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Eigen analysis of functionally graded beams with variable cross-section resting on elastic supports and elastic foundation

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Abstract. The free vibration of functionally graded material (FGM) beams on an elastic foundation and spring supports is investigated. Young's modulus, mass density and width of the beam are assumed to vary in thickness and axial directions respectively following the exponential law. The spring supports are also taken into account at both ends of the beam. An analytical formulation is suggested to obtain eigen solutions of the FGM beams. Numerical analyses, based on finite element method by using a beam finite element developed in this study, are performed in order to show the legitimacy of the analytical solutions. Some results for the natural frequencies of the FGM beams are given considering the effect of various structural parameters. It is also shown that the spring supports show the greatest effect on the natural frequencies of FGM beams.

Keywords: FGM beam, free vibration, closed-form solution, spring support, elastic foundation, FEM

1. Introduction

Many new materials have been developed in recent years, including functionally graded materials (FGM). In 1984 the theory of functionally graded materials was firstly introduced in Japan during the space plane project. Generally, FGM is a material in which the volume fractions of two or more material components are created to vary continuously with position, in particular, along the thickness direction.

The non-uniform cross-section beam is popularly used in various engineering fields. In the past, many studies have been performed for the free vibration of the non-uniform beams. The problem often leads to the solution of fourth-order partial differential equations; however closed-form solutions cannot be found for a number of problems, therefore the use of numerical methods such as the finite element method (FEM), the finite difference method, etc. has been required. Closed-form solutions can be found only for certain special cases, such as those structures with an exponentially varying cross-section (Ece *et al*. 2007, Haasen *et al.* 2011, Lardner 1968, Suppiger and Taleb 1956). By solving the partial differential equations, solutions to free vibration are obtained via trigonometric and hyperbolic functions (Ece *et al*. 2007, Haasen *et al.* 2011, Suppiger

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and Taleb 1956), hypergeometric functions (Caruntu 2008, Bessel functions Lardner 1968). These are analytical solutions which describe the vibration behavior of the beam under different boundary conditions and the determination of the effects of a continuously varying cross-section on the natural frequencies and mode shapes. In addition, a closed-form solution of the free vibration problem has been found for stepped beams (Gutierrez *et al*. 1991). Most previous studies have used numerical techniques to analyze the vibration of variable cross-section beams. For instance, employment of finite element method (Klein 1974, Gupta 1985, Ramalingerswara Rao and Ganesan 1995), Galerkin method (Pakar 2012), approximate method (Tong *et al*. 1995) has been reported in the literature. Several studies have been performed considering the effect of foundation in vibration problems (Zhou 1993, Lee and Lin 1995, Sen and Huel 1990). Furthermore, some authors studied the response of beams subjected to impulsive load (Ramalingerswara Rao and Ganesan 1995, Calm 2008), and moving load (Simsek and Cansiz 2012, Wang 1997, Abu-Hilal and Mohsen 2000).

In recent years, structures made of functionally graded material have drawn some interests by researchers. Haasen *et al*. (2011) investigated the free vibration of a functionally graded beam with exponentially varying cross-section using analytical methods. However, most of the investigations on the vibration of functionally graded beams are performed based on the finite element method (Alshorbagy *et al*. 2011, Ying *et al*. 2008, Chakraborty *et al*. 2003, Mohanty *et al*. 2011). Also this direction, Mohanty *et al*. (2012, 2013) have investigated the free vibration of uniform functionally graded sandwich beam using FEM based on first order Timoshenko beam theory. Employing other schemes, Ke *et al*. (2010) investigated nonlinear vibration of functionally graded beams, and Simsek and Cansiz (2012) studied dynamic responses of an elastically connected double-functionally graded beam under moving harmonic load at constant speed.

The foregoing review clearly shows that the majority of works are performed on the analysis of free and forced vibrations of beams made of various materials. However, the research works concerning the FGM beams on elastic foundations are still limited. In this study, we focus on the dynamic behavior of FGM beams on elastic foundation. In particular, additional effects of varying sectional properties, elastic end-springs and elastic foundation on the dynamic characteristics of FGM beams are explained by using the formulation suggested in this study.

The paper is outlined as follows: In Section 2, a formulation for the solution of free vibration of FGM beams is proposed in the context of analytical methods. Section 3 describes a procedure of using the finite element method for the computation of free vibration of FGM beams. In section 4, analytical and FEM solutions of natural frequencies are addressed for FGM beams with varying cross-sections, which not only rests on an elastic foundation but also with end-spring supports. The natural frequencies of FGM beams and their dependence on the through-thickness material and geometrical properties are investigated in detail.

2. Analytical formulation

2.1 FGM beam model

Let us consider the functionally graded beam in Fig. 1. The parameters of the model FGM beam are as follows: L is the length of the beam, h is the thickness of the beam, and b denotes the width of the beam. For the material parameter of Young's modulus E, the mass density ρ of the beam and the width of the beam b, the following exponential law is assumed with absolute values

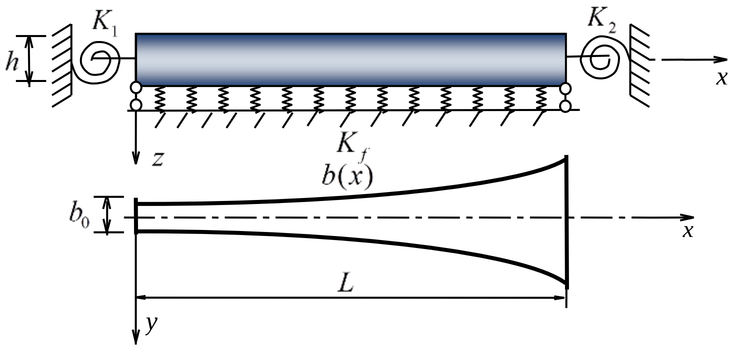


Fig. 1 Model of FGM beams

for z coordinate, which endows the symmetric characteristic to the beam with respect to mid-plane.

$$E(z) = E_0 e^{\beta|z|};\ \rho(z) = \rho_0 e^{\beta|z|};\ b(x) = b_0 e^{\psi x} \tag{1}$$

In Fig. 1, K_1, K_2, and K_f denote the stiffness of the rotational spring supports at both ends and the elastic foundation. E_o, ρ_o are the values of the Young's modulus and mass density at the mid-plane (z=0) of the beam. The parameter β in the exponent in Eq. (1) characterizes the material property variation along the thickness direction. The parameter ψ in Eq. (1), called the non-uniformity parameter, characterizes the variation of the width of the beam $b(x)$ along the axis direction.

2.2 Analytical formulation

We will use the assumption of the Euler-Bernoulli beam theory, and the plane of beam is symmetric with respect to x-y plane, thus, the mid-plane displacement in the x direction is zero and the displacement is obtained by referring to Haasen *et al.* (2011) as follows

$$\begin{cases} \bar{u}(x,z,t) = u(x,t) - z\dfrac{\partial w}{\partial x} \\ \bar{w}(x,z,t) = w(x,t) \end{cases} \tag{2}$$

where $u(x,t)$ and $w(x,t)$ are the displacement components in the mid-plane at time t along x and z directions, respectively.

The bending moment M, with the stiffness coefficient of beam D_{11} and the flexural curvature κ, can be determined as the following equations

$$M = -b(x) D_{11} \kappa \tag{3}$$

$$D_{11} = \int_{-h/2}^{h/2} \frac{E(z)}{1-\nu^2} z^2 dz \tag{4}$$

$$\kappa = \frac{\partial^2 w}{\partial x^2} \tag{5}$$

The governing motion of equation for the beam can be written as follows

$$D_{11}\frac{\partial^2}{\partial x^2}\left(b\frac{\partial^2 w}{\partial x^2}\right)+mb\frac{\partial^2 w}{\partial t^2}+bK_f w=0, \tag{6}$$

which can be rearranged as Eq. (7)

$$\xi\frac{\partial^4 w}{\partial x^4}+2\xi\frac{b^{'}(x)}{b(x)}\frac{\partial^3 w}{\partial x^3}+\xi\frac{b^{"}(x)}{b(x)}\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial t^2}+\frac{K_f}{m}w=0. \tag{7}$$

Here, m is unit weight of the beam, $m=\int_{-h/2}^{h/2}\rho(z)dz$, and v denotes the Poisson's ratio. The parameter ξ is $\xi=\frac{D_{11}}{m}$.

In case of harmonic vibration, the root of Eq. (7) can be assumed as

$$w(x,t)=W(x)e^{i\omega t}, \tag{8}$$

where ω is the natural frequency of the FGM beam.

For the family of cross-sections with exponentially varying width and constant height, Eq. (7) reduces to

$$W^{(4)}+2\psi W^{(3)}+\psi^2 W^{(2)}-\mu^2 W=0. \tag{9}$$

The solution of Eq. (9) can be obtained as follows

$$W(x)=e^{-\frac{\psi}{2}x}\left[C_1\sin(\lambda_1 x)+C_2\cos(\lambda_1 x)+C_3\sinh(\lambda_2 x)+C_4\cosh(\lambda_2 x)\right], \tag{10}$$

where parameters are

$$\lambda_1=\frac{\sqrt{4\mu-\psi^2}}{2}, \lambda_2=\frac{\sqrt{4\mu+\psi^2}}{2}, \mu=\sqrt{\frac{\omega^2-\frac{K_f}{m}}{\xi}}. \tag{11}$$

The boundary conditions relative to displacement, bending moment and rotation angle at the right and left ends of the beam are as follows:

At x=0, at any time t

$$\begin{cases} w(0,t)=0 \\ M\big|_{x=0}=-K_1\left.\frac{\partial w}{\partial x}\right|_{x=0} \end{cases} \tag{12}$$

At x=L, at any time t

$$\begin{cases} w(L,t)=0 \\ M\big|_{x=L}=K_2\left.\frac{\partial w}{\partial x}\right|_{x=L} \end{cases} \tag{13}$$

Substitution of Eq. (10) into Eqs. (8), (12), (13), and then collecting the resulting terms in terms of variables C_1, C_2, C_3, C_4, we obtain the following matrix equation

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{14}$$

The matrix elements A_{ij} are given in detail in the Appendix. For Eq. (14) to be valid, i.e., to have nontrivial solution, the following needs to be satisfied

$$\det(A) = 0 \tag{15}$$

where:

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix}$$

Solving Eq. (15) by using the trial and error method, we can get μ. By the substitution of μ into Eq. (11), we obtain the natural frequency ω .

3. Finite element formulation

In order to validate the analytical formulation and solution in the foregoing section, we also derive the finite element specific to the given problem. The transverse displacement function may be assumed as a cubic polynomial in x, and the shape functions for the four-degree of freedoms beam element are assumed as the Hermite interpolation function (Fig. 2).

The linearly varying width b_e and mass per unit length m_e of the element are

$$b_e = b_1 + \frac{b_2 - b_1}{L_e} x \tag{16}$$

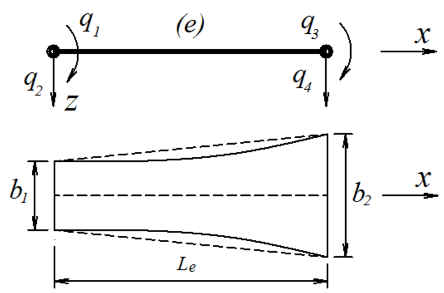


Fig. 2 Beam finite element

$$m_e = mb_e = m\left(b_1 + \frac{b_2 - b_1}{L_e}x\right). \tag{17}$$

The displacement vector of the element, as noted in Fig. 2, is

$$\{q\}_e = \{q_1 \quad q_2 \quad q_3 \quad q_4\}^T \tag{18}$$

then the displacement field is interpolated as

$$w_e = \langle N \rangle \{q\}_e \tag{19}$$

where $\langle N \rangle$ denotes the Hermite shape function vector: $\langle N \rangle = \langle N_1 \quad N_2 \quad N_3 \quad N_4 \rangle$.

In this case, the stiffness of the beam $b_e D_{11}$ is similar to the flexural rigidity EI of the homogenous beam, so the strain energy expression U_e for bending is given as follows

$$U_e = \int_0^{L_e} \frac{b_e D_{11}}{2}\left(\frac{\partial^2 w_e}{\partial x^2}\right)^2 dx \tag{20}$$

The kinematic energy T_e for flexural vibration is

$$T_e = \int_0^{L_e} \frac{m_e \dot{w}_e^2}{2} dx \tag{21}$$

and the potential energy T_f of the elastic foundation is

$$T_{fe} = \int_0^{L_e} b_e K_f \frac{w_e^2}{2} dx . \tag{22}$$

Substituting Eq. (19) into Eqs. (20)-(22), the following can be obtained

$$U_e = \frac{1}{2}\{q\}_e^T \tilde{K}_e \{q\}_e \tag{23}$$

$$T_e = \frac{1}{2}\{\dot{q}\}_e^T M_e \{\dot{q}\}_e \tag{24}$$

$$T_{fe} = \frac{1}{2}\{q\}_e^T \tilde{K}_{fe} \{q\}_e . \tag{25}$$

where

$$\begin{aligned} \tilde{K}_e &= \int_0^{L_e} b_e D_{11} \langle N'' \rangle^T \langle N'' \rangle dx \\ M_e &= m\int_0^{L_e} b_e \langle N \rangle^T \langle N \rangle dx \\ \tilde{K}_{fe} &= K_f \int_0^{L_e} b_e \langle N \rangle^T \langle N \rangle dx \end{aligned} \tag{26}$$

The potential energy T_s due to spring constants is determined as follows

$$T_s = \frac{K_1 q_{1(1)}^2}{2} + \frac{K_2 q_{3(N)}^2}{2}, \tag{27}$$

where $q_{1(1)}$ is the rotation at node one of the first element, $q_{3(N)}$ denotes the rotation at node two of the last element.

The governing differential equations of motion can be derived using Hamilton's principle

$$\delta \int_{t_1}^{t_2} \left(\sum_1^N U_e + \sum_1^N T_{fe} + T_s - \sum_1^N T_e \right) dt = 0 \,, \tag{28}$$

where N denotes the number of finite elements.

Therefore substituting Eqs. (23)-(27) into Eq. (28), the following can be obtained:

$$M\ddot{\underset{\sim}{w}} + K\underset{\sim}{w} = 0 \tag{29}$$

where M, K are mass matrix and stiffness matrix, respectively. The displacement vector $\underset{\sim}{w}$ can be written as following in terms of nodal displacement q

$$\underset{\sim}{w} = \begin{Bmatrix} \{q\}_1 \\ \{q\}_2 \\ \cdots \\ \{q\}_N \end{Bmatrix}, \tag{30}$$

where $\{q\}_1$, $\{q\}_2$,…$\{q\}_N$ are the displacement vectors of elements 1,2,…,N.

For simple harmonic vibrations, we assume that the displacements vector is the same as Eq. (8). Therefore, substituting Eq. (8) into Eq. (29), we can obtain the following

$$\left(K - \omega^2 M\right) \underset{\sim}{W} = 0, \tag{31}$$

where $K = \tilde{K} + \tilde{K}_f + K_s$ and $\tilde{K}$, $\tilde{K}_f$ and K_s are the flexural stiffness matrix, the stiffness matrix of elastic foundation, stiffness matrix of spring support, respectively, which are given in detail in the Appendix. For Eq. (31) to be valid, i.e., to have nontrivial solution, the following needs to be satisfied

$$\det\left(K - \omega^2 M\right) = 0 \tag{32}$$

4. Numerical example

The geometric dimensions of the example FGM beam are: h=0.1 m, L=1.0 m, and b_0=0.1 m. It is assumed that the material properties are E_0=70 GPa, ρ_0=2,780 kg/m^3, and ν=0.33. We also assume for the finite element analysis that the FGM beam is divided into several equal-length finite elements. E_0 denotes the Young's modulus at the mid-surface of the beam and E_1 at the top and bottom surfaces. The beam is homogeneously isotropic in the special case of E_1/E_0=1, i.e.,

Table 1 First three dimensionless natural frequencies ϖ of a uniform FGM beam $(\psi = 0, R_{K_f} = 0, R_{Ki} = 0)$

E_1/E_0	Mode number	Present	Haasen *et al.* (2011)
	1	9.869	9.869
1.0	2	39.478	39.478
	3	88.826	88.826

when β=0 in Eq. (1). In order to demonstrate the results more intuitively, dimensionless variables are defined as follows

$$\alpha_K = \frac{K_1}{K_2}, R_{K_1} = \frac{K_1}{D_{11}},\ R_{K_2} = \frac{K_2}{D_{11}},\ R_{K_f} = \frac{K_f h^3}{D_{11}},\ \varpi = \frac{\omega}{\sqrt{\xi_0}} \tag{33}$$

where $R_{K_1}, R_{K_2}, R_{K_f}$ are the coefficients of rotational and foundation spring, respectively, and ϖ is a dimensionless natural frequency, where ξ_0 is the value of ξ for the case of isotropic homogenous beam (E_1/E_0).

Table 1 provides the first three dimensionless natural frequencies of a uniform FGM beam without any spring support when the modulus ratio E_1/E_0=1.0. The results obtained in this study are exactly the same as given by Hassen *et al.* (2011), showing the adequacy of the proposed formulation.

Table 2 presents the first three dimensionless natural frequencies of FGM beams evaluated based on analytical solution. If we use finite element mesh having more than 30 finite elements, the FE solution also gives the same results with those of analytical solutions only with ignorable amount of errors. The variable parameters in analyses are the non-uniform parameter (ψ), the coefficient of foundation (R_{K_f}), the coefficients of rotational spring support (R_{K_i}) and the ratio of Young's modulus (E_1/E_0). The non-uniform parameter ψ is investigated for five cases of -2, -1, 0, 1 and 2. For each value of the non-uniform parameter, three different cases of Young's modulus ratio, i.e., 0.2, 1.0 and 5.0, are taken into consideration. Here, the stiffness of rotational spring support (R_{K_2}) is assumed to be 1.0 in all the cases. In the case of the coefficient of foundation spring (R_{K_f}), three values of 0.0, 1.0 and 10.0 are employed. From Table 2 it is obvious that when the non-uniform parameter (ψ) increases from -2 to 2, the natural frequency decreases independent of the other parameters. On the contrary, the natural frequency increases as the ratio of the coefficients of rotational spring support increases from $R_{K_1} = 0.2R_{K_2}$ to $R_{K_1} = 10.0R_{K_2}$.

Figs. 3 and 4 show the natural frequency versus mode number with the ratio of Young's modulus (E_1/E_0) of 1 and 5, respectively. Three cases of non-uniform parameter equal to -2, 0 and 2 are shown. There are six curves in each case with various combinations of coefficient of foundation (R_{K_f}) and the coefficients of rotational spring support (R_{K_1}, R_{K_2}). All the cases show that the lowest curves correspond to the cases of $R_{K_f} = 0, R_{K_1} = R_{K_2} = 0$, while the upper most curves correspond to the cases of $R_{K_f} = 1.0, R_{K_1} = R_{K_2} = 1.0$. This result can easily be understood because the natural frequency is proportional to the stiffness of foundation and the coefficients of rotational spring support.

Fig. 5 shows the relationship between the dimensionless natural frequency (ϖ_i) and the coefficient of rotational spring support (R_{K_i}). The coefficient of foundation R_{K_f} is equal to 1 for

Table 2 First three dimensionless natural frequencies of non-uniform FGM beams

ψ	E_1/E_0	R_{k_f}	$R_{k_1}=0.2R_{k_2}$			$R_{k_1}=R_{k_2}$			$R_{k_1}=10R_{k_2}$		
			ϖ_1	ϖ_2	ϖ_3	ϖ_1	ϖ_2	ϖ_3	ϖ_1	ϖ_2	ϖ_3
		0.0	12.584	39.541	81.860	15.453	43.306	86.207	17.506	47.685	93.094
	0.2	1.0	28.155	46.881	85.647	29.549	50.098	89.810	30.673	53.928	96.441
		10.0	80.633	88.921	114.213	81.131	90.658	117.367	81.547	92.829	122.515
		0.0	15.800	49.646	102.281	19.402	54.374	108.238	21.980	59.872	116.886
-2	1.0	1.0	35.350	58.862	107.535	37.100	62.901	112.763	38.512	67.710	121.088
		10.0	101.240	111.646	143.401	101.865	113.827	147.362	102.387	116.553	153.825
		0.0	18.737	58.875	121.886	23.009	64.481	128.358	26.066	71.001	138.613
	5.0	1.0	41.921	69.804	127.525	43.997	74.593	133.724	45.670	80.296	143.596
		10.0	120.060	132.399	170.057	120.800	134.986	174.754	121.419	138.219	182.420
		0.0	12.399	38.534	79.851	14.738	41.823	83.781	16.583	45.834	90.189
	0.2	1.0	28.072	46.035	83.728	29.182	48.821	87.486	30.156	52.299	93.640
		10.0	80.605	88.478	112.781	80.998	89.958	115.598	81.354	91.893	120.323
		0.0	15.567	48.383	100.258	18.504	52.511	105.193	20.822	57.549	113.238
-1	1.0	1.0	35.247	57.801	105.127	36.639	61.298	109.844	37.862	65.665	117.571
		10.0	101.205	111.090	141.604	101.698	112.949	145.140	102.145	115.377	151.073
		0.0	18.462	57.377	118.895	21.945	62.272	124.747	24.692	68.246	134.288
	5.0	1.0	41.799	68.545	124.669	43.450	72.693	130.262	44.900	77.871	139.426
		10.0	120.017	131.740	167.927	120.602	133.944	172.120	121.132	136.824	179.155
		0.0	11.903	36.944	77.144	13.754	39.791	80.695	15.3494	43.4147	86.6325
	0.2	1.0	27.857	44.713	81.151	28.697	47.092	84.534	29.495	50.191	90.219
		10.0	80.530	87.797	110.881	80.824	89.032	113.380	81.111	90.710	117.680
		0.0	14.945	46.386	96.859	17.269	49.960	101.318	19.272	54.510	108.773
0	1.0	1.0	34.977	56.140	101.891	36.031	59.127	106.138	37.033	63.019	113.276
		10.0	101.111	110.235	139.218	101.480	111.786	142.356	101.840	113.892	147.755
		0.0	17.723	55.009	114.864	20.479	59.247	120.152	22.855	64.643	128.993
	5.0	1.0	41.478	66.576	120.831	42.729	70.118	125.868	43.917	74.733	134.332
		10.0	119.906	130.726	165.097	120.344	132.565	168.819	120.771	135.063	175.221
		0.0	10.992	35.209	74.805	12.424	37.687	78.035	13.765	40.977	83.587
	0.2	1.0	27.480	43.290	78.931	28.084	45.329	81.999	28.702	48.099	87.299
		10.0	80.401	87.081	109.267	80.609	88.112	111.503	80.826	89.569	115.457
		0.0	13.802	44.207	93.922	15.600	47.319	97.978	17.283	51.450	104.949
1	1.0	1.0	34.504	54.354	99.104	35.261	56.913	102.954	36.038	60.392	109.610
		10.0	100.948	109.336	137.192	101.210	110.630	139.999	101.483	112.460	144.963
		0.0	16.368	52.425	111.382	18.499	56.115	116.191	20.496	61.014	124.458
	5.0	1.0	40.917	64.457	117.526	41.816	67.492	122.093	42.736	71.618	129.985
		10.0	119.713	129.660	162.694	120.023	131.195	166.023	120.347	133.364	171.910
		0.0	9.846	34.021	73.582	10.960	36.201	76.530	12.094	39.235	81.767
	0.2	1.0	27.042	42.329	77.773	27.468	44.101	80.568	27.939	46.624	85.559
		10.0	80.252	86.607	108.433	80.396	87.487	110.455	80.559	88.785	114.146
		0.0	12.362	42.715	92.387	13.761	45.453	96.089	15.185	49.263	102.664
2	1.0	1.0	33.953	53.147	97.649	34.487	55.372	101.159	35.080	58.539	107.424
		10.0	100.761	108.741	136.145	100.942	109.845	138.683	101.146	111.476	143.318
		0.0	14.660	50.656	109.561	16.319	53.902	113.950	18.008	58.420	121.748
	5.0	1.0	40.264	63.027	115.801	40.898	65.664	119.963	41.601	69.421	127.393
		10.0	119.491	128.955	161.453	119.706	130.264	164.463	119.948	132.198	169.959

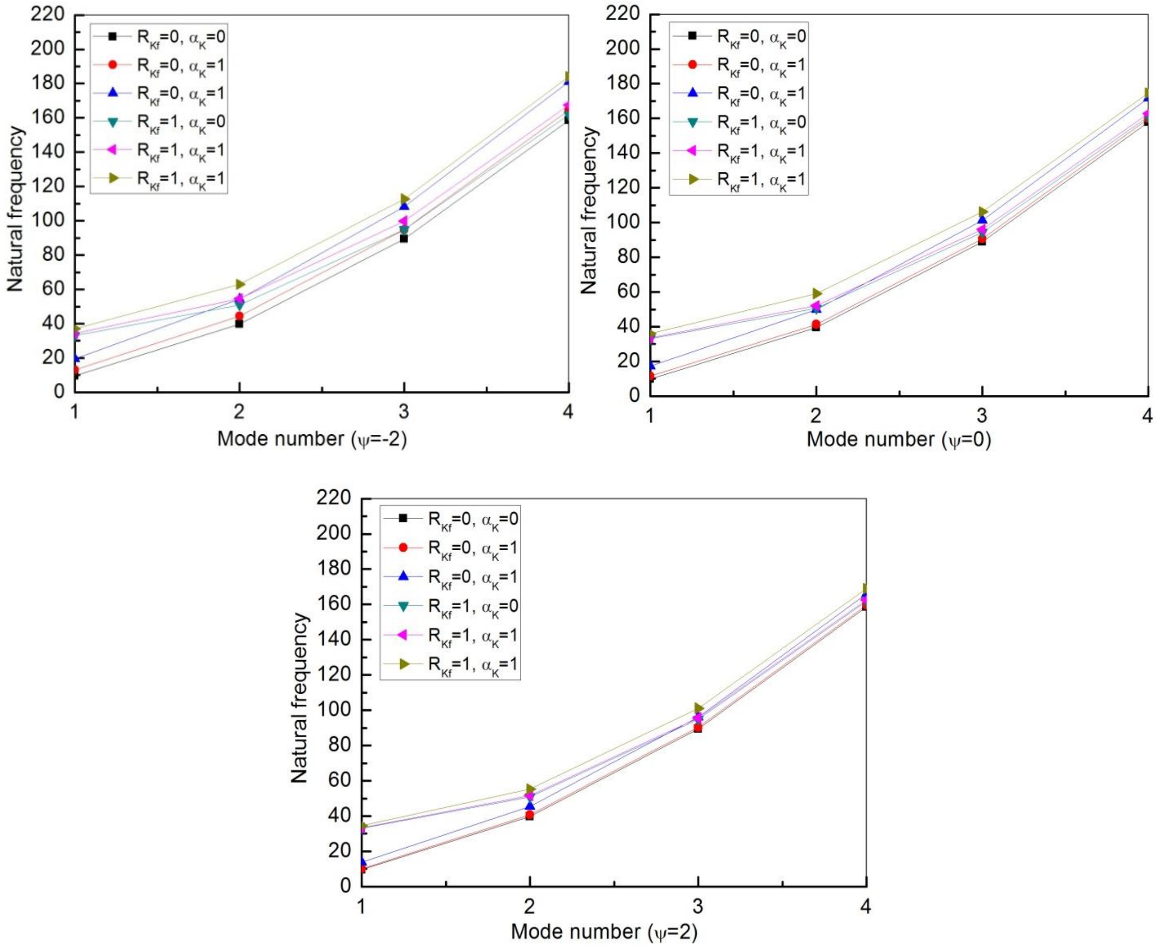


Fig. 3 Dimensionless natural frequencies of FGM beams (E_1/E_0=1)

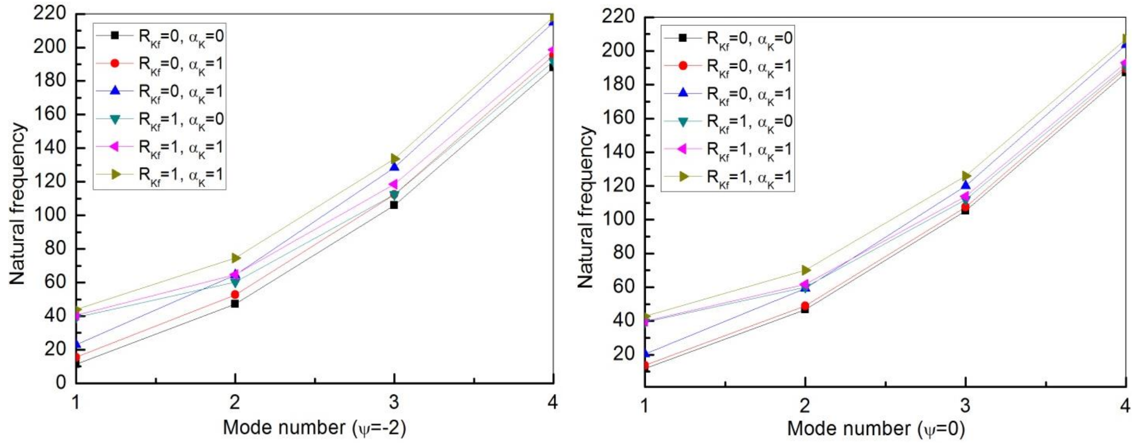


Fig. 4 Dimensionless natural frequencies of FGM beams (E_1/E_0=5)

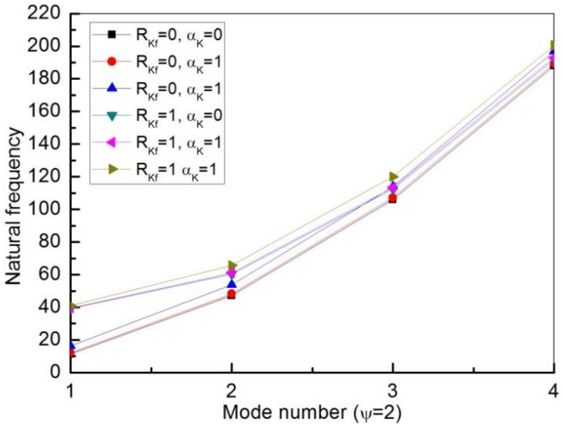


Fig. 4 Continued

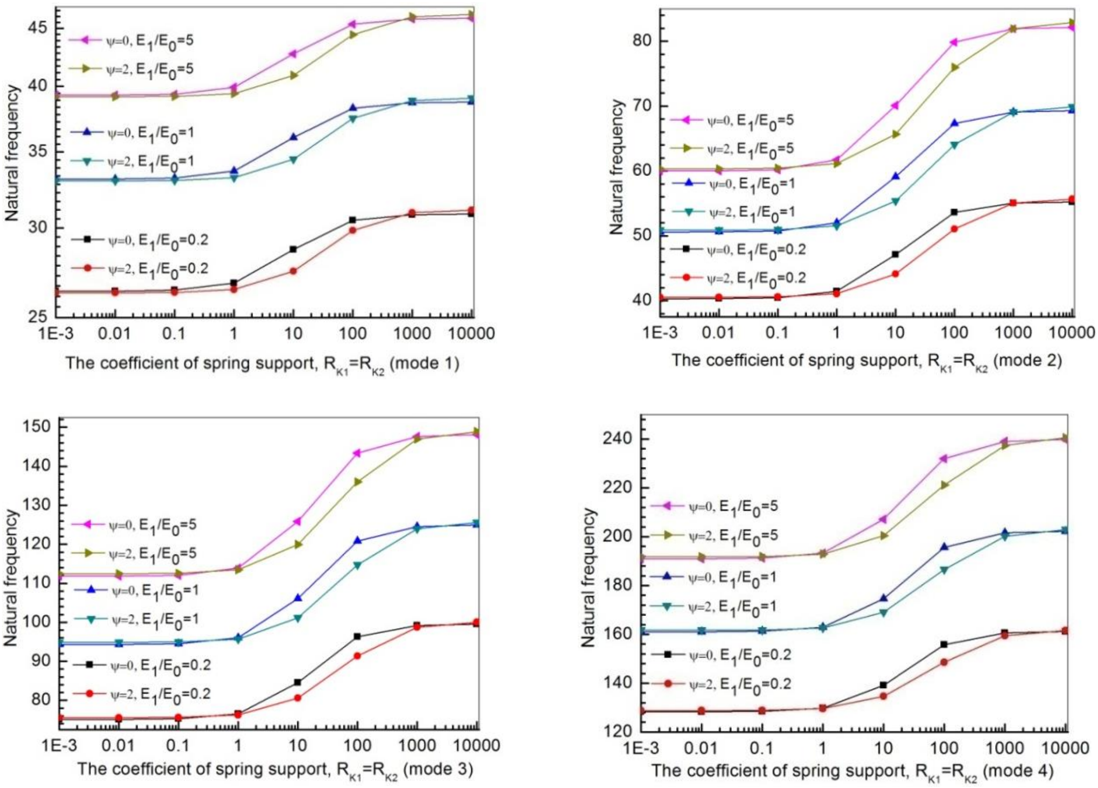


Fig. 5 Natural frequencies with $R_{K_f}=1$

all the cases in Fig. 5. The trend of variation of natural frequencies is observed to be similar to each other regardless of the mode number and of the pair of parameters (ψ, E_1/E_0). It is noteworthy, however, that when the coefficient of spring support is in the range of (0.001, 0.1), i.e., equivalent to hinge – hinge support, and in the range of (1000, 10000), i.e., equivalent to fixed

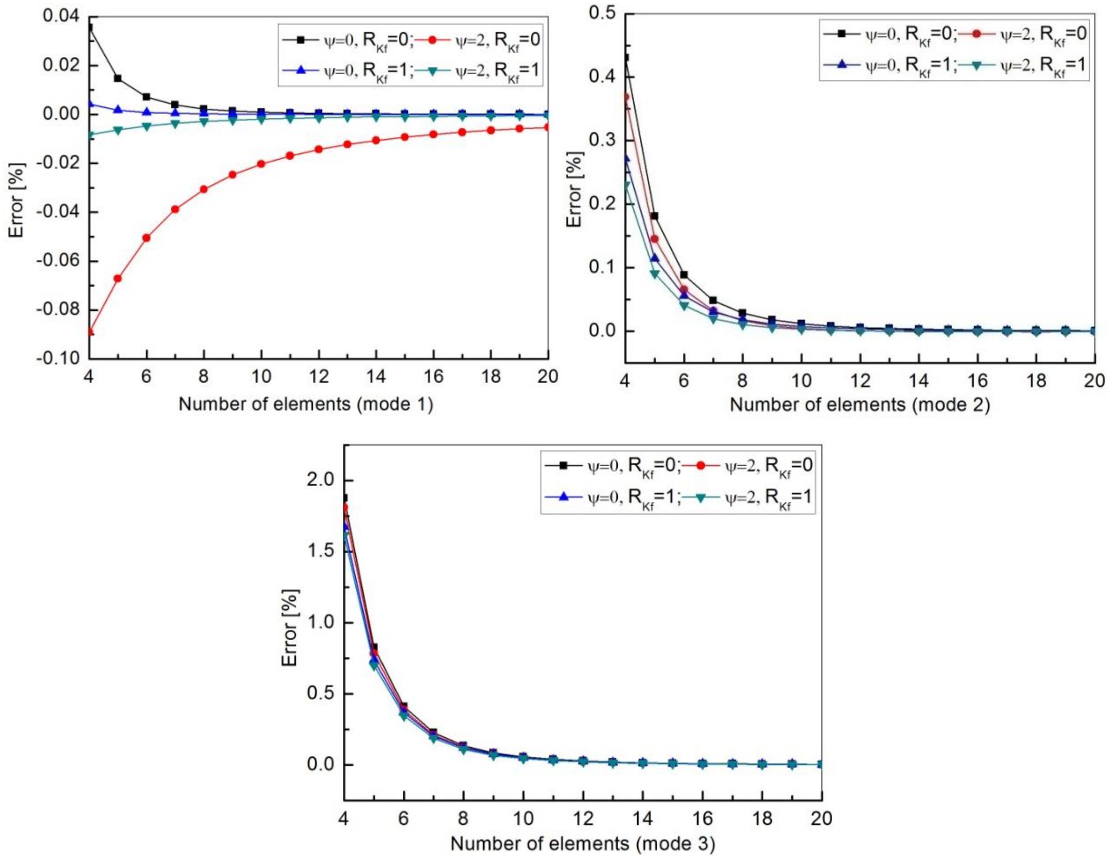


Fig. 6 Error depending on mesh refinement E_1/E_0=1, $R_{K_1} = R_{K_2} = 1$

- fixed support, the natural frequencies are nearly equal to each other for the cases of ψ=0 and ψ=2. This indicates that the effect of rotational spring at the support on the natural frequency for both beams, having a varying width (ψ=2) and a constant width (ψ=0), is not that prominent especially when the stiffness of spring support is extremely high or low. However, when the values of coefficient of spring support are in the moderate range of (0.1, 1000), the results are relatively sensitive to ψ, i.e., the beam of constant width (ψ=0) shows higher value of natural frequency than the beam with varying width (ψ=2). With these results, we can assert that the stiffness for the hinge support or fixed support can be conjectured with respect to the stiffness of the FGM beams.

The discrepancies between analytical and finite element (FE) solutions in the frequencies for the first three modes are shown in Figs. 6 and 7 depending on two pairs of parameters $(E_1/E_0 \text{ and } \mathrm{R}_{K_1} = \mathrm{R}_{K_2})$. In all the cases, the differences between those two results in percentile tend to zero as the finite element mesh is refined. This means not only that the FE solutions are converging to exact solutions but also that the analytical solutions based on the proposed formulation is exact. The discrepancies between the analytical solution and the FE analyses with coarse mesh seem to be caused by the linear approximation on the geometry of the FGM beams in the FE model (see Fig. 2).

The first three normalized mode shapes of the FGM beam having the same ratio of Young's modulus (E_1/E_0=2) and the non-uniformity parameter for the width of the beam (ψ=1) are presented in Fig. 8 with various values of R_{K_1}, R_{K_2}, and R_{Kf}. As noted in the figure, the mode shapes are significantly affected by the stiffness of rotational springs at both ends. Furthermore,

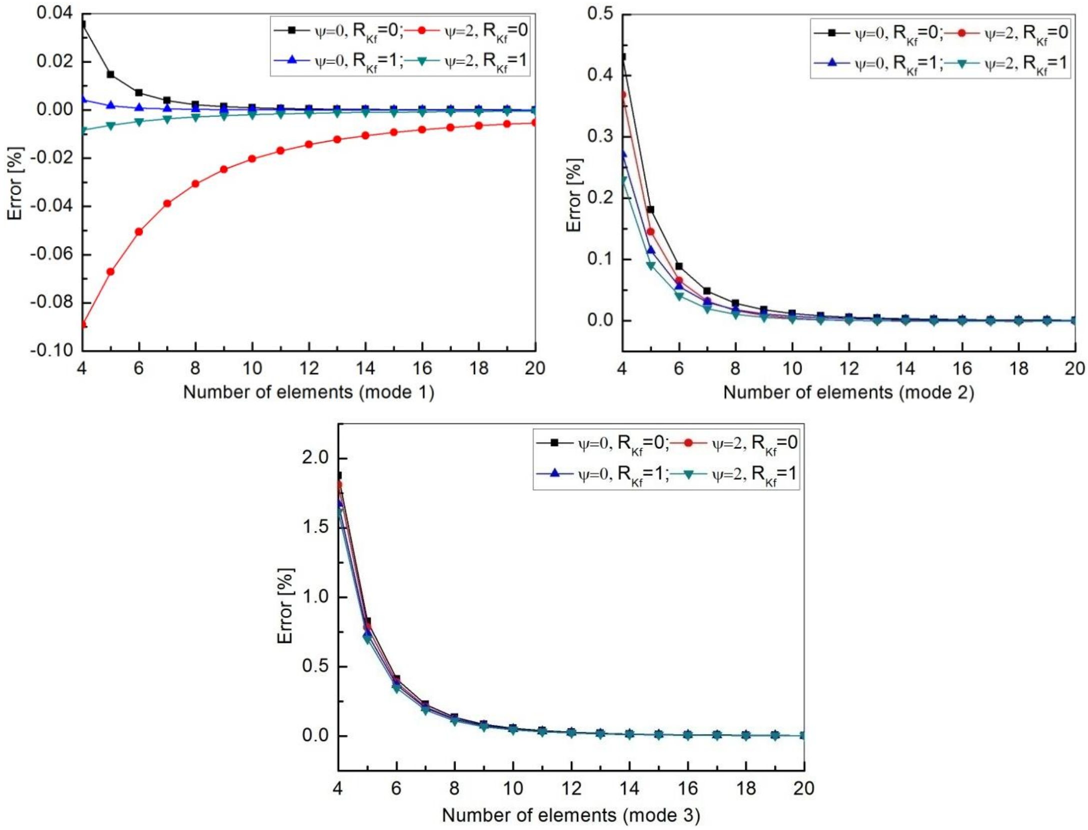


Fig. 7 Error depending on mesh refinement E_1/E_0=5, $R_{K_1} = R_{K_2} = 1$

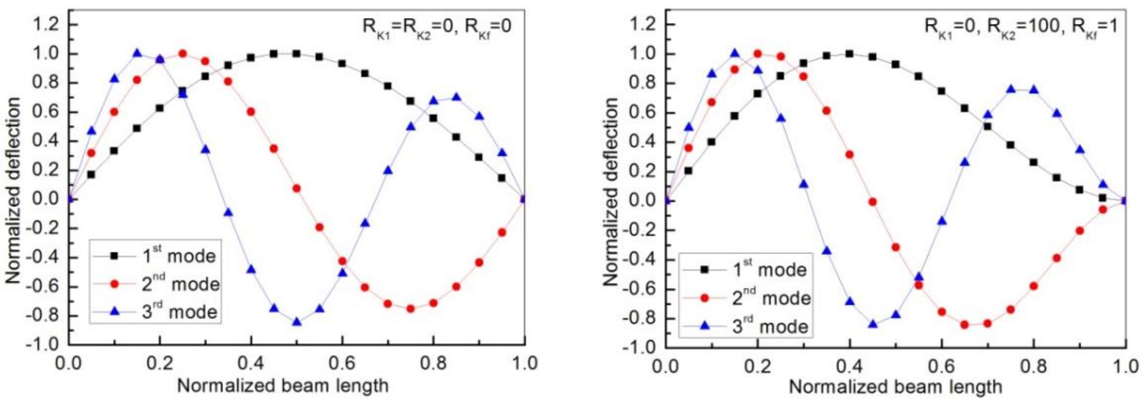
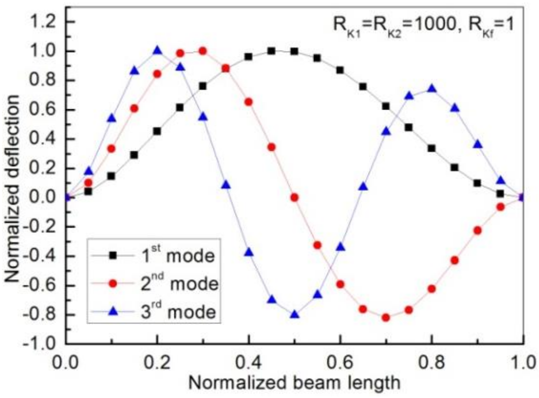


Fig. 8 Normalized mode shapes of the FGM beam when E_1/E_0=2, ψ=1



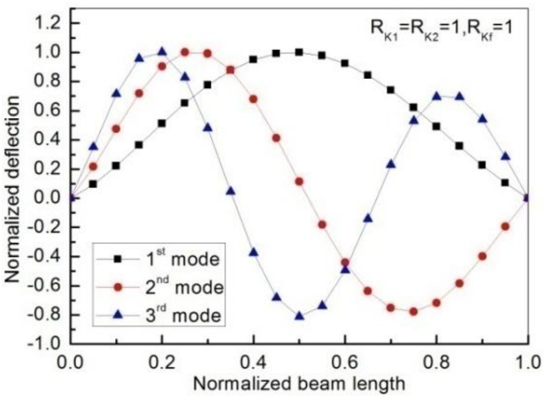


Fig. 8 Continued

due to the non-uniformity parameter ψ, the mode shapes are not symmetric (or anti-symmetric) with respect to the center point of the beam even in the case when $R_{K_1} = R_{K_2}$.

5. Conclusions

In this paper, an analytical solution for the evaluation of natural frequencies of functionally graded material (FGM) beams resting on an elastic foundation and rotational springs, at both ends of the beams, is suggested. The Euler-Bernoulli beam theory and the Winkler elastic foundation hypothesis are employed in modeling the FGM beam. In order to verify the adequacy of the proposed analytical scheme, a formulation for finite beam element analyses employing both the Hermite interpolation functions and Hamilton's principle is also presented. The results of finite element analysis converge to the analytical solutions suggested in this study as the mesh is refined. This fact shows not only the adequacy of the proposed analytical scheme but also the efficacy of the proposed analytical scheme since we need many finite elements for convergence due to non-uniformity of the beam under consideration.

The rotational springs at both ends, representing the practical boundary conditions depending on the values of the spring stiffness, cause a variety of behaviors. In particular, the frequencies of the FGM beam are observed virtually not to be affected by the non-uniformity parameter when the stiffness of end spring is extremely low or high. As to the mode shapes of the FGM beam, we observed that they are affected significantly by not only the non-uniformity parameter but the rotational-springs at both ends as well.

Acknowledgements

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Appendix

Herimite shape functions of beam finite element:

$$N_1 = x\left(1-2\frac{x}{L}+\frac{x^2}{L^2}\right);\ N_2 = 1-3\frac{x^2}{L^2}+2\frac{x^3}{L^3};\ N_3 = x\left(-\frac{x}{L}+\frac{x^2}{L^2}\right);\ N_4 = 3\frac{x^2}{L^2}-2\frac{x^3}{L^3}$$

The term A_{ij} of matrix A in Eq. (14)

$$A_{21} = e^{-\frac{1}{2}\psi L}\sin(\lambda_1 L),\ A_{22} = e^{-\frac{1}{2}\psi L}cos(\lambda_1 L),\ A_{23} = e^{-\frac{1}{2}\psi L}\sinh(\lambda_1 L),\ A_{24} = e^{-\frac{1}{2}\psi L}\cosh(\lambda_1 L)$$

$$A_{31} = K_1\lambda_1 + D_{11}\psi\lambda_1,\ A_{32} = -\frac{K_1\psi}{2} - D_{11}\left(\frac{\psi^2}{4}-\lambda_1^{\ 2}\right),\ A_{33} = K_1\lambda_2 + D_{11}\psi\lambda_2, A_{34} = -\frac{K_1\psi}{2} - D_{11}\left(\frac{\psi^2}{4}+\lambda_2^{\ 2}\right)$$

$$A_{41} = K_2\left(-\frac{\psi}{2}\mathrm{e}^{-\frac{\psi}{2}L}\sin(\lambda_1 L)+\mathrm{e}^{-\frac{\psi}{2}L}\cos(\lambda_1 L)\lambda_1\right)+D_{11}\left(\frac{\psi^2}{4}\mathrm{e}^{-\frac{\psi}{2}L}\sin(\lambda_1 L)-\psi\mathrm{e}^{-\frac{\psi}{2}L}\cos(\lambda_1 L)\lambda_1-\mathrm{e}^{-\frac{\psi}{2}L}\sin(\lambda_1 L)\lambda_1^{\ 2}\right)$$

$$A_{42} = K_2\left(-\frac{\psi}{2}\mathrm{e}^{-\frac{\psi}{2}L}c\mathrm{os}(\lambda_1 L)-\mathrm{e}^{-\frac{\psi}{2}L}\sin(\lambda_1 L)\lambda_1\right)+D_{11}\left(\frac{\psi^2}{4}\mathrm{e}^{-\frac{\psi}{2}L}\cos(\lambda_1 L)+\psi\mathrm{e}^{-\frac{\psi}{2}L}\sin(\lambda_1 L)\lambda_1-\mathrm{e}^{-\frac{\psi}{2}L}\cos(\lambda_1 L)\lambda_1^{\ 2}\right)$$

$$A_{43} = K_2\left(\frac{\psi}{2}\mathrm{e}^{-\frac{\psi}{2}L}\sinh(\lambda_2 L)+\mathrm{e}^{-\frac{\psi}{2}L}\cosh(\lambda_2 L)\lambda_2\right)+D_{11}\left(\frac{\psi^2}{4}\mathrm{e}^{-\frac{\psi}{2}L}\sinh(\lambda_2 L)-\psi\mathrm{e}^{-\frac{\psi}{2}L}\cosh(\lambda_2 L)\lambda_2+\mathrm{e}^{-\frac{\psi}{2}L}\sinh(\lambda_2 L)\lambda_2^{\ 2}\right)$$

$$A_{44} = K_2\left(-\frac{\psi}{2}\mathrm{e}^{-\frac{\psi}{2}L}\cosh(\lambda_2 L)+\mathrm{e}^{-\frac{\psi}{2}L}\sinh(\lambda_2 L)\lambda_2\right)+D_{11}\left(\frac{\psi^2}{4}\mathrm{e}^{-\frac{\psi}{2}L}\cosh(\lambda_2 L)-\psi\mathrm{e}^{-\frac{\psi}{2}L}\sinh(\lambda_2 L)\lambda_2+\mathrm{e}^{-\frac{\psi}{2}L}\cosh(\lambda_2 L)\lambda_2^{\ 2}\right)$$

Element flexural stiffness matrix in Eq. (26)

$$\tilde{K}_e = \frac{D_{11}}{L_e^3}\begin{bmatrix} 6(b_2+b_1) & 2(b_2+2b_1)L & -6(b_2+b_1) & 2(2b_2+b_1)L \\ & (b_2+3b_1)L^2 & -2(b_2+2b_1)L & (b_2+b_1)L^2 \\ & & 6(b_2+b_1) & -2(2b_2+b_1)L \\ Sym. & & & (3b_2+b_1)L^2 \end{bmatrix}$$

Element mass matrix in Eq. (26)

$$M_e = \frac{mL_e}{840}\begin{bmatrix} 24(10b_1+3b_2) & 2(15b_1+7b_2)L & 54(b_1+b_2) & -2(7b_1+6b_2)L \\ & (5b_1+3b_2)L^2 & 2(6b_1+7b_2)L & -3(b_1+b_2)L^2 \\ & & 24(3b_1+10b_2) & -2(7b_1+15b_2)L \\ Sym. & & & (3b_1+5b_2)L^2 \end{bmatrix}$$

Element stiffness matrix of elastic foundation in Eq. (26)

$$\tilde{K}_{fe} = \frac{K_f L_e}{840}\begin{bmatrix} 24(10b_1+3b_2) & 2(15b_1+7b_2)L & 54(b_1+b_2) & -2(7b_1+6b_2)L \\ & (5b_1+3b_2)L^2 & 2(6b_1+7b_2)L & -3(b_1+b_2)L^2 \\ & & 24(3b_1+10b_2) & -2(7b_1+15b_2)L \\ Sym. & & & (3b_1+5b_2)L^2 \end{bmatrix}$$