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Feedback control design for intelligent structures with closely-spaced eigenvalues

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Abstract. Large space structures may have resonant low eigenvalues and often these appear with closely-spaced natural frequencies. Owing to the coupling among modes with closely-spaced natural frequencies, each eigenvector corresponding to closely-spaced eigenvalues is ill-conditioned that may cause structural instability. The subspace to an invariant subspace corresponding to closely-spaced eigenvalues is well-conditioned, so a method is presented to design the feedback control law of intelligent structures with closely-spaced eigenvalues in this paper. The main steps are as follows: firstly, the system with closely-spaced eigenvalues is transformed into that with repeated eigenvalues by the spectral decomposition method; secondly, the computation for the linear combination of eigenvectors corresponding to repeated eigenvalues is obtained; thirdly, the feedback control law is designed on the basis of the system with repeated eigenvalues; fourthly, the system with closely-spaced eigenvalues is regarded as perturbed system on the basis of the system with repeated eigenvalues; finally, the feedback control law is applied to the original system, the first order perturbations of eigenvalues are discussed when the parameter modifications of the system are introduced. Numerical examples are given to demonstrate the application of the present method.

Keywords: feedback control; closely-spaced eigenvalue; intelligent structure; perturbed analysis

1. Introduction

Contemporary structures in many engineering fields may have large amplitude vibration when there is a small external excitation that may cause structural instability. Specially, large space structures may have resonant low eigenvalues and often these appear with closely-spaced natural frequencies (Rao and Pan 1990, Rao 1994). In this case, one important characteristic may arise (Chen 1992, Liu 1999): if the small changes are made on the structural parameters, the eigenvectors corresponding to the multiple eigenvalues may have a jump because in this case eigenvector corresponding to distinct eigenvalues of the real symmetric matrix is well-conditioned and to the closely-spaced eigenvalues may be ill-conditioned which depend on the distinctness of eigenvalues. The vibration control analysis of structures with closely-spaced natural frequencies has been an active research area. Controllability and observability criteria for multivariable linear second-order models are discussed (Laub *et al*. 1984). The controllability and its measurement of

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repeated eigenvalues is discussed (Liu *et al*. 1994) but does not design the feedback control law of the system with repeated eigenvalues. Perturbation analysis of vibration modes with close frequencies is discussed (Chen *et al*. 1993). The condition for the existence of output feedback gain matrices is discussed to arrive at the desired eigenvalue placements (Maghami *et al*. 1997). A method is developed for syntheses of output feedback gains (Srinathkumar *et al*. 1978). A variable gain algorithm for direct velocity feedback to suppress transient response of structures with only two closely spaced natural frequencies is proposed on the basis of perturbation solutions of the initial value problem (Abe 1998). A technique for determining the optimum locations of piezoelectric sensors and actuators of intelligent structures is presented with element sensitivities of the singular-value which are used to measure the observability and controllability of intelligent structures (Chen *et al*. 2000). The active control of the intelligent structures with the uncertainties is studied (Cao *et al*. 2003), but the problem about systems with closely-spaced eigenvalues is not mentioned. In order to get practical conditions of structural controllability, two necessary conditions of controllability of systems with repeated eigenvalues (regular and defective systems) are discussed (Yao *et al*. 2011). An isoparametric element method was constructed to solve 3-D crack problems (Cao *et al*. 2012), this method can be widely used in the numerical analysis of 3-D crack fields in engineering but not mention of vibration modes with closely-spaced natural frequencies. A recursive procedure for designing the feedback controller of the multi-input system with defective repeated eigenvalues was presented (Chen 2007). A method to calculate the equivalent stiffness of the wheel center was obtained with the stiffness of the bushings (Zhao *et al*. 2012), which would be used in suspension structures. An influence of the frequency intensity on the controllability of structures with closely-spaced natural frequencies was investigated (Xie *et al*. 2009). A new method was presented to define close modes based on mode shape sensitivity to structure parameters，which emphasized the impacts of close modes on control (Liu and Hu 2010). A dimensional decomposition method for obtaining probabilistic descriptors of real value eigenvalues of positive semi-definite random matrices was presented (Rahman 2007). A method was developed to optimally locate actuators and sensors for structures with close modes (Liu and Hu 2010).

However, vibration modes with closely-spaced natural frequencies often occur in some structural systems, such as multispan beams, some nearly periodic structures and large space structures. Owing to the coupling among modes with closely-spaced natural frequencies, each eigenvector corresponding to closely-spaced eigenvalues is ill-conditioned and subspace to an invariant subspace corresponding to closely-spaced eigenvalues is well-conditioned. That is why the method to deal with vibration control of structures with distinct eigenvalues can not be used to deal with vibration control of structures with closely-spaced eigenvalues. It is very necessary to develop the analysis approach of vibration modes of intelligent structures with closely spaced eigenvalues. The main idea in this paper is how to design the feedback control law of intelligent structures with closely-spaced eigenvalues.

2. Eigenproblem with closely-spaced eigenvalues

2.1 Motion equations of intelligent structures

The vibration control equations of intelligent structures with distributed sensors and actuators are given (Chen *et al*. 1999) as follows

$$\begin{cases} \mathbf{M}_0\ddot{\mathbf{q}}(t) + \mathbf{C}_0\dot{\mathbf{q}}(t) + \mathbf{K}_0\mathbf{q}(t) = \mathbf{B}_0\mathbf{F}(t) \\ \mathbf{V}_s = \mathbf{D}_0\mathbf{q}(t) \end{cases} \tag{1}$$

where $\mathbf{q}$ is a displacement vector; $\mathbf{M}_0$, $\mathbf{K}_0$ and $\mathbf{C}_0$ are the real symmetric mass, stiffness and damping matrices of the entire system, $\in R^{n\times n}$, respectively; $\mathbf{F}(t)$ is a control force vector, $\in R^{p}$; $\mathbf{B}_0$ is the controllable matrix determined by placements of actuators, $\in R^{n\times p}$; $\mathbf{D}_0$ is the observable matrix determined by placements of sensors, $\in R^{p\times n}$. If the sensor is not only used for measuring the motion, but also for controlling and suppressing the vibration of intelligent structures as the actuator, then $\mathbf{D}_0$ is written as follows

$$\mathbf{D}_0 = d\mathbf{B}_0^T \tag{2}$$

Where d is a constant that is concerned with the dielectric constant and the thickness of intelligent materials as sensors bonded on the surface of the main structure.

2.2 *Transformation of eigenproblem*

If the right side of Eq. (1) and $\mathbf{C}_0$ are equal to zero, the vibration eigenproblem of Eq. (1) can be obtained and written in the partitioned form

$$\begin{cases} \mathbf{K}_0[\Phi_0 \vdots \Phi_A] = \mathbf{M}_0[\Phi_0 \vdots \Phi_A]diag(\Lambda_0, \Lambda_A) \\ [\Phi_0 \vdots \Phi_A]^T \mathbf{M}_0[\Phi_0 \vdots \Phi_A] = \mathbf{I} \end{cases} \tag{3}$$

Where $\mathbf{\Lambda}_0$ is a diagonal of closely-spaced eigenvalues matrix, $\in R^{m\times m}$, $\mathbf{\Phi}_0$ is a modal matrix corresponding to closely-spaced eigenvalues, $\in R^{n\times m}$. $\mathbf{\Lambda}_A$ and $\mathbf{\Phi}_A$ are the remaining distinct eigenvalue matrix and corresponding modal matrix, respectively. By the spectral decomposition, $\mathbf{K}_0$ can be expressed as follows

$$\mathbf{K}_0 = \mathbf{M}_0(\mathbf{\Phi}_0\mathbf{\Lambda}_0\mathbf{\Phi}_0^T)\mathbf{M}_0 + \mathbf{M}_0(\mathbf{\Phi}_A\mathbf{\Lambda}_A\mathbf{\Phi}_A^T)\mathbf{M}_0 \tag{4}$$

Let λ_0 be the average of the closely-spaced eigenvalues

$$\boldsymbol{\lambda}_0 = (\sum_{i=1}^{m} \boldsymbol{\lambda}_{0i})/m \tag{5}$$

thus $\mathbf{\Lambda}_0$ can be expressed as

$$\mathbf{\Lambda}_0 = \boldsymbol{\lambda}_0\mathbf{I} + \varepsilon\delta\mathbf{\Lambda}_0 \tag{6}$$

where

$$\varepsilon\delta\mathbf{\Lambda}_0 = \mathbf{\Lambda}_0 - \boldsymbol{\lambda}_0\mathbf{I} = \mathbf{\Lambda}_0 - ((\sum_{i=1}^{m} \boldsymbol{\lambda}_{0i})/m)\mathbf{I} \tag{7}$$

Substituting Eq. (6) into Eq. (4), we have

$$\mathbf{K}_0 = \underline{\mathbf{K}}_0 + \varepsilon\delta\mathbf{K}_0 \tag{8}$$

where

$$\begin{cases} \underline{\mathbf{K}}_0 = \mathbf{M}_0(\boldsymbol{\lambda}_0\boldsymbol{\Phi}_0\boldsymbol{\Phi}_0^T)\mathbf{M}_0 + \mathbf{M}_0(\boldsymbol{\Phi}_A\boldsymbol{\Lambda}_A\boldsymbol{\Phi}_A^T)\mathbf{M}_0 \\ \varepsilon\delta\mathbf{K}_0 = \mathbf{M}_0(\boldsymbol{\Phi}_0(\varepsilon\delta\boldsymbol{\Lambda}_0)\boldsymbol{\Phi}_0^T)\mathbf{M}_0 \end{cases} \tag{9}$$

For the system with $\underline{\mathbf{K}}_0$ given by Eq. (9) and $\mathbf{M}_0$, the following equation exists

$$\begin{cases} \underline{\mathbf{K}}_0\left[\Phi_0 \vdots \Phi_A\right] = \mathbf{M}_0\left[\Phi_0 \vdots \Phi_A\right] diag\left(\boldsymbol{\lambda}_0\mathbf{I}, \Lambda_A\right) \\ \left[\Phi_0 \vdots \Phi_A\right]^T \mathbf{M}_0\left[\Phi_0 \vdots \Phi_A\right] = \mathbf{I} \end{cases} \tag{10}$$

Eq. (10) indicates that the repeated eigenvalues $\boldsymbol{\lambda}_0$ with multiplicity m and corresponding eigenvector subspace $\boldsymbol{\Phi}_0$ are the eigensolutions of the system ($\underline{\mathbf{K}}_0$, $\mathbf{M}_0$) and $\boldsymbol{\Lambda}_A$ and $\boldsymbol{\Phi}_A$ are also the eigensolutions of the same system.

As can be seen from Eq. (8) that $\mathbf{K}_0$ is equal to the sum of $\underline{\mathbf{K}}_0$ and $\varepsilon\delta\mathbf{K}_0$. That is the system ($\mathbf{K}_0$ and $\mathbf{M}_0$) with closely-spaced eigenvalues is changed into that ($\underline{\mathbf{K}}_0$ and $\mathbf{M}_0$) with repeated eigenvalues. At the same time two systems both with closely-spaced eigenvalues and repeated eigenvalues have the same eigenvector space ($[\boldsymbol{\Phi} \vdots \boldsymbol{\Phi}_A]$). This indicates that the dynamic characteristics of the system with closely-spaced eigenvalues are the same as those of the system with repeated eigenvalue which is equal to the average value of closely-spaced eigenvalues. The method to design the feedback control law of the system with closely-spaced eigenvalues can be transformed into that to design the feedback control law of the system with repeated eigenvalues.

3. Design of the feedback control law of intelligent structures

3.1 Eigenproblem of intelligent structures with repeated eigenvalues

From the above discussion, it can be seen that the eigenproblem of intelligent structures ($\mathbf{K}_0$ and $\mathbf{M}_0$) with closely-spaced eigenvalues can be transformed into that ($\underline{\mathbf{K}}_0$ and $\mathbf{M}_0$) with repeated eigenvalues. Thus the control equation for the system ($\underline{\mathbf{K}}_0$ and $\mathbf{M}_0$) with repeated eigenvalues can be written as

$$\mathbf{M}_0\ddot{\mathbf{q}}(t) + \underline{\mathbf{C}}_0\dot{\mathbf{q}}(t) + \underline{\mathbf{K}}_0\mathbf{q}(t) = \mathbf{B}_0\mathbf{F}(t) \tag{11}$$

where

$$\mathbf{C}_0 = \underline{\mathbf{C}}_0 + \varepsilon\delta\mathbf{C}_0 \tag{12}$$

or

$$\underline{\mathbf{C}}_0 = \mathbf{C}_0 - \varepsilon\delta\mathbf{C}_0 \tag{13}$$

As the above discussion, each eigenvector corresponding to closely-spaced or repeated eigenvalues is ill-conditioned, and the subspace corresponding to repeated eigenvalues is well-conditioned. Although the eigenvectors corresponding to the multiple eigenvalues are not unique, the linear combination of $\boldsymbol{\Phi}_0$, denoted by $\underline{\boldsymbol{\Phi}}_0$, is also the eigenvector associated with $\boldsymbol{\lambda}_0$. The eigenproblem of the system with repeated eigenvalues is as follows

$$\begin{cases} \underline{\mathbf{K}}_0\mathbf{\Phi}_0 = \mathbf{M}_0\mathbf{\Phi}_0[\boldsymbol{\lambda}_0\mathbf{I}, \Lambda_A] \\ \mathbf{\Phi}_0^T\mathbf{M}_0\mathbf{\Phi}_0 = \mathbf{I} \end{cases} \tag{14}$$

By Eq. (8), the following equation can be obtained

$$\begin{cases} \mathbf{K}_0 = \underline{\mathbf{K}}_0 + \varepsilon\delta\mathbf{K}_0 \\ \mathbf{M}_0 = \mathbf{M}_0 \end{cases} \tag{15}$$

Substituting Eq. (15) into Eq. (14) yields

$$\begin{cases} (\underline{\mathbf{K}}_0 + \varepsilon\delta\mathbf{K}_0)\underline{\mathbf{\Phi}}_0 = \mathbf{M}_0\underline{\mathbf{\Phi}}_0\mathbf{\Lambda}_0 \\ \underline{\mathbf{\Phi}}_0^T\mathbf{M}_0\underline{\mathbf{\Phi}}_0 = \mathbf{I} \end{cases} \tag{16}$$

Because ε is a small scalar parameter, the norm of matrix $\varepsilon\delta\mathbf{K}_0$ is significantly smaller than the norm of $\underline{\mathbf{K}}_0$. Thus Eq. (16) can be considered as the perturbation of Eq. (14). Since $\boldsymbol{\lambda}_0$ is the repeated eigenvalue with multiplicity equal to m, and ϕ_{01}, ϕ_{02}, …, ϕ_{0m} are the eigenvectors associated with $\boldsymbol{\lambda}_0$, then the linear combination of ϕ_{0j}(j=1, 2, …, m), denoted by $\underline{\mathbf{\Phi}}_0$, is also an eigenvector associated with $\boldsymbol{\lambda}_0$, i.e.

$$\underline{\mathbf{\Phi}}_0 = \mathbf{\Phi}_0\mathbf{b} \tag{17}$$

where

$$\mathbf{\Phi}_0 = [\phi_{01},\ \phi_{02},\ \cdots,\ \phi_{0m}] \tag{18}$$

$$\mathbf{b}^T\mathbf{b} = 1 \tag{19}$$

where

$$\mathbf{b}^T = [b_1,\ b_2,\ \cdots,\ b_m] \tag{20}$$

Note that $\mathbf{b}$ is arbitrary constant vector. According to the matrix perturbation (Chen1992), we obtain the $m\times m$ eigenproblem

$$\mathbf{W}\mathbf{b} = \mathbf{b}\mathbf{\Lambda}_1 \tag{21}$$

where

$$\begin{cases} \mathbf{\Lambda}_1 = \mathbf{\Lambda}_0 - \boldsymbol{\lambda}_0\mathbf{I} \\ \mathbf{W} = \mathbf{\Phi}_0^T(\varepsilon\delta\mathbf{K}_0)\mathbf{\Phi}_0 \end{cases} \tag{22}$$

Solving the m×m eigenproblem of Eq. (21) yields $\mathbf{b}$. If matrix $\mathbf{W}$ has no repeated eigenvalues, $\mathbf{b}$ can be uniquely determined; if matrix $\mathbf{W}$ has repeated eigenvalues, $\mathbf{b}$ can be determined through the higher order perturbation equations. Here we assume that the matrix $\mathbf{W}$ has no repeated eigenvalues.

This completes the computation for $\underline{\mathbf{\Phi}}_0$. The complete algorithm is summarized as follows

(1) Compute

$$\boldsymbol{\lambda}_0 = (\sum_{i=1}^{m} \boldsymbol{\lambda}_{0i})/m$$

(2) Compute

$$\mathbf{W} = \boldsymbol{\Phi}_0^T(\varepsilon\delta\mathbf{K}_0)\boldsymbol{\Phi}_0$$

(3) Solve the eigenvalue problem

$$\mathbf{Wb} = \mathbf{b}\boldsymbol{\Lambda}_1$$

$$\boldsymbol{\Lambda}_1 = \boldsymbol{\Lambda}_0 - \boldsymbol{\lambda}_0\mathbf{I}$$

$$\mathbf{b}^T\mathbf{b} = 1$$

(4) Compute the new eigenvectors

$$\underline{\boldsymbol{\Phi}}_0 = \boldsymbol{\Phi}_0\mathbf{b}$$

3.2 Feedback control of intelligent structures with repeated frequencies

Transforming Eq. (11) into the modal coordinates through the coordinate transformation

$$\mathbf{q}(t) = [\underline{\boldsymbol{\Phi}}_0 \vdots \boldsymbol{\Phi}_A]\boldsymbol{\eta}(t) \tag{23}$$

yields

$$\begin{cases} \ddot{\boldsymbol{\eta}}(t) + \mathbf{Z}\dot{\boldsymbol{\eta}}(t) + \boldsymbol{\Lambda}\boldsymbol{\eta}(t) = \underline{\mathbf{B}}\mathbf{F}(t) \\ \mathbf{V}_s = \underline{\mathbf{D}}\boldsymbol{\eta}(t) \end{cases} \tag{24}$$

where $\mathbf{Z}=[\underline{\boldsymbol{\Phi}}_0 \vdots \boldsymbol{\Phi}_A]^T\underline{\mathbf{C}}_0[\underline{\boldsymbol{\Phi}}_0 \vdots \boldsymbol{\Phi}_A]$, $\in R^{n\times n}$; $\boldsymbol{\Lambda}=[\underline{\boldsymbol{\Phi}}_0 \vdots \boldsymbol{\Phi}_A]^T\underline{\mathbf{K}}_0[\underline{\boldsymbol{\Phi}}_0 \vdots \boldsymbol{\Phi}_A]$, $\in R^{n\times n}$; $\underline{\mathbf{B}}=[\underline{\boldsymbol{\Phi}}_0 \vdots \boldsymbol{\Phi}_A]^T\mathbf{B}_0$, $\in R^{n\times p}$; $\underline{\mathbf{D}}=\mathbf{D}_0[\underline{\boldsymbol{\Phi}}_0 \vdots \boldsymbol{\Phi}_A]$, $\in R^{p\times n}$. In order to obtain the governing equation corresponding to the repeated eigenvalue in modal coordinates, $\boldsymbol{\eta}_0(t)= \{\eta_{01}(t), \eta_{02}(t), \ldots, \eta_{0m}(t)\}^T$, Eq. (24) can be rewritten in the partitioned form

$$\begin{cases} \begin{bmatrix} \ddot{\boldsymbol{\eta}}_0(t) \\ \cdots \\ \ddot{\boldsymbol{\eta}}_A(t) \end{bmatrix} + \begin{bmatrix} \mathbf{Z}_0 & \\ \cdots & \cdots \\ & \mathbf{Z}_A \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\eta}}_0(t) \\ \cdots \\ \dot{\boldsymbol{\eta}}_A(t) \end{bmatrix} + \begin{bmatrix} \boldsymbol{\lambda}_0\mathbf{I}_m & \\ \cdots & \cdots \\ & \Lambda_A \end{bmatrix} \begin{bmatrix} \boldsymbol{\eta}_0(t) \\ \cdots \\ \boldsymbol{\eta}_A(t) \end{bmatrix} = \begin{bmatrix} \underline{\boldsymbol{\Phi}}_0^T \\ \cdots \\ \boldsymbol{\Phi}_A^T \end{bmatrix} \mathbf{B}_0\mathbf{F}(t) \\ \mathbf{V}_s = \mathbf{D}_0[\underline{\boldsymbol{\Phi}}_0 \vdots \boldsymbol{\Phi}_A] \begin{bmatrix} \boldsymbol{\eta}_0(t) \\ \cdots \\ \boldsymbol{\eta}_A(t) \end{bmatrix} = \mathbf{D}_0\underline{\boldsymbol{\Phi}}_0\boldsymbol{\eta}_0(t) + \mathbf{D}_0\boldsymbol{\Phi}_A\boldsymbol{\eta}_A(t) \end{cases} \tag{25}$$

Where $\boldsymbol{\Lambda}_A=\mathrm{diag}(\omega^2_{m+1}, \omega^2_{m+2}, \ldots, \omega^2_n)$; $\boldsymbol{\omega}_A=\mathrm{diag}(\omega_{m+1}, \omega_{m+2}, \ldots, \omega_n)$; $\mathbf{Z}_0=2\omega_0\xi_0\mathbf{I}_m$; $\mathbf{Z}_A=\mathrm{diag}(2\omega_{m+1}\xi_{m+1}, 2\omega_{m+2}\xi_{m+2}, \ldots, 2\omega_n\xi_n)$; $\mathbf{I}_m$ is the $m\times m$ identity matrix; ξ_i is the modal damping factor. $\underline{\boldsymbol{\Phi}}_0$ is the modal matrix and $\boldsymbol{\lambda}_0=\omega_0^2$, which satisfy

$$\begin{cases} \underline{\mathbf{K}}_0 \underline{\phi}_{0i} = \omega_0^2 \mathbf{M}_0 \underline{\phi}_{0i} \\ \underline{\phi}_{0i}^T \mathbf{M}_0 \underline{\phi}_{0i} = 1 \end{cases} \tag{26}$$

In the next section we will mainly discuss how to design the feedback control law of systems with repeated eigenvalues. Then the observation equation corresponding to repeated eigenvalue modes, denoted by $\mathbf{V}_{s0}$, can be expressed as

$$\mathbf{V}_{s0} = \mathbf{D}_0 \underline{\mathbf{\Phi}}_0 \boldsymbol{\eta}_0(t) = \mathbf{D}\boldsymbol{\eta}_0(t) \tag{27}$$

and the governing equation corresponding to the repeated eigenvalue in term of modal coordinates $\eta_0(t)$ can be rewritten as

$$\ddot{\boldsymbol{\eta}}_0(t) + 2\omega_0\xi_0 \mathbf{I}_m \dot{\boldsymbol{\eta}}_0(t) + \omega_0^2 \mathbf{I}_m \boldsymbol{\eta}_0(t) = \mathbf{BF}(t) \tag{28}$$

where $\mathbf{B}= \underline{\mathbf{\Phi}}_0^{T} \mathbf{B}_0, \ \in R^{m\times p}$; $\mathbf{D}= \mathbf{D}_0\underline{\mathbf{\Phi}}_0, \ \in R^{p\times m}$. Taking the singular value decomposition of **B**, the following equation can be obtained

$$\mathbf{B} = \mathbf{U}_0 \sum \mathbf{V}_0^T \tag{29}$$

Where $\mathbf{U}_0$ and $\mathbf{V}_0$ are left and right singular vectors of **B**, respectively. $\mathbf{U}_0 \in R^{m\times m}$, $\mathbf{V}_0 \in R^{p\times p}$, $\mathbf{U}_0^T\mathbf{U}_0=\mathbf{I}_m$ $\mathbf{V}_0^T\mathbf{V}_0=\mathbf{I}_p$. $\sum = \begin{bmatrix} \Sigma_0 & 0 \\ 0 & 0 \end{bmatrix}_{m\times p}$, $\Sigma_0=\mathrm{diag}(\sigma_{a1}, \sigma_{a2}, \ldots, \sigma_{a\underline{a}})$, in which $\underline{a}$ is the number of controllable modes. σ_{ai} is a measure of the controllability of the ith mode, $\sigma_{ai} >0$. We assume that $\underline{a}=p$ here. Similarly, the singular value decomposition of **D** can be obtained

$$\mathbf{D} = \mathbf{V}_0 \sum{}' \mathbf{U}_0^T \tag{30}$$

where $\sum{}' = \begin{bmatrix} \Sigma_0' & 0 \\ 0 & 0 \end{bmatrix}_{p\times m}$, $\Sigma_0'=\mathrm{diag}(\sigma_{s1}, \sigma_{s2}, \ldots, \sigma_{s\underline{s}})$, in which $\underline{s}$ is the number of controllable modes. σ_{si} is a measure of the controllability of the ith mode, $\sigma_{si} >0$. We assume that $\underline{s}=p$ here.

It can be seen that Eq. (27) does not illustrate the relation between the feedback control force (**F**(t)) and controllability and that Eq. (26) does not illustrate the relation between $\mathbf{V}_s$ and observability. In order to reveal the relation between **F**(t) and controllability and between $\mathbf{V}_s$ and observability, the modal transformation, $\boldsymbol{\eta}_0(t)=\mathbf{U}_0\mathbf{x}(t)$, can be used, Eq. (28) and Eq. (27) can be changed into the following form

$$\ddot{\mathbf{x}}(t) + 2\omega_0\xi_0 \mathbf{I}_m \dot{\mathbf{x}}(t) + \omega_0^2 \mathbf{I}_m \mathbf{x}(t) = \sum\nolimits_0 f(t) \tag{31}$$

$$\mathbf{V}_f = \Sigma_0' \mathbf{x}(t) \tag{32}$$

where $\mathbf{V}_f=\mathbf{V}_0^T \mathbf{V}s_0$, $\mathbf{f}(t)= \mathbf{V}_0^T \mathbf{F}(t)$.

It can be seen from Eq. (31) that the controllability of $\boldsymbol{x_i}$ can be measured by σ_{ai} since **f**(t) is equivalent to **F**(t) in the terms of energy required. The greater σ_{ai} is, the less energy is required to produce the same control effect on $\boldsymbol{x_i}$. Similarly, it can be seen from Eq. (32) that the observability of $\boldsymbol{x_i}$ can be measured by σ_{si} since $\mathbf{V}_s$ is equivalent to $\mathbf{V}_f$. The greater σ_{si} is, the less energy is required to observe $\boldsymbol{x_i}$. For a direct output feedback control system, the control force $\mathbf{f}$(t) of Eq. (31) is assumed to be the following form

$$\mathbf{f}(t) = -\mathbf{G}_1\mathbf{V}_f - \mathbf{G}_2\dot{\mathbf{V}}_f = -\mathbf{G}_1\Sigma_0^{'}\mathbf{x}(t) - \mathbf{G}_2\Sigma_0^{'}\dot{\mathbf{x}}(t) \tag{33}$$

where $\mathbf{G}_1$ and $\mathbf{G}_2$ are the feedback gain matrices of displacement and velocity, $\in R^{p\times p}$. Coefficients of $\mathbf{G}_1$ and $\mathbf{G}_2$ are to be determined. From Eq. (33), it can be seen that $\mathbf{f}$(t) is proportional to $\Sigma_0^{'}$ when $\mathbf{G}_1$, $\mathbf{G}_2$, $\mathbf{x}$(t) and $\dot{\mathbf{x}}$(t) are given. The greater σ_{si} is, the smaller feedback gain is required to produce the same control effect on $\boldsymbol{x_i}$. It is assumed that $\mathbf{G}_1$=diag(g_{11}, g_{12}, …, g_{1p}) and that $\mathbf{G}_2$=diag(g_{21}, g_{22}, …, g_{2p}). Substituting Eq. (33) into Eq. (31), then Eq. (31) can be expressed as

$$\ddot{\mathbf{x}}(t) + (2\omega_0\xi_0\mathbf{I} + \Sigma_0\mathbf{G}_2\Sigma_0^{'})\dot{\mathbf{x}}(t) + (\omega_0^2\mathbf{I} + \Sigma_0\mathbf{G}_1\Sigma_0^{'})\mathbf{x}(t) = 0 \tag{34}$$

Eq. (34) can be uncoupled as

$$\ddot{x}_i(t) + (2\omega_0\xi_0 + \sigma_{ai}\sigma_{si}g_{2i})\dot{x}_i(t) + (\omega_0^2 + \sigma_{ai}\sigma_{si}g_{1i})x_i(t) = 0,\ i = 1,\ 2,\ ...,\ p \tag{35}$$

The key factor of control and suppression vibration is the damping factor. It is assumed that the poles of the closed-loop system are S_{0j} (=-α_j ±$i\beta$), j=1, 2, ..., p, α_j >0. Then we have

$$\begin{cases} 2\omega_0\xi_0 + \sigma_{ai}\sigma_{si}g_{2i} = 2\alpha_i \\ \omega_0^2 + \sigma_{ai}\sigma_{si}g_{1i} = \beta_i^2 + \alpha_i^2 \end{cases},\quad i = 1,\ 2,\ ...,\ p \tag{36}$$

From Eq. (36) we obtain

$$\begin{cases} g_{1i} = (\beta_i^2 + \alpha_i^2 - \omega_0^2)/(\sigma_{ai}\sigma_{si}) \\ g_{2i} = 2(\alpha_i - \omega_0\xi_0)/(\sigma_{ai}\sigma_{si}) \end{cases},\quad i = 1,\ 2,\ ...,\ p \tag{37}$$

Substituting Eq.(37) into Eq.(33), the control force $\mathbf{F}$(t) in Eq.(1) is obtained

$$\mathbf{F}(t) = -\mathbf{V}_0\mathbf{G}_1\Sigma_0^{'}\mathbf{U}_0^T\underline{\Phi}_0^T\mathbf{q}(t) - \mathbf{V}_0\mathbf{G}_2\Sigma_0^{'}\mathbf{U}_0^T\underline{\Phi}_0^T\dot{\mathbf{q}}(t) \tag{38}$$

It can be noted that the design of the feedback control law in Eq.(38) is based on the system with the repeated eigenvalues that are equal to the average of closely-spaced eigenvalues. Eq.(8) indicates that the system with closely-spaced eigenvalues can be referred as the perturbed system on the basis of the system with repeated eigenvalues. Therefore, it is necessary to discuss how the feedback control force affects eigenvalues of original and perturbed system.

4. Perturbation analysis

When the control force $\mathbf{F}$(t) of Eq. (38) is applied to Eq. (11), we have

$$\mathbf{M}_0\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = 0 \tag{39}$$

where $\mathbf{C} = \underline{\mathbf{C}}_0 + \underline{\Phi}_0\mathbf{U}_0\Sigma_0\mathbf{G}_2\Sigma_0^{'}\mathbf{U}_0^T\underline{\Phi}_0^T$; $\mathbf{K} = \underline{\mathbf{K}}_0 + \underline{\Phi}_0\mathbf{U}_0\Sigma_0\mathbf{G}_1\Sigma_0^{'}\mathbf{U}_0^T\underline{\Phi}_0^T$. The eigenvalue problem corresponding to Eq. (39) is

$$(\mathbf{M}_0\mathbf{S}_0^2 + \mathbf{C}\mathbf{S}_0 + \mathbf{K})[\underline{\mathbf{\Phi}}_0 \vdots \mathbf{\Phi}_A] = 0 \tag{40}$$

Let us introduce a state vector

$$\mathbf{u}_0 = \begin{Bmatrix} \mathbf{S}_0[\underline{\mathbf{\Phi}}_0 \vdots \mathbf{\Phi}_A] \\ [\underline{\mathbf{\Phi}}_0 \vdots \mathbf{\Phi}_A] \end{Bmatrix} = \mathbf{T}[\underline{\mathbf{\Phi}}_0 \vdots \mathbf{\Phi}_A] \tag{41}$$

where $\mathbf{T}$ is the state transformation matrix, $\mathbf{T}=[\mathbf{S}_0\mathbf{I}\ \ \mathbf{I}]^T$; $\mathbf{I}$ is the identity matrix. $\mathbf{u}_0$ is the eigenvectors. Hence Eq. (40) becomes

$$(\mathbf{A}_0\mathbf{S}_0 + \mathbf{E}_0)\mathbf{u}_0 = 0 \tag{42}$$

where

$$\mathbf{A}_0 = \begin{bmatrix} 0 & \mathbf{M}_0 \\ \mathbf{M}_0 & \mathbf{C} \end{bmatrix}, \qquad \mathbf{E}_0 = \begin{bmatrix} -\mathbf{M}_0 & 0 \\ 0 & \mathbf{K} \end{bmatrix} \tag{43}$$

If the small changes ($\varepsilon\delta\mathbf{C}_0$ and $\varepsilon\delta\mathbf{K}_0$) are introduced, the corresponding state equation of Eq.(1) is as follows

$$(\mathbf{A}\mathbf{S} + \mathbf{E})\mathbf{u} = 0 \tag{44}$$

where

$$\begin{cases} \mathbf{A} = \mathbf{A}_0 + \varepsilon\mathbf{A}_1 \\ \mathbf{E} = \mathbf{E}_0 + \varepsilon\mathbf{E}_1 \end{cases} \tag{45}$$

where

$$\varepsilon\mathbf{A}_1 = \begin{bmatrix} 0 & 0 \\ 0 & \varepsilon\delta\mathbf{C}_0 \end{bmatrix}, \qquad \varepsilon\mathbf{E}_1 = \begin{bmatrix} 0 & 0 \\ 0 & \varepsilon\delta\mathbf{K}_0 \end{bmatrix} \tag{46}$$

The eigenvalue and eigenvector can be expressed as the power series in ε, that is

$$\mathbf{S} = \mathbf{S}_0 + \varepsilon\mathbf{S}_1 + \varepsilon^2\mathbf{S}_2 + \cdots \tag{47}$$

$$\mathbf{u} = \mathbf{u}_0 + \varepsilon\mathbf{u}_1 + \varepsilon^2\mathbf{u}_2 + \cdots \tag{48}$$

where $S_{0j}==-\alpha_j \pm i\beta$. According to the matrix perturbation theory (Chen 1992), the first order perturbation of eigenvalues is obtained as follows

$$\mathbf{S}_1^{(i)} = -\mathbf{u}_{0i}^T(\varepsilon\mathbf{E}_1 + \mathbf{S}_0^{(i)}\varepsilon\mathbf{A}_1)\mathbf{u}_{0i} \tag{49}$$

It is well known that the change of the jth damping factor of the closed-loop system only concerns with the real part of $\mathbf{S}_1^{(i)}$. So we define δ_{min}=min $Re(\mathrm{S}_1)$, δ_{max}=max $Re(\mathrm{S}_1)$, then the jth modal damping factor (η_j) of the closed-loop system of intelligent structures satisfies the following condition

$$\alpha_j + \delta_{\min} \le \eta_j \le \alpha_j + \delta_{\max} \tag{50}$$

It can be seen from Eq. (50) that if α_j is suitably selected, intelligent structures will have enough dynamic stability tolerance. When the small modifications ($\varepsilon\mathbf{K}_1$, $\varepsilon\mathbf{M}_1$, and $\varepsilon\mathbf{C}_1$) are introduced to the matrices ($\mathbf{K}_0$, $\mathbf{M}_0$, and $\mathbf{C}_0$), respectively, the state equation of the closed-loop eigenproblem of intelligent structures like Eq. (42) becomes as follows

$$[(\mathbf{A}_0 + \varepsilon\underline{\mathbf{A}}_1)\mathbf{S}' + (\mathbf{E}_0 + \varepsilon\underline{\mathbf{E}}_1)]\mathbf{u}' = 0 \tag{51}$$

where

$$\varepsilon\underline{\mathbf{A}}_1 = \begin{bmatrix} 0 & \varepsilon\mathbf{M}_1 \\ \varepsilon\mathbf{M}_1 & \varepsilon\delta\mathbf{C}_0 + \varepsilon\mathbf{C}_1 \end{bmatrix}, \qquad \varepsilon\underline{\mathbf{E}}_1 = \begin{bmatrix} -\varepsilon\mathbf{M}_1 & 0 \\ 0 & \varepsilon\delta\mathbf{K}_0 + \varepsilon\mathbf{K}_1 \end{bmatrix} \tag{52}$$

It is obvious that Eq. (52) is similar to Eq. (46). There is no difficulties to obtain the first order perturbations (S'_1) of the eigenproblem when parameters of intelligent structures have small modifications. Similarly, we can completely obtain the jth modal damping factor (η'_1) of the perturbed system of intelligent structures under the feedback control force. The jth modal damping factor (η'_1) satisfies the following condition

$$\alpha_j + \delta'_{\min} \le \eta_j \le \alpha_j + \delta'_{\max} \tag{53}$$

Where $\delta'_{\min}$=min $Re(S'_1)$, $\delta'_{\max}$=max $Re(S'_1)$.

Obviously, when α_j is selected large enough in design of the feedback control force of intelligent structures in Eq. (53), perturbed system of intelligent structures may have the dynamic stability we need.

5. Numerical example

The numerical example of a simply supported plate is given to illustrate the application of the method presented in this paper. A rectangular composite plate simply supported with distributed piezoelectric polyvinylidene fluoride (PVDF) layers bonded on the lower surface of the main structure as sensors and upper surface of the main structure as actuators was used, as shown in Fig. 1, modeled by 8×8 elements. The mass density of the main structure is 7800 Kg/m^3, and Young's modulus E_1=E_2=0.80E+10 N/m^2, and Poison ratio μ_{12}=μ_{21}=0.29. The mass density of the PVDF is 1680Kg/m^3, and Young's modulus E_1=E_2=0.20E+10 N/m^2, and Poison ratio μ_{12}=μ_{21}=0.28. With Z direction being the poling direction, dielectric constants ε_{11}=ε_{22}=ε_{33}=0.1062E-9 and piezoelectric constants e_{31}=e_{32}= 0.046, e_{33}= e_{24}= e_{15}=0.0. The plate lamina of the main structure is 6 mm thick and two PVDF layers, each is 0.05 mm thick.

Four sensors and actuators are put at the 19th, 22th, 43th and 46th elements. The thickness of the tenth element is 1% more than that of other elements. To design the feedback control law of intelligent structures with closely-spaced eigenvalues, three steps for the simply supported plate are used. The first step is the eigenvalue analysis with the finite element method, as shown in Table 1. In Table 1, Λ_0 denotes the eigenvalues of the original open-loop system, λ_0 denotes the

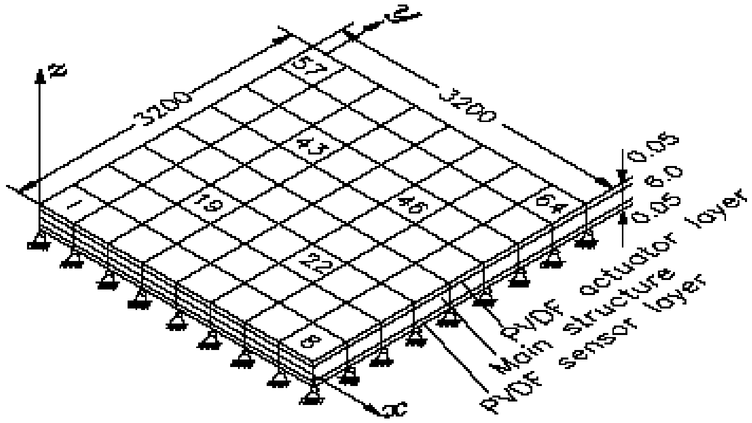

Fig. 1 the simply supported plate with S/As

Table 1 The eigenvalues and its perturbation analysis

mode	eigenvalues			$\varepsilon\delta\lambda_0$	$\mathbf{S}_1$
	Λ_0	λ_0	$\underline{\Lambda}_0$		
1	35.89367	35.89367	35.89367	0	0
2	227.45967	227.48772	227.48772	-0.02805	0.4148E-04-0.02805i
3	227.51577	227.48772	288.54541	0.02805	-0.4655E-04+0.02805i
4	624.99954	624.99954	624.99954	0	0
5	928.36787	928.55237	928.55237	-0.184	0.5409E-03-0.1840i
6	928.73679	928.55237	1079.147	0.184	-0.5824E-03+0.1840i

corresponding repeated eigenvalues, $\underline{\Lambda}_0$ is the eigenvalues of the closed-loop system, $\mathbf{S}_1$ is the first order perturbations of eigenvalues. It is clear from Table 1 that the 2nd and 3rd modes form a closely-spaced eigenvalue subspace and that the 5th, 6th modes form a closely-spaced eigenvalue subspace, where $\varepsilon\delta\lambda_{02}$=-0.02805, $\varepsilon\delta\lambda_{03}$=0.02805, $\varepsilon\delta\lambda_{05}$=-0.184, $\varepsilon\delta\lambda_{06}$=0.184. It is assumed that the damping coefficient of each mode ξ_0 of the main structure is equal to 0.001. The second step is to design feedback gain matrices in order to obtain the control force of intelligent structures with repeated eigenvalues. It is assumed that α_j is equal to 0.4 and β_2^2 =227.48772, β_3^2 =288.54541, β_5^2 =928.55237, β_6^2 =1079.1476. The third step is to discuss (1) the first order perturbations of original system, listed in Table 1, (2) the perturbations of perturbed systems when the thickness of each element is added ±5% and ± 10% to those of the original system, listed in Table 2.

Similarly, it is assumed that α_j is equal to 0.6. We can obtain the first order perturbations of perturbed systems when the thickness of each element of the simply supported plate is added ± 5% and ± 10% to those of the original system, respectively, listed in Table 3.

When there is a velocity impulse at the central point of the simply supported plate, transient displacement responses of the central point of the structure are calculated with Newmar's method after the feedback control law is applied. In Table 1, when the eigenproblem of Eq. (1) with closely-spaced eigenvalues is substituted by the eigenproblem of repeated eigenvalues, the first order perturbation of the eigenvalues (S_1) is very small. In Fig. 2, it can be seen that the responses of the system with closely-spaced eigenvalues is close to those with repeated eigenvalues when the same feedback control law of Eq. (38) is used to both systems. These results illustrate that the feedback control law of intelligent structures with closely-spaced eigenvalues can be replaced by the feedback control law of Eq. (38) designed on the basis of the system with repeated eigenvalues.

It can be seen from Tables 2 and 3 that absolute values of real or imaginary parts of the first order perturbations of eigenvalues increase as structural parameter modifications increase. When the thickness of the simply supported plate decrease, the real parts of the first order perturbations of eigenvalues increase, which make modal damping factors smaller than that of the original system. From Fig. 3, we can see that, consequently, the effect of the control and suppression vibration becomes worse. When the thickness of the plate simply supported increases, the real

Table 2 The first order perturbations of Eigenvalues (α_j=0.4)

Mode	$\mathbf{S}'_{1\ -5\%}$	$\mathbf{S}'_{1\ +5\%}$	$\mathbf{S}'_{1\ -10\%}$	$\mathbf{S}'_{1\ +10\%}$
1	0.00776-0.236i	-0.00776+0.236i	0.0155-0.47i	-0.0155+0.47i
2	0.00744-0.609i	-0.00736+0.553i	0.0148-1.19i	-0.0148+1.13i
3	0.00723-0.621i	-0.00733+0.677i	0.0145-1.27i	-0.0146+1.33i
4	0.00664-0.915i	-0.00664+0.915i	0.0133-1.83i	-0.0133+1.83i
5	0.00661-1.250i	-0.00552+0.891i	0.0127-2.32i	-0.0116+1.96i
6	0.00520-0.951i	-0.00636+1.310i	0.0110-2.08i	-0.0121+2.44i

Table 3 The first order perturbations of Eigenvalues (α_j=0.6)

Mode	$\mathbf{S}'_{1\ -5\%}$	$\mathbf{S}'_{1\ +5\%}$	$\mathbf{S}'_{1\ -10\%}$	$\mathbf{S}'_{1\ +10\%}$
1	0.0176-0.356i	-0.0176+0.356i	0.0353-0.712i	-0.0353+0.712i
2	0.0171-0.910i	-0.0171+0.854i	0.0342-1.790i	-0.0341+1.740i
3	0.0169-0.961i	-0.0170+1.020i	0.0338-1.950i	-0.0339+2.010i
4	0.0160-1.420i	-0.0160+1.420i	0.0319-2.830i	-0.0319+2.830i
5	0.0156-1.860i	-0.0146+1.500i	0.0307-3.540i	-0.0297+3.180i
6	0.0141-1.610i	-0.0153+1.970i	0.0288-3.400i	-0.0299+3.760i

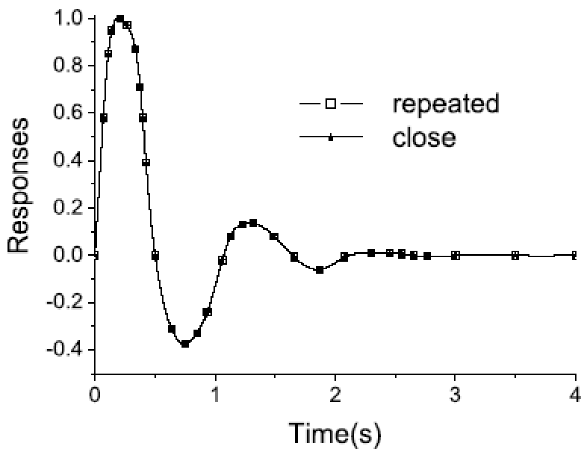

Fig. 2 Responses of intelligent structures

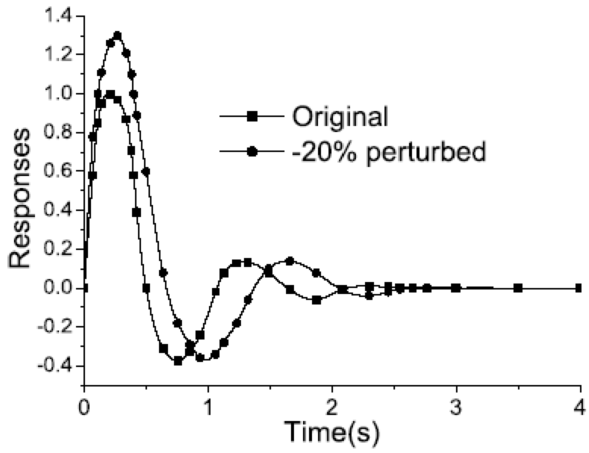

Fig. 3 Closed loop responses of the original system and the perturbed system using the feedback control law designed

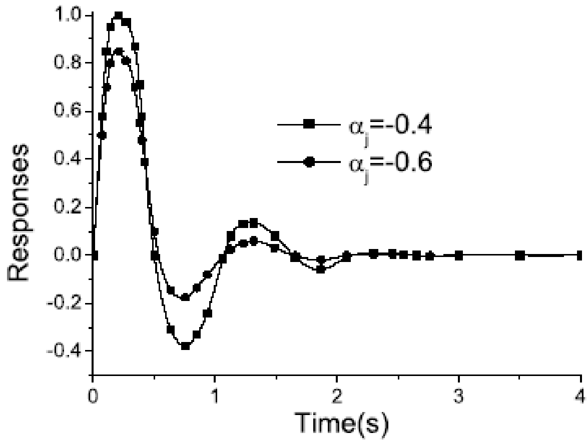

Fig. 4 Closed loop responses of intelligent structures different damping factors

parts of the first order perturbations of eigenvalues decrease, which makes modal damping factors larger than that of the original system. Effect of the control and suppression vibration becomes better. It can be seen from Fig. 4, the larger α_j is, the larger effect of the control vibration is.

When different feedback controls of systems designed according to with distinct eigenvalues and repeated eigenvalues are applied to the system with closely-spaced eigenvalues at the same feedback gain (G=150), transient displacement responses of the central point of the structure are calculated with Newmark's method, respectively, shown in Figs. 5 and 6. Fig. 5 denotes transient displacement responses when feedback control of systems designed according to with distinct eigenvalues is applied to the system with closely-spaced eigenvalues. Fig. 6 denotes transient displacement responses when feedback control of systems designed according to with repeated eigenvalues is applied to the system with closely-spaced eigenvalues. It is seen from Fig. 5 and Fig. 6 that the feedback control effect of transient displacement responses to the excitation is obvious improvement on control by using the method for repeated eigenvalues compared with

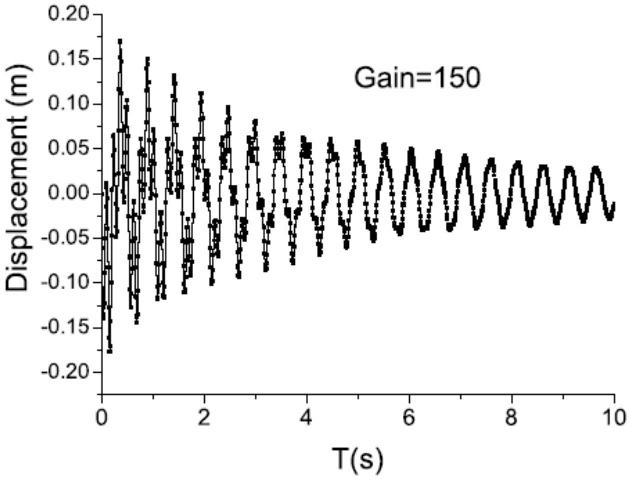

Fig. 5 Displacement responses of intelligent structures with no perturbed analysis

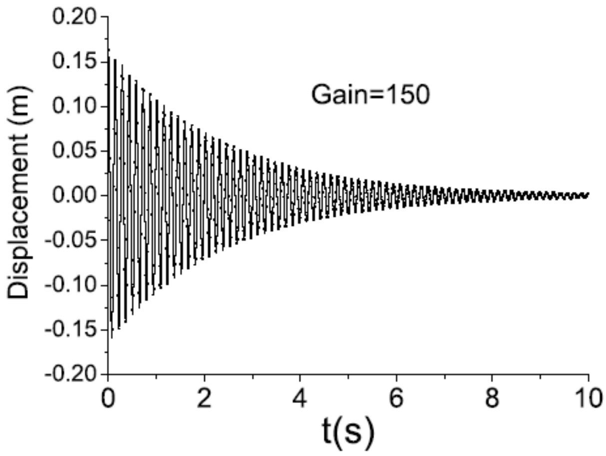

Fig. 6 Displacement responses of intelligent structures with perturbed analysis

responses to excitation by using the method for distinct eigenvalues. The displacement decays more rapidly and the vibration of the structure can be effectively suppressed when the feedback control is designed according to the repeated eigenvalue system.

6. Conclusions

In this paper, a method is presented to design the feedback control law of intelligent structures with closely-spaced eigenvalues. To design the feedback control law of intelligent structures with closely-spaced eigenvalues, following steps are used. Firstly, the system with closely-spaced eigenvalues is transformed into that with repeated eigenvalues by the spectral decomposition method. Secondly, the computation for the linear combination of Φ_0, denoted as $\underline{\Phi}_0$, is completed. Thirdly, the feedback control law is designed on the basis of the system with repeated eigenvalues.

Finally, the feedback control law is applied to original system and perturbed system, this paper discusses the dynamic stability of the closed-loop system of intelligent structures when small modifications of structural parameters are introduced. It can be seen that the feedback control law of intelligent structures with closely-spaced eigenvalues can be designed on the basis of that with repeated eigenvalues. The numerical results prove that the present method is effective.

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