1\cosh(D_3\frac{L_2}{L})\ S_{67}=-B_3\sin(D_4\frac{L_2}{L}), S_{68}=B_3\cos(D_4\frac{L_2}{L}), S_{75}=1, S_{76}=0$$

$$S_{77}=1, S_{78}=0, S_{85}=A_1, S_{86}=0, S_{87}=A_3, S_{88}=0$$

for $\gamma>0$, $\Delta_1<0$ and $\Delta_2>0$

$$S_{35}=-\cos(D_3\frac{L_2}{L}), S_{36}=-\sin(D_3\frac{L_2}{L}), S_{37}=-\cosh(D_4\frac{L_2}{L}), S_{38}=-\sinh(D4\frac{L_2}{L})$$

$$S_{45}=\frac{D_3}{L}\sin(D_3\frac{L_2}{L}), S_{46}=-\frac{D_3}{L}\cos(D_3\frac{L_2}{L}), S_{47}=-\frac{D_4}{L}\sinh(D_4\frac{L_2}{L}),$$

$$S_{48}=-\frac{D_4}{L}\cosh(D_4\frac{L_2}{L}), S_{55}=-A_4\cos(D_3\frac{L_2}{L}), S_{56}=-A_4\sinh(D_3\frac{L_2}{L})$$

$$S_{57}=-A_2\cosh(D_4\frac{L_2}{L}), S_{58}=-A_2\sinh(D_4\frac{L_2}{L}), S_{65}=-B_4\sin(D_3\frac{L_2}{L})$$

$$S_{66}=-B_4\cos(D_3\frac{L_2}{L}), S_{67}=-B_2\sinh(D_4\frac{L_2}{L}), S_{68}=-B_2\cosh(D_4\frac{L_2}{L}), S_{75}=1, S_{76}=0$$

$$S_{77}=1, S_{78}=0, S_{85}=A_4, S_{86}=0, S_{87}=A_2, S_{88}=0$$

for $\gamma>0$, $\Delta_1<0$ and $\Delta_2<0$

$$S_{35}=-\cos(D_3\frac{L_2}{L}), S_{36}=-\sin(D_3\frac{L_2}{L}), S_{37}=-\cos(D_4\frac{L_2}{L}), S_{38}=-\sin(D4\frac{L_2}{L})$$

$$S_{45}=\frac{D_3}{L}\sin(D_3\frac{L_2}{L}), S_{46}=-\frac{D_3}{L}\cos(D_3\frac{L_2}{L}), S_{47}=-\frac{D_4}{L}\sin(D_4\frac{L_2}{L}),$$

$$S_{48}=-\frac{D_4}{L}\cos(D_4\frac{L_2}{L}), S_{55}=-A_4\cos(D_3\frac{L_2}{L}), S_{56}=-A_4\sin(D_3\frac{L_2}{L}),$$

$$S_{57}=-A_3\cos(D_4\frac{L_2}{L}), S_{58}=-A_3\sin(D_4\frac{L_2}{L}), S_{65}=-B_4\sin(D_3\frac{L_2}{L}),$$

$$S_{66}=-B_4\cos(D_3\frac{L_2}{L}), S_{67}=-B_3\sin(D_4\frac{L_2}{L}), S_{68}=-B_3\cos(D_4\frac{L_2}{L}), S_{75}=1, S_{76}=0,$$

$$S_{77}=1, S_{78}=0, S_{85}=A_4, S_{86}=0, S_{87}=A_3, S_{88}=0$$

for $\gamma < 0$

$$S_{35} = \cosh(D_3 \frac{L_2}{L})\cos(D_4 \frac{L_2}{L}),\, S_{36} = \sinh(D_3 \frac{L_2}{L})\cos(D_4 \frac{L_2}{L}),$$

$$S_{37} = \cosh(D_3 \frac{L_2}{L})\sin(D_4 \frac{L_2}{L})$$

$$S_{38} = \sinh(D_3 \frac{L_2}{L})\sin(D_4 \frac{L_2}{L})$$

$$S_{45} = -\frac{D_3}{L}\sinh(D_3 \frac{L_2}{L})\cos(D_4 \frac{L_2}{L}) + \frac{D_4}{L}\cosh(D_3 \frac{L_2}{L})\sin(D_4 \frac{L_2}{L})$$

$$S_{46} = -\frac{D_3}{L}\cosh(D_3 \frac{L_2}{L})\cos(D_4 \frac{L_2}{L}) + \frac{D_4}{L}\sinh(D_3 \frac{L_2}{L})\cos(D_4 \frac{L_2}{L})$$

$$S_{47} = -\frac{D_3}{L}\sinh(D_3 \frac{L_2}{L})\sin(D_4 \frac{L_2}{L}) - \frac{D_4}{L}\cosh(D_3 \frac{L_2}{L})\cos(D_4 \frac{L_2}{L})$$

$$S_{48} = -\frac{D_3}{L}\cosh(D_3 \frac{L_2}{L})\sin(D_4 \frac{L_2}{L}) - \frac{D_4}{L}\sinh(D_3 \frac{L_2}{L})\cos(D_4 \frac{L_2}{L})$$

$$S_{55} = -A_5 \cosh(D_3 \frac{L_2}{L})\cos(D_4 \frac{L_2}{L}) - A_6 \sinh(D_3 \frac{L_2}{L})\sin(D_4 \frac{L_2}{L})$$

$$S_{56} = -A_5 \sinh(D_3 \frac{L_2}{L})\cos(D_4 \frac{L_2}{L}) - A_6 \cosh(D_3 \frac{L_2}{L})\sin(D_4 \frac{L_2}{L})$$

$$S_{57} = -A_5 \cosh(D_3 \frac{L_2}{L})\sin(D_4 \frac{L_2}{L}) + A_6 \sinh(D_3 \frac{L_2}{L})\cos(D_4 \frac{L_2}{L})$$

$$S_{58} = -A_5 \sinh(D_3 \frac{L_2}{L})\sin(D_4 \frac{L_2}{L}) + A_6 \cosh(D_3 \frac{L_2}{L})\cos(D_4 \frac{L_2}{L})$$

$$S_{65} = -(B_5 D_3 + 2U_3 D_3 D_4^2)\sinh(D_3 \frac{L_2}{L})\cos(D_4 \frac{L_2}{L})$$
$$+ (B_5 D_3 - 2U_3 D_3^2 D_4)\cosh(D_3 \frac{L_2}{L})\sin(D_4 \frac{L_2}{L})$$

$$S_{66} = -(B_5 D_3 + 2U_3 D_3 D_4^2)\cosh(D_3 \frac{L_2}{L})\cos(D_4 \frac{L_2}{L})$$
$$+ (B_5 D_3 - 2U_3 D_3^2 D_4)\sinh(D_3 \frac{L_2}{L})\sin(D_4 \frac{L_2}{L})$$

$$S_{67} = -(B_5 D_3 + 2U_3 D_3 D_4^2)\sinh(D_3 \frac{L_2}{L})\sin(D_4 \frac{L_2}{L})$$
$$- (B_5 D_3 - 2U_3 D_3^2 D_4)\cosh(D_3 \frac{L_2}{L})\sin(D_4 \frac{L_2}{L})$$

$$S_{68} = -(B_5 D_3 + 2U_3 D_3 D_4^2)\cosh(D_3 \frac{L_2}{L})\sin(D_4 \frac{L_2}{L})$$

$$-(B_5D_3 - 2U_3D_3^2D_4)\sinh(D_3\frac{L_2}{L})\cos(D_4\frac{L_2}{L})$$

$$S_{75}=1, S_{76}=0, S_{77}=0, S_{78}=0, S_{85}=A_5, S_{86}=0, S_{87}=0, S_{88}=-A_6$$

6. Numerical analysis

A thin-walled circular steel pile with outer diameter of 355,6 mm and thickness of 32 mm is considered for numerical analysis. The buckling loads of the semi-rigid connected pile partially embedded in soil having modulus of subgrade reaction of 15.000kN/m^2 are calculated for support conditions given in Fig. 3 by a computer program having an iteration algorithm and prepared by the writer.

The characteristics of the steel pile used numerical analysis are presented in the following:

I=246,63*10^{-6} m^4; A=17,1*10^{-3} m^2; EI=51792,3 kN/m^2 ; AG=1382670 kN; $\bar{k}$ =2,0

Buckling loads and relative stiffness values (α) of the thin-walled steel pile are calculated by taking pile lengths of the first and the second regions (L_1 and L_2), stiffness of the rotational spring (C_θ) from Table 2 and by using DTM and analytical method for L_2/L=0.25, L_2/L=0.50, L_2/L=0.75, f=0.25 and f=0.75.

Euler critical buckling load of piles are calculated using $N_E=\pi^2 EI/(L_b)^2$ by neglecting the effects of modulus of subgrade reaction, shear deformation and stiffness of rotational spring and by taking $L_b=L$ for both ends simply supported pile.

$N_r=N/N_E$ values are calculated according to α, L_2/L, f and series size (n) values using DTM and according to α, L_2/L and f values by using analytical method; and the values obtained are presented Table 2(a),(b).

Buckling loads ($\bar{N}$) and relative stiffness values of the pile are calculated by neglecting the effects of shear deformation by using analytical method for for L_2/L=0.25, L_2/L=0.50, L_2/L=0.75, f=0.25 and f = 0.75. $\frac{\bar{N}}{N}$ values obtained are presented Table 3(a),(b).

Table 2 Values of L with respect to α, values of L_1 and L_2 with respect to L_2/L and values of C_θ with respect to f, L, E, I

L(m.)	$\alpha = C_sL^4/EI$	L_2/L=	0.25	L_2/L=	0.50	L_2/L=	0.75	f=0.25	f=0.75
		L_1(m)	L_2(m)	L_1(m)	L_2(m)	L_1(m)	L_2(m)	C_θ kNm/rd	C_θ kNm/rd
1.36	1	1.020	0.340	0.680	0.680	0.340	1.020	38083	342743
2.42	10	1.815	0.605	1.210	1.210	0.605	1.815	21402	102616
4.31	100	3.232	1.078	2.155	2.155	1.078	3.232	12017	108151
7.66	1000	5.745	1.915	3.830	3.830	1.915	5.745	6761	60853
13.63	10000	10.220	3.410	6.815	6.815	3.410	10.220	3800	34199
24.24	100000	18.180	6.060	12.120	12.120	6.060	18.180	2137	19230
43.10	1000000	32.320	10.770	21.550	21.550	10.770	32.320	1202	10815

Table 2(a) N_r values for the first, second, third modes of the pile and f=0,25

α	Method	n	L_2/L=0.25			L_2/L=0.5			L_2/L=0.75		
			1st. Mode	2nd. Mode	3rd. Mode	1st. Mode	2nd. Mode	3rd. Mode	1st. Mode	2nd. Mode	3rd. Mode
1.0	DTM	4	0.8323964	1.6132217	1.9959288	0.8368092	1.6088700	1.9961459	0.8415466	1.6087995	2.1185852
		6	0.8323532	1.6073521	1.9959288	0.8367862	1.6078526	1.9961459	0.8406512	1.6087995	2.1509392
		8	0.8323532	1.6074565	1.9965785	0.8367862	1.6078506	1.9961919	0.8406496	1.6087995	2.1698455
		10	0.8323532	1.6074598	1.9960621	0.8367862	1.6078506	1.9962313	0.8406496	1.6088816	1.9971236
		12	0.8323532	1.6074598	1.9960460	0.8367862	1.6078506	1.9962315	0.8406496	1.6088820	1.9964410
		14	0.8323532	1.6074598	1.9960459	0.8367862	1.6078506	1.9962315	0.8406496	1.6088820	1.9964350
		16	0.8323532	1.6074598	1.9960459	0.8367862	1.6078506	1.9962315	0.8406496	1.6088820	1.9964350
		20	0.8323532	1.6074598	1.9960459	0.8367862	1.6078506	1.9962315	0.8406496	1.6088820	1.9964350
		24	0.8323532	1.6074598	1.9960459	0.8367862	1.6078506	1.9962315	0.8406496	1.6088820	1.9964350
		32	0.8323532	1.6074598	1.9960459	0.8367862	1.6078506	1.9962315	0.8406496	1.6088820	1.9964350
Analytic	Method		0.8323532	1.6075598	1.9960459	0.8367862	1.6078506	1.9962315	0.8406496	1.6088820	1.9964350
10.0	DTM	4	1.0503351	2.9141826	4.3062926	1.0939307	2.7972091	4.3078965	1.1350488	2.8022353	4.9155951
		6	1.0503018	2.7896213	4.3062926	1.0939072	2.7936564	4.3078965	1.1338980	2.8022353	4.7360670
		8	1.0503018	2.7889705	4.3062926	1.0939072	2.7936466	4.3084253	1.1338963	2.8023086	4.4712353
		10	1.0503018	2.7889694	4.3065234	1.0939072	2.7936466	4.3085427	1.1338963	2.8025057	4.3121396
		12	1.0503018	2.7889694	4.3065561	1.0939072	2.7936466	4.3085433	1.1338963	2.8025062	4.3105213
		14	1.0503018	2.7889694	4.3065566	1.0939072	2.7936466	4.3085433	1.1338963	2.8025062	4.3105081
		16	1.0503018	2.7889694	4.3065566	1.0939072	2.7936466	4.3085433	1.1338963	2.8025062	4.3105081
		20	1.0503018	2.7889694	4.3065566	1.0939072	2.7936466	4.3085433	1.1338963	2.8025062	4.3105081
		24	1.0503018	2.7889694	4.3065566	1.0939072	2.7936466	4.3085433	1.1338963	2.8025062	4.3105081
		32	1.0503018	2.7889694	4.3065566	1.0939072	2.7936466	4.3085433	1.1338963	2.8025062	4.3105081
Analytic	Method		1.0503018	2.7889694	4.3065566	1.0939072	2.7936466	4.3085433	1.1338963	2.8025062	4.3105081
			1st. Mode	2nd. Mode	3rd. Mode	1st. Mode	2nd. Mode	3rd. Mode	1st. Mode	2nd. Mode	3rd. Mode
100.0	DTM	4	1.2273294	4.0908940	6.8061934	1.6501371	3.7447645	6.8258172	2.0709243	3.8066576	9.2046450
		6	1.2272931	3.6776056	6.8061934	1.6501353	3.7385576	6.8258172	2.0706481	3.8066576	7.7146879
		8	1.2272931	3.6756850	6.8061934	1.6501353	3.7385394	6.8265422	2.0706463	3.8073844	7.0736514
		10	1.2272931	3.6756850	6.8067894	1.6501353	3.7385394	6.8267675	2.0706463	3.8076206	6.8497528
		12	1.2272931	3.6756850	6.8069384	1.6501353	3.7385394	6.8267675	2.0706463	3.8076206	6.8469346
		14	1.2272931	3.6756850	6.8069384	1.6501353	3.7385394	6.8267675	2.0706463	3.8076206	6.8469128
		16	1.2272931	3.6756850	6.8069384	1.6501353	3.7385394	6.8267675	2.0706463	3.8076206	6.8469128
		20	1.2272931	3.6756850	6.8069384	1.6501353	3.7385394	6.8267675	2.0706463	3.8076206	6.8469128
		24	1.2272931	3.6756850	6.8069384	1.6501353	3.7385394	6.8267675	2.0706463	3.8076206	6.8469128
		32	1.2272931	3.6756850	6.8069384	1.6501353	3.7385394	6.8267675	2.0706463	3.8076206	6.8469128
Analytic	Method		1.2272931	3.6756850	6.8069384	1.6501353	3.7385394	6.8267675	2.0706463	3.8076206	6.8469128
1000.0	DTM	4	1.9156797	10.4453785	8.6985619	7.7863958	7.7182812	79.3330134	5.9596015	9.3747834	10.9504437
		6	1.9156624	4.7008935	8.6985619	3.7057824	7.7216330	8.9051785	5.8068774	9.8880820	10.7705637
		8	1.9156567	4.6905225	8.6985619	3.7057881	7.7208812	8.9078013	5.8034568	9.1067616	10.6205893
		10	1.9156567	4.6904651	8.7006224	3.7057881	7.7208812	8.9079391	5.8034453	9.0151042	10.6187871
		12	1.9156567	4.6904651	8.7011733	3.7057881	7.7208812	8.9079391	5.8034453	9.0137497	10.6187642
		14	1.9156567	4.6904651	8.7011733	3.7057881	7.7208812	8.9079391	5.8034453	9.0137382	10.6187642
		16	1.9156567	4.6904651	8.7011733	3.7057881	7.7208812	8.9079391	5.8034453	9.0137382	10.6187642

Table 2(a) Continued

	20	1.9156567	4.6904651	8.7011733	3.7057881	7.7208812	8.9079391	5.8034453	9.0137382	10.6187642
	24	1.9156567	4.6904651	8.7011733	3.7057881	7.7208812	8.9079391	5.8034453	9.0137382	10.6187642
	32	1.9156567	4.6904651	8.7011733	3.7057881	7.7208812	8.9079391	5.8034453	9.0137382	10.6187642
Analytic Method		1.9156567	4.6904651	8.7011733	3.7057881	7.7208812	8.9079391	5.8034453	9.0137382	10.6187642
		1st. Mode	2nd. Mode	3rd. Mode	1st. Mode	2nd. Mode	3rd. Mode	1st. Mode	2nd. Mode	3rd. Mode
	4	2.9011019	16.4312969	14.4102166	5.6148530	21.2643103	21.3440523	32.3480907	20.9447782	32.3480907
	6	2.8949235	7.7384764	14.4102166	5.3737651	13.1418284	20.6104681	26.2926154	20.9447782	26.2926154
	8	2.8949053	7.5659538	14.4102166	5.1950492	13.1542579	20.6140523	25.7778279	20.9447782	25.7778279
	10	2.8949053	7.5647363	14.4102166	5.1949946	13.1526769	20.6205942	12.1243557	25.7749931	25.6258212
	12	2.8949053	7.5647363	14.4158135	5.1949946	13.1526769	20.6214301	12.1209394	20.9667478	25.6258212
10000.0 DTM	14	2.8949053	7.5647363	14.4188482	5.1949946	13.1526769	20.6214301	12.1209213	20.9542457	25.6258212
	16	2.8949053	7.5647363	14.4188663	5.1949946	13.1526769	20.6214301	12.1209213	20.9541367	25.6323630
	20	2.8949053	7.5647363	14.4188663	5.1949946	13.1526769	20.6214301	12.1209213	20.9541367	25.6333443
	24	2.8949053	7.5647363	14.4188663	5.1949946	13.1526769	20.6214301	12.1209213	20.9541367	25.6333443
	32	2.8949053	7.5647363	14.4188663	5.1949946	13.1526769	20.6214301	12.1209213	20.9541367	25.6333443
Analytic Method		2.8949053	7.5647363	14.4188663	5.1949946	13.1526769	20.6214301	12.1209213	20.9541367	25.6333443
	4	3.3536565	8.7820615	72.5559057	89.9351928	17.5755603	89.9351928	269.0902588	87.8987627	269.0902588
	6	3.2930216	8.7820615	17.1962329	18.9567718	18.9434379	78.6472715	48.6279341	69.9768817	64.4282414
	8	3.2934814	8.7965449	17.1962329	17.3335379	17.9182866	35.7402275	48.6261642	62.9461327	64.4282414
	10	3.2934814	8.8148216	17.2208892	6.5096605	17.6992970	33.6912274	47.9936395	57.8987627	64.4282414
	12	3.2934814	8.8149940	17.2221536	6.5045453	17.6009637	33.7086419	47.7123499	57.8987627	64.4282414
100000.0DTM	14	3.2934814	8.8149940	17.2196247	6.5046028	17.6007338	33.7191022	18.8149264	51.4524856	64.7355615
	16	3.2934814	8.8149940	17.2195673	6.5046028	17.6007338	33.7191596	18.7380838	48.0181358	64.6437183
	20	3.2934814	8.8149940	17.2195673	6.5046028	17.6007338	33.7191596	18.7406701	47.9327297	64.4914052
	24	3.2934814	8.8149940	17.2195673	6.5046028	17.6007338	33.7191596	18.7406701	47.9327297	64.4544495
	32	3.2934814	8.8149940	17.2195673	6.5046028	17.6007338	33.7191596	18.7406701	47.9327297	64.4544495
Analytic Method		3.2934814	8.8149940	17.2195673	6.5046028	17.6007338	33.7191596	18.7406701	47.9327297	64.4544495
		1st. Mode	2nd. Mode	3rd. Mode	1st. Mode	2nd. Mode	3rd. Mode	1st. Mode	2nd. Mode	3rd. Mode
	4	6.9413775	10.3859012	206.4769995	92.1821184	36.1132606	138.8346366	640.8970840	122.1302322	640.8970840
	6	3.6941821	9.6303845	190.0077174	18.3386317	32.3881891	123.1612090	54.5790836	119.7395797	480.4233817
	8	3.6941821	9.6303845	176.4309484	15.8358692	28.3785724	104.9550380	53.8655401	118.2336341	473.8085225
	10	3.5866146	9.7141491	19.4788854	15.5333354	20.4144586	92.1821184	52.8739926	116.4618584	427.7372689
	12	3.5931559	9.7146942	19.3060128	14.1903764	20.4144586	86.1259972	51.6709441	113.6400273	386.3373903
1000000.0DTM	14	3.5929742	9.7146942	19.1871798	7.4952050	20.4661797	40.0472893	50.8222143	101.0595301	210.1586442
	16	3.5929742	9.7146942	19.1857261	7.3939970	20.4109423	40.2140917	49.9598569	74.4768997	206.8576651
	20	3.5929742	9.7146942	19.1857261	7.3992664	20.4125776	40.1299637	48.0617984	66.8664956	206.6944967
	24	3.5929742	9.7146942	19.1857261	7.3992664	20.4125776	40.1297820	24.1888891	66.5990304	124.4770944
	32	3.5929742	9.7146942	19.1857261	7.3992664	20.4125776	40.1297820	24.1605436	66.9591636	126.9607782
Analytic Method		3.5929742	9.7146942	19.1857261	7.3992664	20.4125776	40.1297820	24.1605436	66.9591636	126.9607782

Table 2(b) N_r values for the first, second, third modes of the pile and f=0,75

α	Method	n	L_2/L=0.25			L_2/L=0.5			L_2/L=0.75		
			1st. Mode	2nd. Mode	3rd. Mode	1st. Mode	2nd. Mode	3rd. Mode	1st. Mode	2nd. Mode	3rd. Mode
1.0	DTM	4	1.0577092	1.7364917	2.0579838	1.0602643	1.7355074	2.4141278	1.0645418	1.7381200	2.2452462
		6	1.0558298	1.7364917	2.0579838	1.0601084	1.7367931	2.0581286	1.0624891	1.7381200	2.2614031
		8	1.0558238	1.7364917	2.0628315	1.0601084	1.7368239	2.0581286	1.0624831	1.7381200	2.2737935
		10	1.0558238	1.7365929	2.0582838	1.0601084	1.7368241	2.0582022	1.0624831	1.7382179	2.0638094
		12	1.0558238	1.7365934	2.0580853	1.0601084	1.7368241	2.0582029	1.0624831	1.7382210	2.0585941
		14	1.0558238	1.7365934	2.0580826	1.0601084	1.7368241	2.0582029	1.0624831	1.7382210	2.0585407
		16	1.0558238	1.7365934	2.0580826	1.0601084	1.7368241	2.0582029	1.0624831	1.7382210	2.0585403
		20	1.0558238	1.7365934	2.0580826	1.0601084	1.7368241	2.0582029	1.0624831	1.7382210	2.0585403
		24	1.0558238	1.7365934	2.0580826	1.0601084	1.7368241	2.0582029	1.0624831	1.7382210	2.0585403
		32	1.0558238	1.7365934	2.0580826	1.0601084	1.7368241	2.0582029	1.0624831	1.7382210	2.0585403
Analytic	Method		1.0558238	1.7365934	2.0580826	1.0601084	1.7368241	2.0582029	1.0624831	1.7382210	2.0585403
10.0	DTM	4	1.3350553	3.0820119	4.6283513	1.3796741	3.0864245	4.5288995	1.4156149	3.0990827	5.3642292
		6	1.3338902	3.0820119	4.5454065	1.3795418	3.0848349	4.5288995	1.4124425	3.0990827	5.6299042
		8	1.3338873	3.0821884	4.5281823	1.3795418	3.0848509	4.5288995	1.4124333	3.0990827	5.5250372
		10	1.3338873	3.0822949	4.5278246	1.3795418	3.0848509	4.5292088	1.4124333	3.0993450	4.6400000
		12	1.3338873	3.0822955	4.5277790	1.3795418	3.0848509	4.5292105	1.4124333	3.0993490	4.6320449
		14	1.3338873	3.0822955	4.5277750	1.3795418	3.0848509	4.5292105	1.4124333	3.0993490	4.6319687
		16	1.3338873	3.0822955	4.5277750	1.3795418	3.0848509	4.5292105	1.4124333	3.0993490	4.6319682
		20	1.3338873	3.0822955	4.5277750	1.3795418	3.0848509	4.5292105	1.4124333	3.0993490	4.6319682
		24	1.3338873	3.0822955	4.5277750	1.3795418	3.0848509	4.5292105	1.4124333	3.0993490	4.6319682
		32	1.3338873	3.0822955	4.5277750	1.3795418	3.0848509	4.5292105	1.4124333	3.0993490	4.6319682
Analytic	Method		1.3338873	3.0822955	4.5277750	1.3795418	3.0848509	4.5292105	1.4124333	3.0993490	4.6319682
			1st. Mode	2nd. Mode	3rd. Mode	1st. Mode	2nd. Mode	3rd. Mode	1st. Mode	2nd. Mode	3rd. Mode
100.0	DTM	4	1.7358441	4.3641192	7.8357166	2.1928936	4.3972471	7.5380888	4.4332677	10.5448581	10.9094304
		6	1.7316286	4.3641192	7.5880387	2.1926374	4.3901553	7.5380888	3.6132323	10.0116302	7.5951869
		8	1.7316141	4.3648224	7.5257331	2.1926374	4.3901607	7.5381252	2.4980566	4.5581769	7.5727952
		10	1.7316141	4.3651840	7.5282279	2.1926374	4.3901607	7.5390882	2.4980566	4.5592126	7.5727952
		12	1.7316141	4.3651840	7.5268142	2.1926374	4.3901607	7.5390918	2.4980566	4.5592217	7.5718654
		14	1.7316141	4.3651840	7.5267997	2.1926374	4.3901607	7.5390918	2.4980566	4.5592217	7.5716383
		16	1.7316141	4.3651840	7.5267997	2.1926374	4.3901607	7.5390918	2.4980566	4.5592217	7.5716383
		20	1.7316141	4.3651840	7.5267997	2.1926374	4.3901607	7.5390918	2.4980566	4.5592217	7.5716383
		24	1.7316141	4.3651840	7.5267997	2.1926374	4.3901607	7.5390918	2.4980566	4.5592217	7.5716383
		32	1.7316141	4.3651840	7.5267997	2.1926374	4.3901607	7.5390918	2.4980566	4.5592217	7.5716383

Table 2(b) Continued

Analytic	Method		1.7316141	4.3651840	7.5267997	2.1926374	4.3901607	7.5390918	2.4980566	4.5592217	7.5716383
		4	2.7027567	5.3563740	21.2571168	5.1241532	7.7877561	9.8286221	10.8956074	9.759754	11.3709579
		6	2.6737902	5.3555219	11.3557654	5.0944005	7.7328592	9.8286221	10.7672870	9.7519754	11.3709579
		8	2.6736008	5.3543082	9.8664389	5.0942857	7.7324517	9.8272774	10.5611870	9.7519754	11.3709579
		10	2.6736008	5.3541634	9.6874000	5.0942857	7.7324517	9.8271957	6.6877354	9.7541486	11.3689601
		12	2.6736008	5.3541691	9.6836063	5.0942857	7.7324517	9.8271015	6.6877354	9.7454076	11.3446425
1000.0	DTM	14	2.6736008	5.3541691	9.6835661	5.0942857	7.7324517	9.8271015	6.6877354	9.7453330	11.3443556
		16	2.6736008	5.3541691	9.6835661	5.0942857	7.7324517	9.8271015	6.6877354	9.7453330	11.3443556
		20	2.6736008	5.3541691	9.6835661	5.0942857	7.7324517	9.8271015	6.6877354	9.7453330	11.3443556
		24	2.6736008	5.3541691	9.6835661	5.0942857	7.7324517	9.8271015	6.6877354	9.7453330	11.3443556
		32	2.6736008	5.3541691	9.6835661	5.0942857	7.7324517	9.8271015	6.6877354	9.7453330	11.3443556
Analytic	Method		2.6736008	5.3541691	9.6835661	5.0942857	7.7324517	9.8271015	6.6877354	9.7453330	11.3443556
			1st. Mode	2nd. Mode	3rd. Mode	1st. Mode	2nd. Mode	3rd. Mode	1st. Mode	2nd. Mode	3rd. Mode
		4	8.1357291	33.6725269	33.9650557	14.1366954	21.2294568	31.4799780	32.4070643	26.2994117	32.4070643
		6	7.1405168	29.6207353	33.9059612	7.8259916	14.3275517	28.3259980	25.7808626	21.3156636	25.7808626
		8	4.0848642	8.4607854	16.0202337	7.0898176	14.3030941	21.3082859	25.7767558	21.2964071	25.7767558
		10	4.0849006	8.4683630	15.9875246	7.0898358	14.3066376	21.2894055	16.2375315	21.2964071	25.7393174
		12	4.0849006	8.4685084	15.9875246	7.0898358	14.3066740	21.2887876	15.3115354	21.2964071	25.7392174
100 00.0	DTM	14	4.0849006	8.4685084	15.9967013	7.0898358	14.3066740	21.2887876	15.3115354	21.2948206	25.7393174
		16	4.0849006	8.4685084	15.9964106	7.0898358	14.3066740	21.2887876	15.3115354	21.2946207	25.7393039
		20	4.0849006	8.4685084	15.9964106	7.0898358	14.3066740	21.2887876	15.3115354	21.2946026	25.7392311
		24	4.0849006	8.4685084	15.9964106	7.0898358	14.3066740	21.2887876	15.3115354	21.2946026	25.7392311
		32	4.0849006	8.4685084	15.9964106	7.0898358	14.3066740	21.2887876	15.3115354	21.2946026	25.7392311
Analytic	Method		4.0849006	8.4685084	15.9964106	7.0898358	14.3066740	21.2887876	15.3115354	21.2946026	25.7392311
		4	9.9347570	72.4833737	77.3208326	13.5810129	22.0419105	63.7450498	222.5992031	50.3011695	222.5992031
		6	8.5448669	9.9226875	19.6099621	19.0912606	42.0733283	73.9661415	54.6277433	51.4686358	69.4442715
		8	4.6252654	9.8907895	18.9949326	15.3792546	27.2259942	37.0028130	53.8847789	50.4971695	69.4100746
		10	4.6253804	9.8905698	18.9949326	8.8497082	19.2651299	36.3505401	52.3972982	50.4971695	64.6731891
		12	4.6253804	9.8899849	18.9903365	8.8489492	19.2620729	36.3413465	50.3919207	50.4971695	64.6731891
100000.0	DTM	14	4.6253804	9.8899849	18.9879801	8.8420642	19.2617280	36.3269206	23.8765332	50.4942937	64.6731891
		16	4.6253804	9.8899849	18.9880951	8.8420642	19.2617280	36.3266907	23.7100890	50.4845686	64.6731891
		20	4.6253804	9.8899849	18.9880951	8.8420642	19.2617280	36.3266907	23.7125029	50.3317456	64.6114681
		24	4.6253804	9.8899849	18.9880951	8.8420642	19.2617280	36.3266907	23.7125029	50.3318031	64.5707766
		32	4.6253804	9.8899849	18.9880951	8.8420642	19.2617280	36.3266907	23.7125029	50.3318031	64.5707766
Analytic	Method		4.6253804	9.8899849	18.9880951	8.8420642	19.2617280	36.3266907	23.7125029	50.3318031	64.5707766
			1st. Mode	2nd. Mode	3rd. Mode	1st. Mode	2nd. Mode	3rd. Mode	1st. Mode	2nd. Mode	3rd. Mode
		4	8.2645246	536.8996153	190.0077174	20.2746659	38.6452771	113.2599069	364.1948285	127.6098176	393.0779872
		6	5.1279922	11.9003869	176.4313118	17.6725124	35.9290148	89.4538638	59.7426894	125.2069911	365.7685490
		8	5.0264208	11.0683737	21.2412664	16.9102727	35.6553717	72.9702272	59.0027991	123.6926871	364.1948285
		10	5.0360510	10.8828560	21.2412664	16.7387461	30.0698540	50.2774719	58.9551932	121.9238188	364.0988899
		12	5.0358693	10.8774049	21.2412664	13.6352770	22.2741588	43.5635415	57.9071364	119.4252354	330.5928705
1000 000.0	DTM	14	5.0358693	10.8774049	21.1119484	10.1302466	22.2684813	43.5635415	56.6245024	108.8376450	215.6307798
		16	5.0358693	10.8774049	21.1188531	9.9981492	22.2921025	43.2628685	54.8625387	84.3760214	212.3188986

Table 2(b) Continued

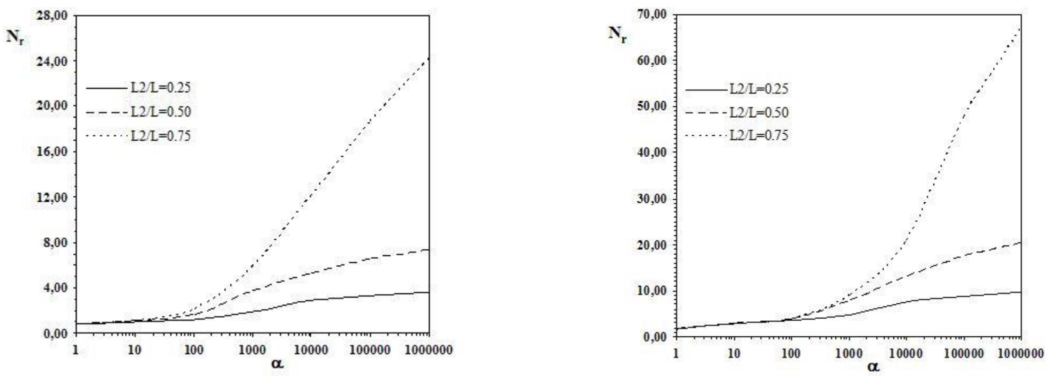
20	5.0358693	10.8774049	21.1190348	10.0048722	22.2895587	43.1592984	52.5498364	70.5055344	211.6593205
24	5.0358693	10.8774049	21.1190348	10.0048722	22.2895587	43.1592984	30.2619130	70.3793392	134.5357504
32	5.0358693	10.8774049	21.1190348	10.0048722	22.2895587	43.1592984	30.2190313	70.5837539	132.4034781
Analytic Method	5.0358693	10.8774049	21.1190348	10.0048722	22.2895587	43.1592984	30.2190313	70.5837539	132.4034781

Table 3(a) $\overline{N}/N_E$ values for the first, second, third modes of the pile and *f*=0,25

α	L_2/L=0.25			L_2/L=0.5			L_2/L=0.75		
	1st. Mode	2nd. Mode	3rd. Mode	1st. Mode	2nd. Mode	3rd. Mode	1st. Mode	2nd. Mode	3rd. Mode
1	1.1764	4.1947	9.1991	1.1807	4.1953	9.1993	1.1845	4.1962	9.1996
10	1.1860	4.2005	9.2022	1.2291	4.2051	9.2049	1.2694	4.2134	9.2067
100	1.2763	4.2533	9.2363	1.6978	4.3172	9.2566	2.1165	4.3870	9.2770
1000	1.9468	4.9037	9.6329	3.8477	7.8514	9.8166	6.0148	9.8717	10.7165
10000	2.9366	7.8211	15.20161	5.3207	13.6942	21.4281	12.5312	21.7770	25.9056
1000000	3.4025	8.9199	17.6560	6.6210	18.0238	35.1280	19.2193	50.2092	66.2099
10000000	3.7776	9.8800	19.4694	7.5588	20.6413	40.9920	24.700	68.8694	132.7988

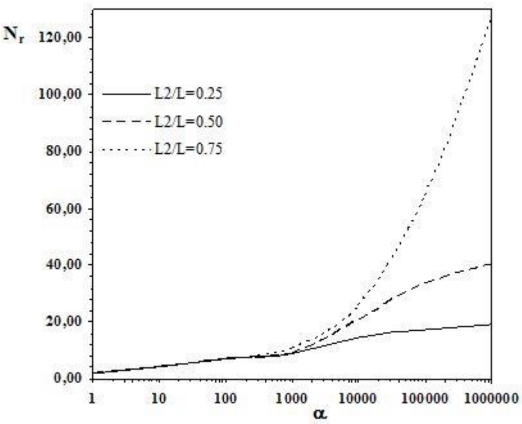
Table 3(b) $\overline{N}/N_E$ values for the first, second, third modes of the pile and *f*=0,75

α	L_2/L=0.25			L_2/L=0.5			L_2/L=0.75		
	1st. Mode	2nd. Mode	3rd. Mode	1st. Mode	2nd. Mode	3rd. Mode	1st. Mode	2nd. Mode	3rd. Mode
1	1.5061	4.1149	9.3578	1.5108	4.1152	9.3578	1.6139	4.1170	9.3581
10	1.5481	4.7578	9.8520	1.5939	4.7606	9.8538	1.6296	4.7743	9.8557
100	1.8262	5.1574	10.3934	2.2909	5.1807	10.4079	2.6107	5.3493	10.4370
10000	2.7273	5.6292	10.7900	5.3445	7.8606	10.9185	6.9515	10.5696	11.7634
100000	4.1577	8.7515	16.9215	7.2687	14.8863	22.1259	15.8167	22.2131	25.9347
1000000	4.6899	10.0234	19.4032	9.0119	19.7710	37.7948	24.2770	52.6001	66.2099
10000000	5.2306	11.0424	21.5035	10.1753	22.6764	43.8992	30.8024	72.6471	138.3200



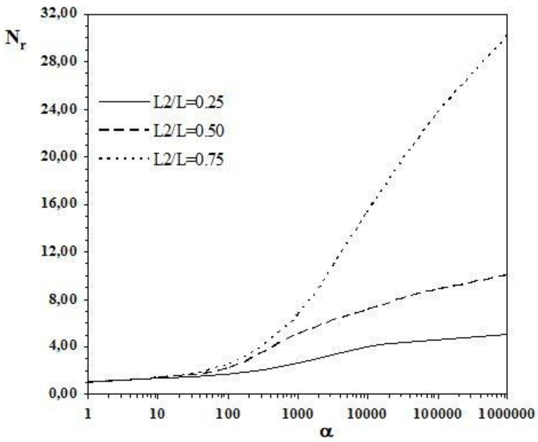
(a) (b)

Fig. 4 Variation of *Nr* value with relative stiffness for *f*=0.25 (a) the first mode, (b) the second mode, (c) the third mode

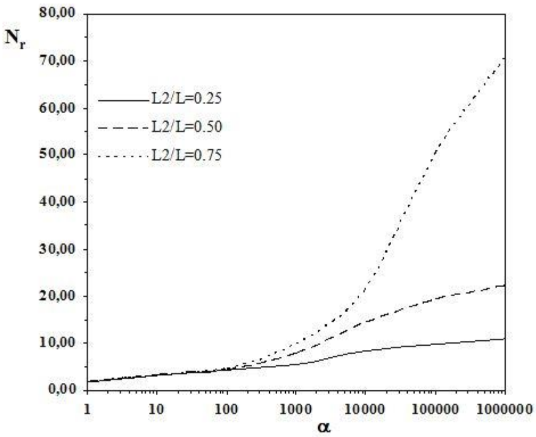


(c)

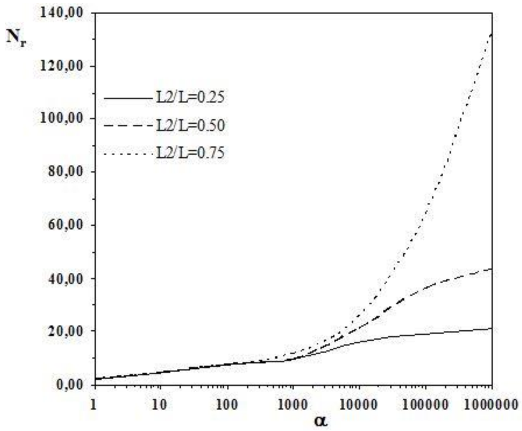
Fig. 4 Continued



(a)



(b)



(c)

Fig. 5 Variation of *Nr* value with relative stiffness for *f*=0.75 (a) the first mode, (b) the second mode, (c) the third mode

Variation of N_r=N /N_E and α according to L_2/L=0.25, L_2/L=0.50, L_2/L=0.75, f=0.25, f=0.75 and series size n=32 are shown in Fig. 4(a), (b), (c) and Fig. 5(a), (b), (c) for the pile.
Fig. 4 and Fig. 5 that give the variation between relative stiffness and N_r values of the pile partially embedded in the soil indicates that N_r values of the pile having relative stiffness between 100 and 1.000.000 increases as L_2/L values increase for all modes f=0.25 and f=0.75. N_r values of the pile having relative stiffness between 1 and 100 are same for L_2/L=0.25, L_2/L=0.50, L_2/L=0.75, f=0.25 and f=0.75.

7. Conclusions

The buckling loads for the first three modes of the both ends simply supported and upper end semi-rigid connected pile calculated by using DTM and analytical method according modulus of subgrade reactions and variation of L_2/L and f values.

In the analytical method, the boundary conditions of the pile are used for obtaining closed-form solution function of the buckling load and the calculation of following derivates necessary in these boundary conditions become more difficult when the order of derivates increases. However calculation of high-order derivates necessary in the analytical method are calculated easier while the DTM is being applied for critical buckling load of the pile, because Taylor series is used as solution function.

Buckling loads of pile values obtained for the first mode, L_2/L=0.25, L_2/L=0.50, L_2/L=0.75, f=0.25 and f=0.75 and relative stiffness between 1 and 100.000 using DTM for series size n=6 and n>6 are same. DTM results indicate that Nr values of the first mode are very fast converging for L_2/L value and f=0.25 and f=0.75 values, and that converging speed decrease as the number of modes increase.

It is seen from Table 2(a), (b) that all buckling loads obtained by using analytical method and DTM for n=32 overlap. Also, the results of DTM and analytical method for higher modes obtained by author for n=32 overlap.

The results in Table 3(a), (b) and Table 2(a), (b) indicate that the buckling loads of the pile are calculated by neglecting the effects of shear deformation are over than the buckling loads of the pile are calculated by taking shear deformations. It is seen form these tables that the shear deformation effect is more important especially in case of short piles.

The results of DTM and analytical method in Table 2(a), (b) indicate that the DTM can be applied for buckling problem of partially embedded and semi-rigid connected piles.

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