Structural Engineering and Mechanics, *Vol. 50, No. 6 (2014) 853-862*
DOI: http://dx.doi.org/10.12989/sem.2014.50.6.853

Forced nonlinear vibration by means of two approximate analytical solutions

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(Received February 27, 2014, Revised April 24, 2014, Accepted April 25, 2014)

Abstract. In this paper, two approximate analytical methods have been applied to forced nonlinear vibration problems to assess a high accurate analytical solution. Variational Iteration Method (VIM) and Perturbation Method (PM) are proposed and their applications are presented. The main objective of this paper is to introduce an alternative method, which do not require small parameters and avoid linearization and physically unrealistic assumptions. Some patterns are illustrated and compared with numerical solutions to show their accuracy. The results show the proposed methods are very efficient and simple and also very accurate for solving nonlinear vibration equations.

Keywords: Variational Iteration Method (VIM); Perturbation Method (PM); nonlinear oscillators; nonlinear spring

1. Introduction

Generally, nonlinear partial differential equations which occur in physical phenomena are often too complicated to be solved exactly. And also if they have exact solutions, the required calculations may be too complicated to be practical, or it might be difficult to interpret the outcome. Recently, some promising approximate analytical solutions are proposed, such as: variational iteration method (Wazwaz 2007, He 1999b), homotopy perturbation method (He 1999c, Shou 2009) energy balance method (Ganji *et al*. 2009), max-min approach (Zeng 2009), amplitude frequency-formulation (Ren *et al*. 2011), parameter expansion method (Kaya *et al*. 2009). Variational approach (He 2007, Shahidi *et al*. 2011) and other methods (Bayat *et al*. 2011a, b, c, d, e, f, 2012a, b, 2013a, b, c, 2014a, b, c, Pakar *et al*. 2011, 2012a, b, 2013a, b, Filobello-Nino *et al*. 2012, Behiry *et al*. 2007, Qian *et al*. 2012, Javanmard *et al*. 2013).

VIM is to construct correction functionals using general Lagrange multipliers identified optimally via the variational theory, and the initial approximations can be freely chosen with unknown constants. This method is the most effective and convenient one for both linear and nonlinear equations. This method has been shown to effectively, easily and accurately solve a large class of linear and nonlinear problems with components converging rapidly to accurate solutions.

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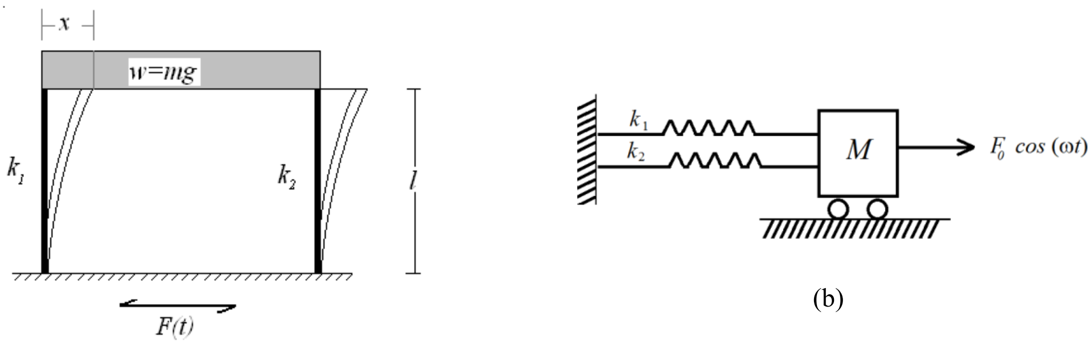
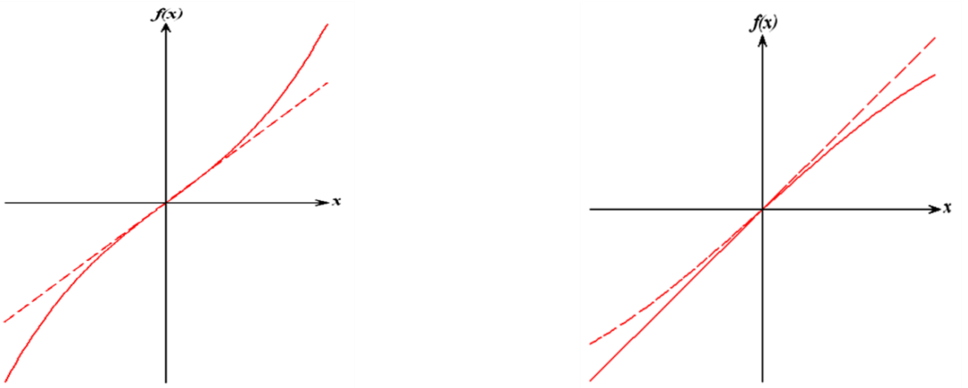
http://www.techno-press.org/?journal=sem&subpage=8 ISSN: 1225-4568 (Print), 1598-6217 (Online)



(a) (b)

Fig. 1 (a) Schematic view of a un damped structure under harmonic load, (b) The dynamic model of a un damped structure under harmonic load



(a) (b)

Fig. 2 (a) Hard spring stiffness nonlinear behavior, (b) Soft spring stiffness nonlinear behavior

VIM was first proposed by He (1999b). The aim of this work is to employ VIM and PM to obtain the analytic solutions of strongly nonlinear oscillators equations, which arises in a number of different fields in natural science.

The Fig. 1 shows the analytical model of a structure under harmonic load. Fig. 2 shows the spring's behavior. In the present paper, we consider a nonlinear oscillator in the form (William *et al*. 2000)

$$u'' + \omega_n^2 u + \mu u^3 = F_0 \cos(\omega t) \tag{1}$$

With the initial condition:

$$u(0) = A, \;\; u'(0) = 0 \tag{2}$$

2. Basic idea of Variational Iteration Method (VIM)

To clarify the basic ideas of VIM, we consider the following differential equation

$$Lu + Nu = g(t) \tag{3}$$

Where L is a linear operator, N a nonlinear operator and $g(t)$ an inhomogeneous term.

According to VIM, we can write down a correction functional as follows

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda\left(Lu_n(\tau) + N\tilde{u}_n(\tau) - g(\tau)\right)d\tau \tag{4}$$

Where λ is a general Lagrange multiplier which can be identified optimally via the variational theory (He 2006).The subscript n indicates the nth approximation and $\tilde{u}_n$ is considered as a restricted variation (He 2006), i.e., $\delta\tilde{u}_n = 0$.

3. Basic idea of Runge-Kutta's algorithm

For such a boundary value problem given by boundary condition, some numerical methods have been developed. Here we apply the fourth-order RK algorithm to solve governing equations subject to the given boundary conditions. RK iterative formulae for the second-order differential equations are

$$\begin{aligned} u'_{(i+1)} &= u'_i + \frac{\Delta t}{6}\left(h_1 + 2h_2 + 2h_3 + h_4\right), \\ u_{(i+1)} &= u_i + \Delta t \left[u'_i + \frac{\Delta t}{6}\left(h_1 + h_2 + h_3\right)\right], \end{aligned} \tag{5}$$

Where Δt is the increment of the time and h_1, h_2, h_3 and h_4 are determined from the following formulas

$$\begin{aligned} h_1 &= f\left(t_i, u_i, u'_i\right), \\ h_2 &= f\left(t_i + \frac{\Delta t}{2},\ u_i + \frac{\Delta t}{2},\ u'_i + \frac{\Delta t}{2}h_1\right), \\ h_3 &= f\left(t_i + \frac{\Delta t}{2},\ u_i + \frac{\Delta t}{2}u'_i,\ \frac{1}{4}\Delta t^2 h_1,\ u'_i + \frac{\Delta t}{2}h_2\right), \\ h_4 &= f\left(t_i + \Delta t,\ u_i + \Delta t u'_i,\ \frac{1}{2}\Delta t^2 h_2,\ u'_i + \Delta t\, h_3\right). \end{aligned} \tag{6}$$

The numerical solution starts from the boundary at the initial time, where the first value of the displacement function and its first-order derivative is determined from the initial conditions. Then, with a small time increment [Δt], the displacement function and its first-order derivative at the new position can be obtained using (5). This process continues to the end of time.

4. Application

In this section, VIM has been applied to the mentioned problem.

4.1 Application of variational iteration method

Supposing that the angular frequency of Eq. (1) is ω, we have the following linearized equation

$$u'' + \omega^2 u = 0 \tag{7}$$

So we can rewrite Eq. (1) in the form

$$u'' + \omega^2 u + g(u) = 0 \tag{8}$$

Where $g(u) = \left(\omega_n^2 - \omega^2\right)u - \mu u^3 - F_0 \cos\left(\omega t\right)$

Applying the variational iteration method, we can construct the following functional equation

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda\left(u_n''(\tau) + \omega^2 u_n(\tau) + \tilde{g}(u_n(\tau))\right) d\tau \tag{9}$$

where $\tilde{g}$ is considered as a restricted variation, i.e., $\delta\tilde{g} = 0$. Calculating variation with the respect to u_n, and noting that $\delta\tilde{g}(u_n) = 0$, we have the following stationary conditions

$$\begin{aligned} &\lambda''(\tau) + \omega^2 \lambda(\tau) = 0 \\ &\lambda(\tau)\big|_{\tau=t} = 0 \\ &1 - \lambda'(\tau)\big|_{\tau=t} = 0 \end{aligned} \tag{10}$$

The lagrangian multiplier can there be identified as

$$\lambda = \frac{1}{\omega}\sin\omega(\tau - t) \tag{11}$$

Substituting the identified multiplier into Eq. (9) results in the following iteration formula

$$u_{n+1}(t) = u_n(t) + \frac{1}{\omega}\int_0^t \sin(\tau - t)(u''(\tau) + \omega_n^2 u(\tau) + \mu u^3(\tau) - F_0\cos(\omega t))d\tau \tag{12}$$

Assuming its initial approximate solution has the form

$$u_0(t) = A\cos(\omega t) \tag{13}$$

and substituting Eq. (13) into Eq. (1) leads to the following residual

$$R_0(t) = -A\omega^2\cos(\omega t) + \omega_n^2 A\cos(\omega t) + \mu A^3\cos^3(\omega t) - F_0\cos(\omega t) \tag{14}$$

By the formulation (12), we have

$$u_1(t) = A\cos(\omega t) + \frac{1}{\omega}\int_0^t R_0(\tau)\sin\left(\tau - t\right) d\tau \tag{15}$$

In order to ensure that no secular terms appear in u_1, resonance must be avoided. To do so, the coefficient of $\cos(\omega t)$ in Eq. (14) requires to be zero

$$\omega = \frac{1}{2}\frac{\sqrt{A\left(4\omega_n^2 A + 3\mu A^3 - 4F_0\right)}}{A} \tag{16}$$

Therefore

$$u_0(t) = A\cos\left(\frac{1}{2}\frac{\sqrt{A\left(4\omega_n^2 A + 3\mu A^3 - 4F_0\right)}}{A}t\right) \tag{17}$$

We obtain the following approximate period:

$$T = \frac{2\pi}{\frac{1}{2}\frac{\sqrt{A\left(4\omega_n^2 A + 3\mu A^3 - 4F_0\right)}}{A}} \tag{18}$$

So, from Eq. (15), and Eq. (16) we have the following first-order approximate solution

$$u_1(t) = A\cos\left(\frac{1}{2}\frac{\sqrt{A\left(4\omega_n^2 A + 3\mu A^3 - 4F_0\right)}}{A}t\right) + \frac{1}{9}\frac{\mu A^4(-1 + \frac{\sqrt{A\left(4\omega_n^2 A + 3\mu A^3 - 4F_0\right)}}{A}t)}{4\omega_n^2 A + 3\mu A^3 - 4F_0} \tag{19}$$

And so on, in the same way the rest of the components of the iteration formula can be obtained.

4.2 Application of Perturbation Method (PM)

To solve Eq. (1) by means of Perturbation Method, we consider the following process. First we change the Eq. (1) to following form

$$u'' + \omega_n^2 u + \mu u^3 - F_0\cos(\omega t) = 0 \tag{20}$$

We can assume that the solution of Eq. (22) can be written as a power series in *u*, as following

$$u(t) = u_0(t) + \mu u_1(t) + \mu^2 u_2(t) + \mu^3 u_3(t) + \ldots \tag{21}$$

Substituting Eq. (21) in to Eq. (20) and rearranging the resultant equation based on powers of μ-terms, one has

$$\mu^0 : u_0'' + \omega_n^2 u_0 + F_0\cos(\omega t) \tag{22}$$

$$\mu^1 : u_1'' + \omega_n^2 u_1 + u_0^3 = 0 \tag{23}$$

$$\mu^2 : u_2'' + \omega_n^2 u_2 + 3u_0^2 u_1 = 0 \tag{24}$$

$$\mu^3 : u_3'' + \omega_n^2 u_3 + 3u_0^2 u_2 + 3u_0 u_1^2 = 0 \tag{25}$$

$u(t)$ may be written as follows by solving the Eq. (22) and Eq. (23),With the initial condition $u(0)=A=1$

$$u_0(t) = -\frac{\cos(\omega_n t)\left(-\omega_n^2 + \omega^2 + F_0\right)}{\left(\omega_n^2 + \omega^2\right)} - \frac{F_0\cos(\omega t)}{\left(-\omega_n^2 + \omega^2\right)} \tag{26}$$

And we have

$$u_1(t)=\frac{\cos(\omega_n t)}{32}\Big(-2205\omega_n{}^4F_0{}^3\omega^2-567\omega_n{}^2F_0{}^3\omega^4+2298\omega_n{}^6F_0\omega^4-590\omega^2\omega_n{}^{10}+45\omega_n{}^{12}$$
$$-1098\omega_n{}^4F_0{}^2\omega^4+2160\omega_n{}^6F_0{}^2\omega^2-1080\omega_n{}^2F_0{}^2\omega^6-771\omega_n{}^2F_0\omega^8-1629\omega_n{}^8F_0\omega^2$$
$$-186\omega_n{}^4F\omega^6-225\omega_n{}^8F_0{}^2+1955\omega_n{}^8\omega^4-590\omega^{10}\omega_n{}^2+153\omega^6F_0{}^3$$
$$-2820\omega^6\omega_n{}^6+1955\omega^8\omega_n{}^4+135\omega^{10}F_0+45\omega^{12}+243\omega^8F_0{}^2+315\omega_n{}^6F_0{}^3$$
$$+153\omega_n{}^{10}F_0\Big)\Big/\omega_n{}^2\left(-118\omega^2\omega_n{}^{10}-564\omega^6\omega_n{}^6+391\omega_n{}^8\omega^4+391\omega^8\omega_n{}^4-118\omega^{10}\omega_n{}^2+9\omega^{12}+9\omega_n{}^{12}\right)$$
$$+\frac{1}{32}\frac{\left(-288\omega^6\omega_n-2912\omega^2\omega_n{}^5+2912\omega^4\omega_n{}^3+288\omega_n{}^7\right)}{\left(\omega^2-\omega_n{}^2\right)^4\omega_n{}^2\left(9\omega^2-\omega_n{}^2\right)\left(\omega^2-9\omega_n{}^2\right)}$$
$$\times\sin(\omega_n t)\left(\int_0^t\left(-\cos(\omega_n\_zl)\left((\omega_n{}^2-\omega^2-F_0)\cos(\omega_n\_zl)+F_0\cos(\omega\_zl)\right)^3\right)d\_zl\right)$$
$$-144\cos(\omega_n t)\left(\frac{1}{12}(\omega^2-9\omega^2{}_n)(\omega-\omega_n)\omega_nF_0{}^3(\frac{1}{3}\omega_n+\omega)(\omega_n+\omega)\cos((-\omega_n+\omega)t)\right.$$
$$\left(-\frac{3}{8}(\omega^2-9\omega^2{}_n)(F_0+\omega^2-\omega_n{}^2)\omega_nF_0{}^2(\frac{1}{3}\omega_n+\omega)(\omega_n+\omega)\cos(2(-\omega_n+\omega)t)+\right.$$
$$\left(\frac{3}{4}(\omega^2-\omega^2{}_n)\left(F_0+\omega^2-\omega_n{}^2\right)^2\omega_nF_0(\frac{1}{3}\omega_n+\omega)(\omega_n+\omega)\cos\left((\omega-3\omega_n)t\right)+\right.$$
$$\left(-\frac{3}{4}(\omega-\omega_n)\left(F_0+\omega^2-\omega_n{}^2\right)^2\omega_nF_0(\frac{1}{3}\omega_n+\omega)(\omega_n+\omega)\cos\left((\omega+3\omega_n)t\right)\right.$$
$$+(\omega+3\omega_n)\left(\frac{3}{2}\left(\frac{1}{2}\omega_n{}^4+(-\omega^2-F_0)\omega_n^2+\frac{1}{2}\omega^4+F_0^2+\omega^2F_0\right)\omega_nF_0(\frac{1}{3}\omega_n+\omega)\right.$$
$$(\omega+\omega_n)\cos\left((\omega-\omega_n)t\right)+(\omega+\omega_n)\left(-\frac{1}{12}F_0^3\omega_n(\omega+\omega_n)\cos\left((\omega_n+3\omega)t\right)\right)+$$
$$(\frac{1}{3}\omega_n+\omega)\left(\frac{3}{8}F_0^2\omega_n(-\omega_n^2+\omega^2+F_0)\cos\left((2\omega_n+2\omega)t\right)\right)-$$
$$\frac{3}{2}\left(\frac{1}{2}\omega_n^4+(-F_0-\omega^2)\omega_n^2+\frac{1}{2}\omega^4+F_0+\omega^2F_0\right)\omega_nF_0\cos\left((\omega_n+\omega)t\right)+$$
$$\left(\frac{1}{4}\omega_n^4+\left(-\frac{1}{2}F_0-\frac{1}{2}\omega^2\right)\omega_n^2+\frac{1}{4}\omega^4+F_0^2+\frac{1}{2}\omega^2F_0\right)\cos(2\omega_nt)+$$
$$\left.\left.\left.\left.\left.\left.\frac{1}{16}\cos(4\omega_nt)(-\omega_n^2+\omega^2+F_0)^2\right)\times(-\omega_n^2+\omega^2+F_0)(\omega_n+\omega)\right)\right)\right)(\omega-3\omega_n)\right)(\omega-\frac{1}{3}\omega_n)\right)\right) \tag{27}$$

In the same manner, the rest of components were obtained using the Maple package. According to the Perturbation, we can conclude that

$$u(t)=u_0(t)+\mu u_1(t)+... \tag{28}$$

5. Results and discussions

In order to assess the accuracy of the variational iteration method and perturbation method, the results are compared with the numerical solution using Runge-Kutta's algorithm.

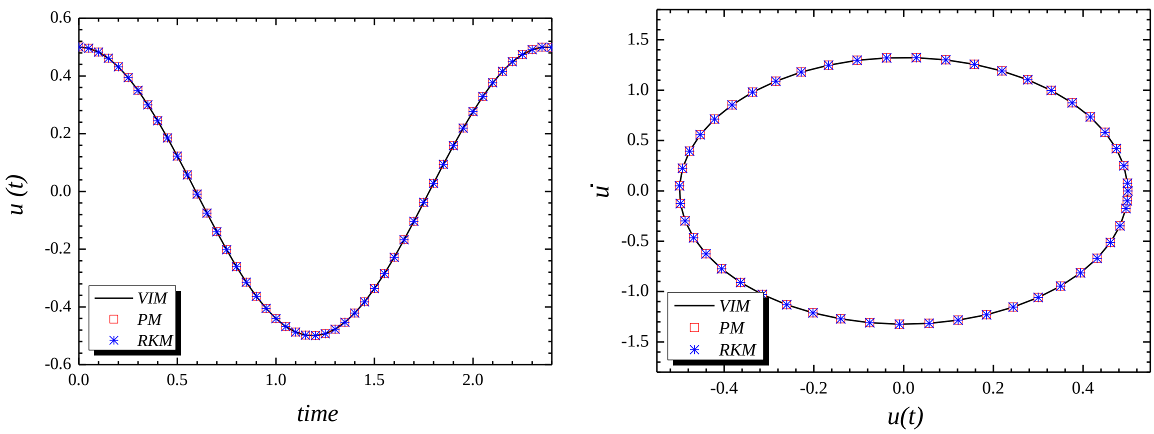
The Table 1 is the point value of the problem per different time in comparison of numerical solution. It can be seen from the table that the results are very close together in a period of motion.

Figs. 3 to 5 are the time history and phase plan of the problem for different cases. It is obvious that the motion of the problem is periodic and it is a function of initial condition.

Fig. 6 is a sensitive analysis of the problem in which we have considered the amplitude, μ and frequency. By increasing the amplitude and μ, the frequency of the system is increased and its top point is when the amplitude and μ are in their maximum value. Perturbation method and variational iteration method are compared with numerical solution and they have an excellent agreement. The variational iteration method is able to solve high nonlinear problem is we choose or obtain the weight factor or the general Lagrange multiplier λ correctly.

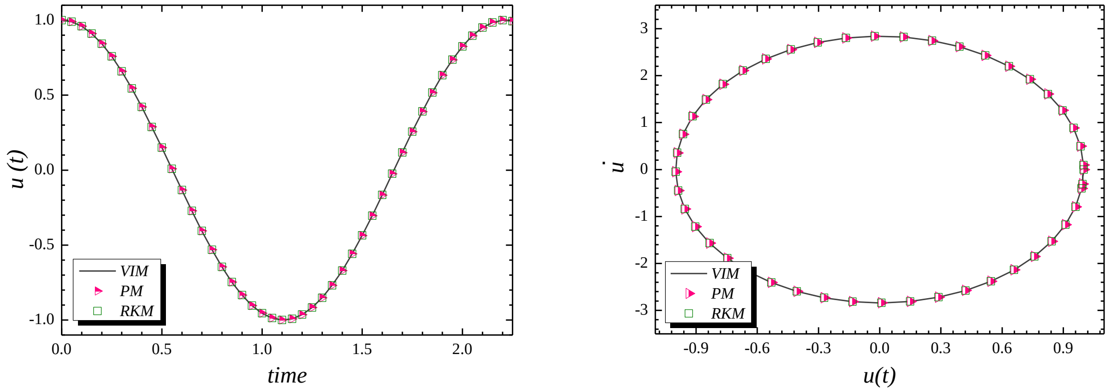
Table1 Comparison of time history response of VIM, PM, RKM

Time	VIM	PM	RKM
0	1	1	1
0.2	0.8428	0.8446	0.8463
0.4	0.4206	0.4218	0.4227
0.6	-0.1338	-0.1335	-0.1337
0.8	-0.6461	-0.6468	-0.6481
1	-0.9554	-0.9570	-0.9589
1.2	-0.9642	-0.9664	-0.9684
1.4	-0.6699	-0.6723	-0.6736
1.6	-0.1650	-0.1669	-0.1672
1.8	0.3918	0.3909	0.3917
2	0.8254	0.8259	0.8276
2.2	0.9995	1.0014	1.0034



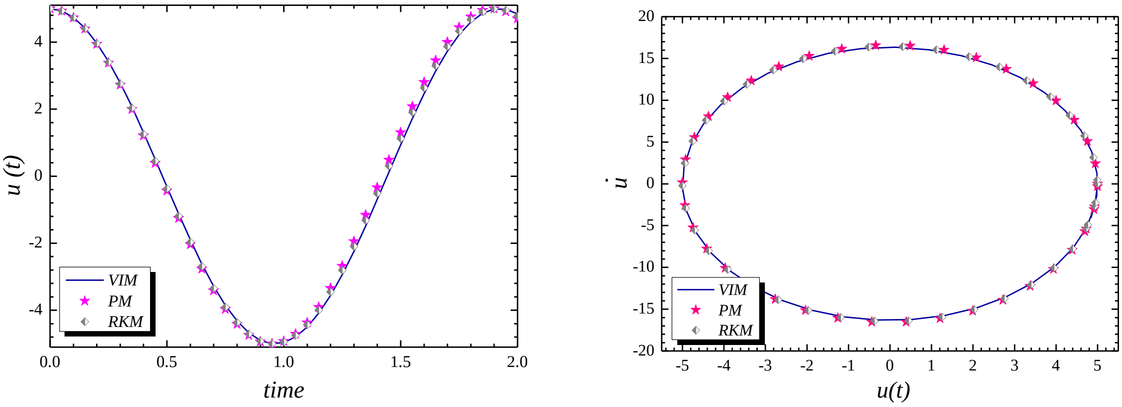
(a) (b)

Fig. 3 The comparison of the variational iteration solution with the perturbation solution and numerical solution, (a) time history response (b) phase plan for F_0=1, μ=0.1, ω_n=3, A=0.5



(a) (b)

Fig. 4 The comparison of the variational iteration solution with the perturbation solution and numerical solution (a) time history response (b) phase plan for F_0=1, μ=0.1, ω_n=3, A=1



(a) (b)

Fig. 5 The comparison of the variational iteration solution with the perturbation solution and numerical solution (a) time history response (b) phase plan for F_0=1, μ=0.1, ω_n=3, A=5

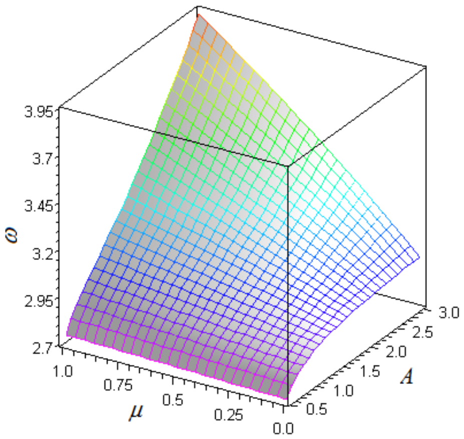


Fig. 6 Sensitivity analysis of frequency

6. Conclusions

In this paper, we studied forced nonlinear vibration problem through variational iterational method (VIM) and perturbation method. Variational iteration method which does not require small parameters, whereas the perturbation technique dose. The result shows that the variational iterational method can give much better analytical approximation for nonlinear oscillators equations than perturbation methods solutions. This mainly because this technique is base on general weighted residual methods. The weight factor or the general Lagrange multiplier λ can be determinate by variational theory; the more exact λ is, the more it leads to rapid convergence to exact and numerical solutions. The variational iterational method could be a strong mathematical tool for solving high nonlinear equations.

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