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# Time harmonic analysis of dam-foundation systems by perfectly matched layers

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**Abstract.** Perfectly matched layers are employed in time harmonic analysis of dam-foundation systems. The Lysmer boundary condition at the truncation boundary of the PML region has been incorporated in the formulation of the dam-foundation FE model (including PML). The PML medium is defined in a way that the formulation of the system can be transformed into time domain. Numerical experiments show that applying Lysmer boundary conditions at the truncation boundary of the PML area reduces the computational cost and make the PML approach a more efficient technique for the analysis of dam-foundation systems.

**Keywords:** wave propagation; semi-infinite domain; perfectly matched layers; dam-foundation system

## 1. Introduction

The reliable simulation of processes in which wave propagations are involved is an interesting subject for physicists and engineers. Particularly, in analyzing dam-foundation systems, one has to deal with elastic waves propagating in a semi-infinite medium (Fig. 1). In numerical models of such systems, the near-field part of the foundation adjacent to the dam body (with probable nonlinear behavior) is often discretized by commonly used finite elements. Meanwhile, through decades of investigations, many approaches such as Lysmer boundary conditions (Lysmer and Kohlemeyer 1963), hyperelements (Lotfi *et al*. 1987), infinite elements (Kim and Yun 2000, Yun *et al*. 2000), rational boundary conditions (Feltrin 1997), Dirichlet to Neuman mappings (Givoli 1999), the boundary element method (Yazdchi *et al*. 1999), the scaled boundary element method (Song and Wolf 2000), discontinuous Galerkin methods (Park and Tassoulas 2000, Park and Antin 2004) and high order non-reflecting boundary conditions (Givoli 2004) have been presented for taking into account the propagation of elastic waves towards infinity in the analysis. Nevertheless, researches still continue in order to find methods for applying the radiation condition as completely and efficiently as possible. The present study is focused on utilizing perfectly matched layers in the time harmonic dynamic analysis of dam-foundation rock systems.

A perfectly matched layer is an absorbing layer which can absorb propagating waves perfectly if it is defined properly. Berenger (1994) introduced perfectly matched layers for solving

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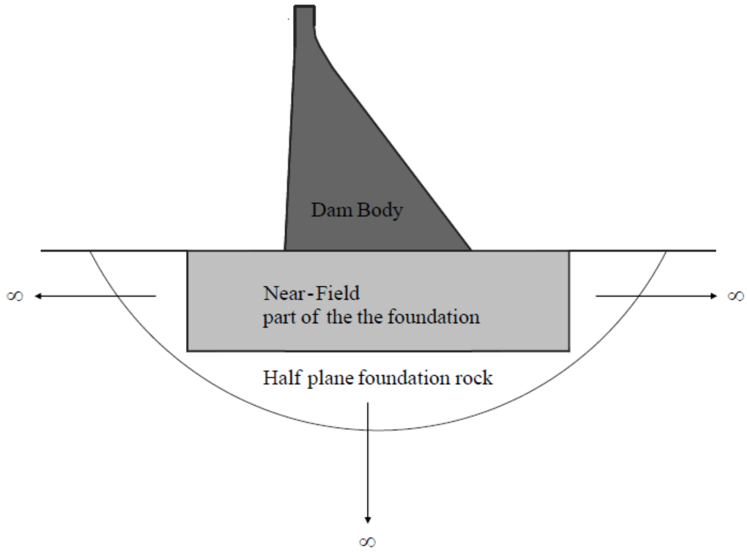



Fig. 1 A typical dam-foundation rock system containing the dam body, the near-field part of the foundation, the semi-infinite foundation rock

unbounded electromagnetic problems with the finite-difference time-domain method. Petropoulos (1998) investigated the effect of truncating the PML medium by local boundary conditions to solve unbounded electromagnetic problems. Hasting (1996) seems to be the first researcher who used perfectly matched layers in problems including elastic waves. He split potential functions corresponding to primary and secondary waves and utilized a finite-difference time-domain (FDTD) approach to solve the resultant equations in 2D domains. Chew *et al*. (1997) introduced a change of variables to transform Maxwell's equations in PML media into ordinary-looking Maxwell's equations in a complex coordinate system. They indicated that many existing closed-form solutions can be easily mapped into solutions in these complex coordinate systems. Chew and Liu (1996) employed complex coordinates to define perfectly matched layers and showed that the resultant medium could absorb propagating waves. Using complex coordinates, Liu (1999) developed perfectly matched layers (PML) in cylindrical and spherical coordinates in time domain. Issac Harari *et al*. (2000) presented a finite element formulation to use PML in time harmonic analysis of acoustic waves in exterior domains. Collino and Tsogka (2001) indicated how to establish a PML model using the split-field approach for a general hyperbolic system. They implemented their theory in the linear elastodynamic problem in an anisotropic medium. Zeng *et al*. (2001) extended the PML approach to truncate unbounded poroelastic media for numerical solutions using a finite-difference method. They adopted the method of complex coordinates to formulate PMLs for poroelastic media. Zheng and Huang (2002) developed anisotropic PMLs for elastic waves in Cartesian, cylindrical and spherical coordinates. Their formulation avoided field splitting and could be used in the FEM directly, and in the FDTD method too. Becache *et al*. (2003) investigated well-posedness and stability of using perfectly matched layers for anisotropic elastic waves from a theoretical point of view. Basu and Chopra (2003) defined perfectly matched layers by employing complex coordinates to solve time harmonic elastodynamic equations by finite element implementation. Furthermore, they transformed the frequency domain equations

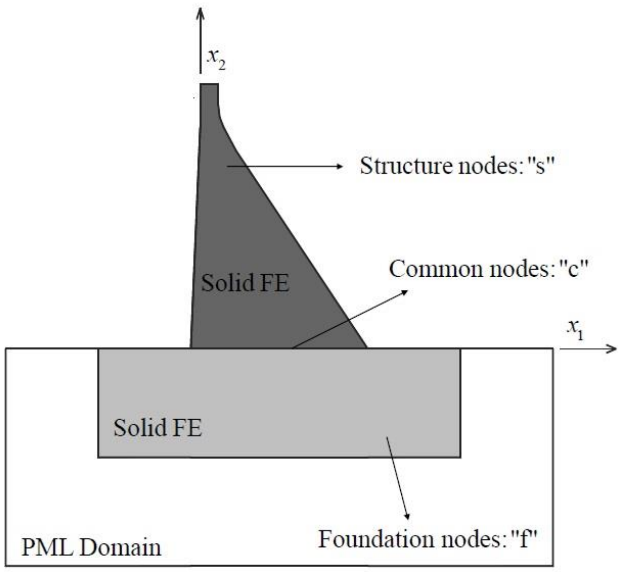



Fig. 2 Dam-foundation FE model, the dam body and the near-field part of the foundation discretized by solid finite elements, the far-field part of the foundation rock is discretized by PML finite elements

into time domain and presented an approach to solve the resultant equations (Basu and Chopra 2004, Basu 2004). Khazaee and Lotfi employed perfectly matched layers in the dynamic analysis of dam-reservoir systems. They introduced proper boundary conditions into the formulation of the PML area in the reservoir. Their results show that the PML approach is a very efficient method for the time harmonic and transient analysis of dam-reservoir systems if boundary conditions of the PML domain are included (Khazaee and Lotfi 2014). Katsibas and Antonopoulos (2004) implemented a FDTD-PML technique to solve stress-velocity acoustic equations. They derived general PML equations governing both lossless and lossy media. Moreover, they used the stretched coordinates idea to introduce further dissipation into the PML area. Appelo and Kreiss (2006) utilized the formulation of a modal PML to the equations of linear elasticity. They indicated that their PML model has better stability properties than previous split-field models. Harari and Albocher (2006) conducted a parametric study on PML used in time harmonic analysis of elastodynamics in an unbounded region by the finite element method and presented some guidelines for choosing PML parameters. Ma and Liu (2006) presented an easy implementation of perfectly matched layers (PML) in the explicit finite element method by using the one-point integration scheme. Zhen *et al*. (2009) introduced auxiliary variables to divide the PML wave equation in the frequency domain into two parts: normal terms and attenuated terms. Using the auxiliary variables, they avoided convolution operations in equations after transforming them into time domain and utilized the finite difference method to propose a novel numerical implementation approach for PML absorbing boundary conditions with simple calculation equations, small memory requirements, and easy programming. Liu *et al*. (2009) utilized the Crank−Nicolson scheme together with several algorithms to calculate the first-order spatial derivatives of the SH wave equations. Furthermore, they investigated how the absorbing boundary width and the algorithms affect the PML results of a homogeneous isotropic medium and a multi-layer medium with a cave. Kim and Pasciak (2012) developed a Cartesian perfectly matched layer for solving Helmholtz equation on an unbounded domain in 2D space. Lancioni (2011) compared the performance of the PML approach and high order non-reflecting boundary conditions in a one dimensional dispersive problem and expressed their merits and drawbacks.

In the analysis of a dam-foundation rock system by perfectly matched layers, the near-field part of the foundation rock is discretized by common finite elements and a PML region is modeled to absorb waves propagating towards infinity (Fig. 2). In previous studies on employing PML in the dam-foundation rock systems, no boundary conditions are adopted at the external boundary of the PML domain (Basu 2004). In the present study, the Lysmer boundary condition at the truncation boundary of the PML region has been incorporated in the formulation of the dam-foundation FE model (including PML). Several numerical experiments are carried out and the effect of applying Lysmer boundary conditions at the truncation boundary of the PML area is investigated.

## 2. The formulation of dam-foundation rock systems

The "added motion" formulation that is commonly used for soil-structure systems is adopted for the analysis of the dam-foundation rock system (Wilson 2002). If the nodes on the dam-foundation interface are identified with superscript "*c*" and the other nodes of the dam body and the foundation rock are identified with superscript "*s*" and "*f*", respectively (Fig. 2), the equation of motion of the system in terms of absolute displacements can be expressed as follows

$$\begin{bmatrix} \mathbf{M}_{ss} & \mathbf{M}_{sc} & \mathbf{0} \\ \mathbf{M}_{cs} & \mathbf{M}_{cc} & \mathbf{M}_{cf} \\ \mathbf{0} & \mathbf{M}_{fc} & \mathbf{M}_{ff} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{U}}_s \\ \ddot{\mathbf{U}}_c \\ \ddot{\mathbf{U}}_f \end{bmatrix} + \begin{bmatrix} \mathbf{C}_{ss} & \mathbf{C}_{sc} & \mathbf{0} \\ \mathbf{C}_{cs} & \mathbf{C}_{cc} & \mathbf{C}_{cf} \\ \mathbf{0} & \mathbf{C}_{fc} & \mathbf{C}_{ff} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{U}}_s \\ \dot{\mathbf{U}}_c \\ \dot{\mathbf{U}}_f \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{ss} & \mathbf{K}_{sc} & \mathbf{0} \\ \mathbf{K}_{cs} & \mathbf{K}_{cc} & \mathbf{K}_{cf} \\ \mathbf{0} & \mathbf{K}_{fc} & \mathbf{K}_{ff} \end{bmatrix} \begin{bmatrix} \mathbf{U}_s \\ \mathbf{U}_c \\ \mathbf{U}_f \end{bmatrix} = \mathbf{0} \tag{1}$$

in which **M**, **C** and **K** are mass, damping and stiffness matrices and **U** is the vector of nodal absolute displacements. The mass, damping and stiffness at the interface nodes are the sum of parts corresponding to the structure and the foundation rock and are given by

$$\mathbf{M}_{cc} = \mathbf{M}_{cc}^{s} + \mathbf{M}_{cc}^{f} \tag{2a}$$

$$\mathbf{C}_{cc} = \mathbf{C}_{cc}^{s} + \mathbf{C}_{cc}^{f} \tag{2b}$$

$$\mathbf{K}_{cc} = \mathbf{K}_{cc}^{s} + \mathbf{K}_{cc}^{f} \tag{2c}$$

Absolute displacements can be expressed in terms of the free-field motion and relative displacements to apply ground motions to the system, thus

$$\begin{bmatrix} \mathbf{U}_s \\ \mathbf{U}_c \\ \mathbf{U}_f \end{bmatrix} = \begin{bmatrix} \mathbf{U}_s^{ff} \\ \mathbf{U}_c^{ff} \\ \mathbf{U}_f^{ff} \end{bmatrix} + \begin{bmatrix} \mathbf{U}_s^{r} \\ \mathbf{U}_c^{r} \\ \mathbf{U}_f^{r} \end{bmatrix} \tag{3}$$

where free-field and relative displacements are identified with superscript "*ff*" and "*r*" respectively. Eq. (1) can now be written as

$$\begin{bmatrix} \mathbf{M}_{ss} & \mathbf{M}_{sc} & \mathbf{0} \\ \mathbf{M}_{cs} & \mathbf{M}_{cc} & \mathbf{M}_{cf} \\ \mathbf{0} & \mathbf{M}_{fc} & \mathbf{M}_{ff} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{U}}_s^r \\ \ddot{\mathbf{U}}_c^r \\ \ddot{\mathbf{U}}_f^r \end{bmatrix} + \begin{bmatrix} \mathbf{C}_{ss} & \mathbf{C}_{sc} & \mathbf{0} \\ \mathbf{C}_{cs} & \mathbf{C}_{cc} & \mathbf{C}_{cf} \\ \mathbf{0} & \mathbf{C}_{fc} & \mathbf{C}_{ff} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{U}}_s^r \\ \dot{\mathbf{U}}_c^r \\ \dot{\mathbf{U}}_f^r \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{ss} & \mathbf{K}_{sc} & \mathbf{0} \\ \mathbf{K}_{cs} & \mathbf{K}_{cc} & \mathbf{K}_{cf} \\ \mathbf{0} & \mathbf{K}_{fc} & \mathbf{K}_{ff} \end{bmatrix} \begin{bmatrix} \mathbf{U}_s^r \\ \mathbf{U}_c^r \\ \mathbf{U}_f^r \end{bmatrix} = \mathbf{R} \tag{4a}$$

$$\mathbf{R} = -\begin{bmatrix} \mathbf{M}_{ss} & \mathbf{M}_{sc} & \mathbf{0} \\ \mathbf{M}_{cs} & \mathbf{M}_{cc} & \mathbf{M}_{cf} \\ \mathbf{0} & \mathbf{M}_{fc} & \mathbf{M}_{ff} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{U}}_s^{ff} \\ \ddot{\mathbf{U}}_c^{ff} \\ \ddot{\mathbf{U}}_f^{ff} \end{bmatrix} - \begin{bmatrix} \mathbf{C}_{ss} & \mathbf{C}_{sc} & \mathbf{0} \\ \mathbf{C}_{cs} & \mathbf{C}_{cc} & \mathbf{C}_{cf} \\ \mathbf{0} & \mathbf{C}_{fc} & \mathbf{C}_{ff} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{U}}_s^{ff} \\ \dot{\mathbf{U}}_c^{ff} \\ \dot{\mathbf{U}}_f^{ff} \end{bmatrix} - \begin{bmatrix} \mathbf{K}_{ss} & \mathbf{K}_{sc} & \mathbf{0} \\ \mathbf{K}_{cs} & \mathbf{K}_{cc} & \mathbf{K}_{cf} \\ \mathbf{0} & \mathbf{K}_{fc} & \mathbf{K}_{ff} \end{bmatrix} \begin{bmatrix} \mathbf{U}_s^{ff} \\ \mathbf{U}_c^{ff} \\ \mathbf{U}_f^{ff} \end{bmatrix} \tag{4b}$$

If the free-field displacement $\mathbf{U}_c^{ff}$ is constant on the dam-foundation rock interface, the vector $\mathbf{U}_s^{ff}$ is the rigid motion of the dam body. Therefore, we have

$$\begin{bmatrix} \mathbf{K}_{ss} & \mathbf{K}_{sc} \\ \mathbf{K}_{cs} & \mathbf{K}_{cc}^s \end{bmatrix} \begin{bmatrix} \mathbf{U}_s^{ff} \\ \mathbf{U}_c^{ff} \end{bmatrix} = \mathbf{0} \tag{5}$$

Furthermore, the dynamic equilibrium of the foundation rock results in

$$\begin{bmatrix} \mathbf{M}_{cc}^f & \mathbf{M}_{cf} \\ \mathbf{M}_{fc} & \mathbf{M}_{ff} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{U}}_c^{ff} \\ \ddot{\mathbf{U}}_f^{ff} \end{bmatrix} + \begin{bmatrix} \mathbf{C}_{cc}^f & \mathbf{C}_{cf} \\ \mathbf{C}_{fc} & \mathbf{C}_{ff} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{U}}_c^{ff} \\ \dot{\mathbf{U}}_f^{ff} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{cc}^f & \mathbf{K}_{cf} \\ \mathbf{K}_{fc} & \mathbf{K}_{ff} \end{bmatrix} \begin{bmatrix} \mathbf{U}_c^{ff} \\ \mathbf{U}_f^{ff} \end{bmatrix} = \mathbf{0} \tag{6}$$

Thus, the Eq. (4b) can be simplified by utilizing Eqs. (5) and (6)

$$\mathbf{R} = -\begin{bmatrix} \mathbf{M}_{ss} & \mathbf{M}_{sc} & \mathbf{0} \\ \mathbf{M}_{cs} & \mathbf{M}_{cc}^s & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{U}}_s^{ff} \\ \ddot{\mathbf{U}}_c^{ff} \\ \ddot{\mathbf{U}}_f^{ff} \end{bmatrix} - \begin{bmatrix} \mathbf{C}_{ss} & \mathbf{C}_{sc} & \mathbf{0} \\ \mathbf{C}_{cs} & \mathbf{C}_{cc}^s & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{U}}_s^{ff} \\ \dot{\mathbf{U}}_c^{ff} \\ \dot{\mathbf{U}}_f^{ff} \end{bmatrix} \tag{7}$$

If the excitation is assumed to be harmonic $(\mathbf{a}_g(t) = \bar{\mathbf{a}}_g e^{i\omega t}, \mathbf{v}_g(t) = \frac{1}{i\omega}\bar{\mathbf{a}}_g e^{i\omega t})$, displacements will be harmonic too. Substituting the harmonic term $\mathbf{U}(t) = \bar{\mathbf{U}} e^{i\omega t}$ in Eqs. (4) and (7), one can express the equation of motion of dam-foundation systems under harmonic excitations as below

$$\left(-\omega^2 \mathbf{M} + i\omega \mathbf{C} + \mathbf{K}\right)\bar{\mathbf{U}} = \bar{\mathbf{R}} \tag{8a}$$

$$\bar{\mathbf{R}} = -\begin{bmatrix} \mathbf{M}_{ss} & \mathbf{M}_{sc} & \mathbf{0} \\ \mathbf{M}_{cs} & \mathbf{M}_{cc}^s & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{J}\bar{\mathbf{a}}_g - \begin{bmatrix} \mathbf{C}_{ss} & \mathbf{C}_{sc} & \mathbf{0} \\ \mathbf{C}_{cs} & \mathbf{C}_{cc}^s & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{J}\left(\frac{1}{i\omega}\bar{\mathbf{a}}_g\right) \tag{8b}$$

where **J** is a matrix which applies the ground acceleration to nodes of the model and the interaction matrix. In the analysis of a half-plane foundation, one can truncate the foundation at a distance far enough from the dam body, discretize the foundation rock area and determine displacements using the finite element method. At the truncation boundary, it is necessary to apply a boundary condition which can transmit elastic waves to obtain a good estimation of the response of the system. The Lysmer boundary condition is one of the most common approaches used at the far end of the foundation (Lysmer and Kohlemeyer 1963).

## 3. Perfectly matched layers (PML)

Perfectly matched layers are media defined in a way that they have two essential properties:

• Waves can pass through boundaries separating perfectly matched layers without any reflections.

• In PML media, wave amplitudes decay as they propagate along some specific directions.

PML can be utilized in the analysis of the half-plane foundation by defining complex stretched coordinates in two perpendicular directions along which elastic waves are to propagate toward infinity. Let us define

$$\tilde{x}_i = \int_0^{x_i} \lambda_i(s)\mathrm{d}s \tag{9}$$

where $\lambda_i(x_i)$ is a complex function called a stretching function which should be continuous in the whole area of the problem. Expressing governing relations of elastic media in terms of $\tilde{x}_i$ will yield the equations defining perfectly matched layers in the foundation (Basu 2004)

$$(\boldsymbol{\sigma}\tilde{\boldsymbol{\Lambda}})\nabla = -\omega^2 \rho[\lambda_1(x_1)\lambda_2(x_2)]\mathbf{u} \tag{10a}$$

$$\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon} \tag{10b}$$

$$\boldsymbol{\varepsilon} = \tfrac{1}{2}\left[(\mathbf{u}\nabla^{\mathrm{T}})\boldsymbol{\Lambda} + \boldsymbol{\Lambda}^{\mathrm{T}}(\mathbf{u}\nabla^{\mathrm{T}})^{\mathrm{T}}\right] \tag{10c}$$

where we have

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}, \quad \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{bmatrix}, \quad \mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}, \quad \nabla = \begin{Bmatrix} \dfrac{\partial}{\partial x_1} \\ \dfrac{\partial}{\partial x_2} \end{Bmatrix} \tag{11a}$$

$$\tilde{\boldsymbol{\Lambda}} = \begin{bmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{bmatrix}, \quad \boldsymbol{\Lambda} = \begin{bmatrix} \dfrac{1}{\lambda_1} & 0 \\ 0 & \dfrac{1}{\lambda_2} \end{bmatrix} \tag{11b}$$

Obviously, perfectly matched layers can be treated like a continuous medium as long as the stretching function ($\lambda_i(x_i)$) is continuous at boundaries. On the other hand, it can be proven that primary and secondary waves decay as they propagate along directions of stretched coordinates. Therefore, a medium in which displacements satisfy Eq. (10) has the aforementioned properties of perfectly matched layers.

### *3.1 Formulation of a PML medium in the foundation*

To solve Eq. (10a) utilizing the finite element method, one can employ the weighted residual approach. Integrating the multiplication of Eq. (10a) by an arbitrary function $\mathbf{W}^{\mathrm{T}}$ and using Green's identity principle will result in the weak form of the equation

$$\int_{\Omega} \tilde{\boldsymbol{\varepsilon}} \boldsymbol{\sigma} \mathrm{d}\Omega - \omega^2 \int_{\Omega} \rho\, f_m \mathbf{w}^{\mathrm{T}} \mathbf{u} \mathrm{d}\Omega = \int_{\Gamma} \mathbf{w}^{\mathrm{T}} \boldsymbol{\sigma} \tilde{\boldsymbol{\Lambda}} \mathbf{n} \mathrm{d}\Gamma \tag{12}$$

where **n** is the outward normal vector at the boundary and $\tilde{\boldsymbol{\varepsilon}}$ is defined as below

$$\tilde{\boldsymbol{\varepsilon}} = \tfrac{1}{2}\left[ (\mathbf{w}\nabla^{\mathrm{T}})\tilde{\boldsymbol{\Lambda}} + \tilde{\boldsymbol{\Lambda}}^{\mathrm{T}} (\mathbf{w}\nabla^{\mathrm{T}})^{\mathrm{T}} \right] \tag{13}$$

Interpolating **u** and **w** in terms of PML elements' nodal values will lead to

$$(\mathbf{K}_{\mathrm{ff}}^{\text{PML-e}} - \omega^2 \mathbf{M}_{\mathrm{ff}}^{\text{PML-e}}) \overline{\mathbf{U}}_{\mathrm{f}}^{\text{PML-e}} = \int_{\Gamma^{\mathrm{e}}} \mathbf{Q} \boldsymbol{\sigma} \tilde{\boldsymbol{\Lambda}} \mathbf{n} \mathrm{d}\Gamma \tag{14a}$$

$$\mathbf{K}_{\mathrm{ff}}^{\text{PML-e}} = \int_{\Omega^{\mathrm{e}}} \tilde{\mathbf{B}}^{\mathrm{T}} \mathbf{D} \mathbf{B} \mathrm{d}\Omega \tag{14b}$$

$$\mathbf{M}_{\mathrm{ff}}^{\text{PML-e}} = \int_{\Omega^{\mathrm{e}}} \rho g_m \mathbf{Q} \mathbf{Q}^{\mathrm{T}} \mathrm{d}\Omega \tag{14c}$$

where $g_m$ is the multiplication of stretching functions ($g_m = \lambda_1(x_1)\lambda_2(x_2)$). **B** and $\tilde{\mathbf{B}}$ are matrices relating the vector-form of $\boldsymbol{\varepsilon}$ and $\tilde{\boldsymbol{\varepsilon}}$ to the vector of elements' nodal displacements **u** and **w** respectively

$$\mathbf{B} = \begin{pmatrix} & \frac{1}{\lambda_1} f_{m,x_1} & 0 & \\ \cdots & 0 & \frac{1}{\lambda_2} f_{m,x_2} & \cdots \\ & \frac{1}{\lambda_2} f_{m,x_2} & \frac{1}{\lambda_1} f_{m,x_1} & \end{pmatrix} \tag{15a}$$

$$\tilde{\mathbf{B}} = \begin{pmatrix} & \lambda_2 f_{m,x_1} & 0 & \\ \cdots & 0 & \lambda_1 f_{m,x_2} & \cdots \\ & \lambda_1 f_{m,x_2} & \lambda_2 f_{m,x_1} & \end{pmatrix} \tag{15b}$$

Moreover, **Q** is a matrix containing shape functions in a form that interpolates the vector of displacement based on nodal ones and **D** is the rigidity matrix

$$\mathbf{Q} = \begin{pmatrix} \cdots & f_m & 0 & \cdots \\ & 0 & f_m & \end{pmatrix}^{\mathrm{T}} \tag{16a}$$

$$\mathbf{D} = \begin{pmatrix} \kappa + 4\mu/3 & \kappa - 2\mu/3 & 0 \\ \kappa - 2\mu/3 & \kappa + 4\mu/3 & 0 \\ 0 & 0 & \mu \end{pmatrix} \tag{16b}$$

where the medium is assumed to be in a plane strain state for the definition of the rigidity matrix. The vector in the right side of Eq. (14a) is only calculated at boundaries of PML area. In previous applications of PML used in the analysis of dam-foundation rock systems, the right side of the equation is assumed to be zero conveniently (Basu 2004). In the present study, the effect of applying a local boundary condition at the truncation boundary of the foundation rock on the response of the system is investigated.

One can rewrite the right side of Eq. (14a) in an expanded form as below

$$\int_{\Gamma^e} \mathbf{Q}\boldsymbol{\sigma}\tilde{\boldsymbol{\Lambda}}\mathbf{n}\mathrm{d}\Gamma = \int_{\Gamma^e} \mathbf{Q}\begin{pmatrix} \lambda_2\sigma_{11}n_{x_1} + \lambda_1\sigma_{12}n_{x_2} \\ \lambda_2\sigma_{12}n_{x_1} + \lambda_1\sigma_{22}n_{x_2} \end{pmatrix}\mathrm{d}\Gamma \tag{17}$$

where $n_{x_1}$ and $n_{x_2}$ are components of the normal vector at the boundary $\Gamma^e$. If the truncation boundary is selected normal to $x_1$ and $x_2$ axes, the Lysmer boundary condition in stretched coordinates can be applied by following relations

$$\sigma_{11}n_{x_1} = -\rho c_\mathrm{p}(\mathrm{i}\omega\bar{u}_1) \text{ and } \sigma_{12}n_{x_1} = -\rho c_\mathrm{s}(\mathrm{i}\omega\bar{u}_2) \text{ on boundaries where } n_{x_2} = 0. \tag{18a}$$

$$\sigma_{22}n_{x_2} = -\rho c_\mathrm{p}(\mathrm{i}\omega\bar{u}_2) \text{ and } \sigma_{21}n_{x_2} = -\rho c_\mathrm{s}(\mathrm{i}\omega\bar{u}_1) \text{ on boundaries where } n_{x_1} = 0. \tag{18b}$$

in which $c_s$ and $c_p$ are the velocity of secondary and primary waves, respectively. Substituting Eq. (18) in Eq. (17) results in

$$\int_{\Gamma^e} \mathbf{Q}\boldsymbol{\sigma}\tilde{\boldsymbol{\Lambda}}\mathbf{n}\mathrm{d}\Gamma = \mathrm{i}\omega\mathbf{C}_{\mathrm{ff}}^{\mathrm{PML-e}}\mathbf{U}_{\mathrm{f}}^{\mathrm{PML-e}} \tag{19a}$$

$$\mathbf{C}_{\mathrm{ff}}^{\mathrm{PML-e}} = \int_{\Gamma^e} \mathbf{Q}\begin{pmatrix} \rho\left(c_\mathrm{p}\lambda_2|n_{x_1}| + c_\mathrm{s}\lambda_1|n_{x_2}|\right) & 0 \\ 0 & \rho\left(c_\mathrm{p}\lambda_1|n_{x_2}| + c_\mathrm{s}\lambda_2|n_{x_1}|\right) \end{pmatrix}\mathbf{Q}^\mathrm{T}\mathrm{d}\Gamma \tag{19b}$$

Replacing the right hand side of Eq. (14a) by Eq. (19a) will yield the governing equation of the PML medium in the foundation in an element level

$$(\mathbf{K}_{\mathrm{ff}}^{\mathrm{PML-e}} + \mathrm{i}\omega\mathbf{C}_{\mathrm{ff}}^{\mathrm{PML-e}} - \omega^2\mathbf{M}_{\mathrm{ff}}^{\mathrm{PML-e}})\overline{\mathbf{U}}_{\mathrm{f}}^{\mathrm{PML-e}} = \mathbf{0} \tag{20}$$

## 4. Numerical experiments

Several numerical experiments have been carried out to evaluate the accuracy and the efficiency of using PML media in analysis of dam-foundation rock systems under a harmonic excitation.

The system being analysed incorporates an ideal triangle concrete dam with an empty reservoir on a flexible foundation rock (Fig. 3). The dam body is assumed to be in the plane stress state, meanwhile, the foundation rock is assumed to be a half plane in the plane strain state. Material properties of the dam body and the foundation rock are summarized in Table 1. The transfer function of the horizontal acceleration of the dam crest due to harmonic horizontal and vertical ground motions (in terms of the ratio of the excitation frequency to the dam's first natural frequency) has been captured by utilizing different approaches. The frequency range of the excitation is chosen between 0 to 12 Hz (a common frequency range for earthquakes).

The dam body and the foundation are discretized by quadratic isotropic elements. The size of elements has been selected in such a way that there are at least two to four elements in every wave length so that all waves can pass through the elements. The response of the system has been also determined by modelling a relatively large part of the foundation rock ($L_f$=8$B$) and using the

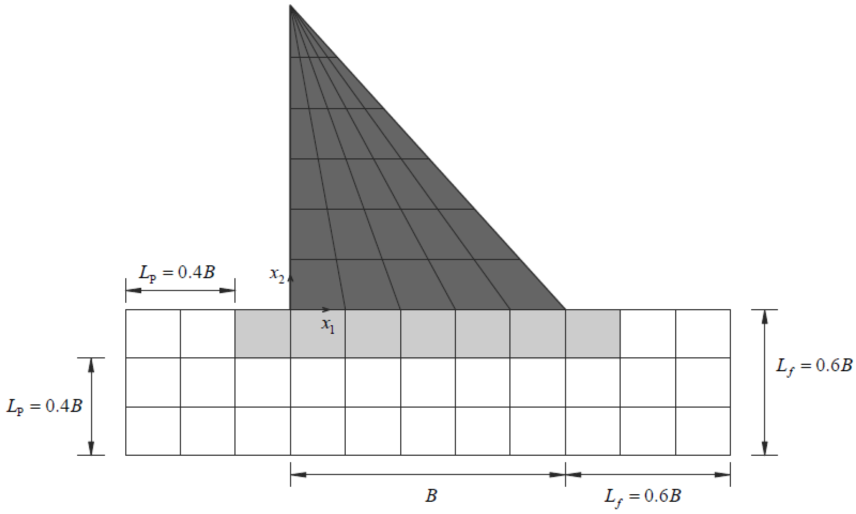



Fig. 3 Numerical model of the ideal dam-foundation rock system

Table 1 Material properties of the model

| | |
|---|---|
| Concrete modulus of elasticity | 27.5 GPa |
| Concrete Poisson's ratio | 0.2 |
| Unit weight of concrete | 24 kN/m$^3$ |
| Foundation rock Poisson's ratio | 0.333 |
| Unit weight of foundation rock | 26 kN/m$^3$ |

Lysmer boundary condition at the truncation boundary. The change of the response due to increasing the size of the foundation rock extension beyond 8B is negligible, as a result, aside from errors related to FE discretization, the solution corresponding to the model in which the dimension of the foundation rock extension equals to 8B may be considered exact. Hence, in the present study, responses obtained by other methods are all compared to the one determined through modelling a large part of the foundation rock domain.

### *4.1 Results*

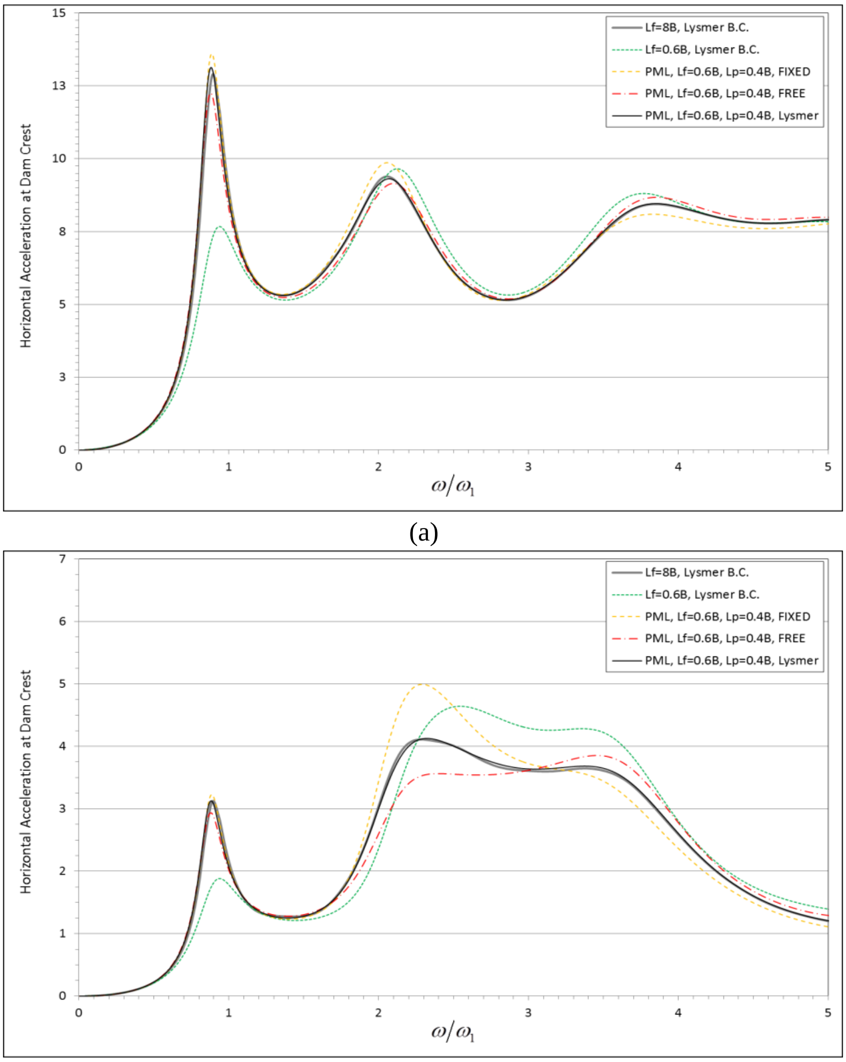
The near-field part of the foundation rock adjacent to the dam body has been discretized by conventional solid elements and the rest has been modeled by perfectly matched layers. The stretching functions are selected in the form recommended by Basu (2004)

$$\lambda_i(\tilde{x}_i) = 1 + f^{\mathrm{e}}\left(\frac{\langle x_i + L_{\mathrm{f}} - L_p\rangle}{L_p}\right)^m - \mathrm{i} f^{\mathrm{p}}\left(\frac{\langle x_i + L_{\mathrm{f}} - L_p\rangle}{L_p}\right)^m \quad \text{for } x_i \le 0 \tag{20a}$$

$$\lambda_i(\tilde{x}_i) = 1 + f^{\mathrm{e}}\left(\frac{\langle -x_i + B + L_{\mathrm{f}} - L_p\rangle}{L_p}\right)^m - \mathrm{i} f^{\mathrm{p}}\left(\frac{\langle -x_i + B + L_{\mathrm{f}} - L_p\rangle}{L_p}\right)^m \quad \text{for } x_i > 0 \tag{20b}$$

Table 2 Parameters of stretching functions (Eq. (20)) used in defining PML ($a_0 = \omega B/2c_s$)

| | | | |
|---|---|---|---|
| Function #1 ($E_s \leq E_f$) | $f^{\,e}=1$ | $f^{\,p}=4.25/a_0$ | $m=1$ |
| Function #2 ($E_s \geq E_f$) | $f^{\,e}=1$ | $f^{\,p}=6.00/a_0$ | $m=1$ |



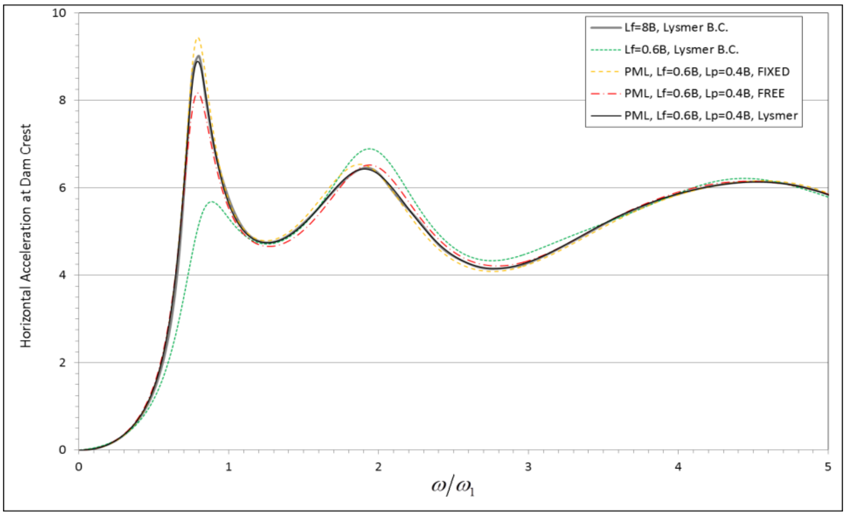

(a)

(b)

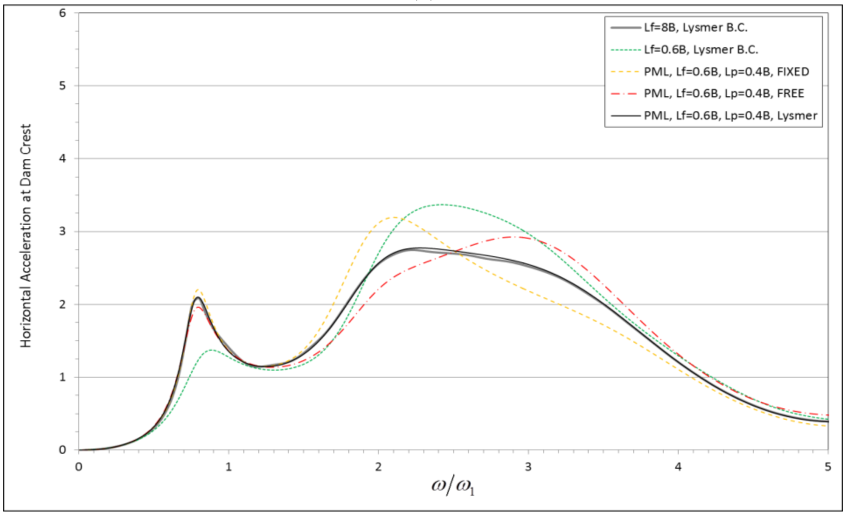
Fig. 4 Horizontal acceleration at dam crest due to harmonic ground motions, for the case $E_f/E_s$=2.00 with different boundary conditions of PML region, (a) Horizontal ground motion, (b) Vertical ground motion

$$\langle x \rangle = \left| \frac{x - |x|}{2} \right| \tag{20c}$$

After several numerical experiments, proper coefficients have been chosen and the smallest PML domain necessary to obtain an acceptable approximation of the response of the system has




(a)

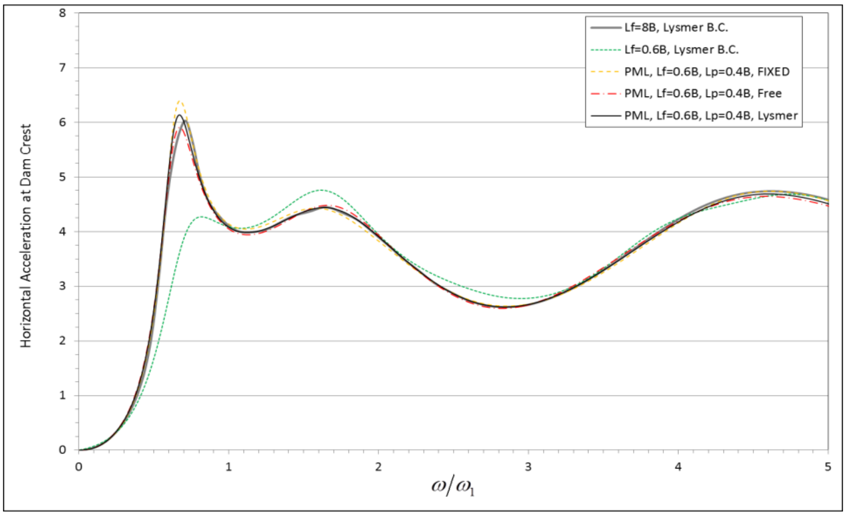



(b)

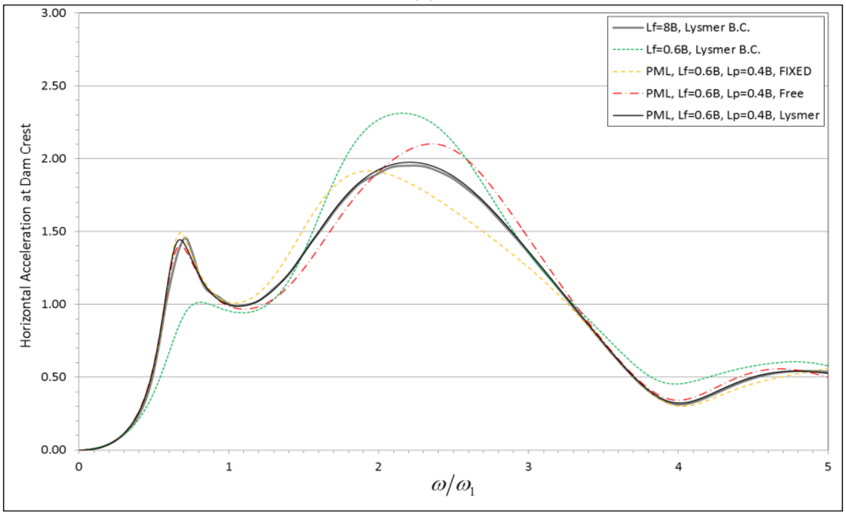
Fig. 5 Horizontal acceleration at dam crest due to harmonic ground motions, for the case $E_f/E_s$=1.00 with different boundary conditions of PML region, (a) Horizontal ground motion, (b) Vertical ground motion

been found. It should be mentioned that the size of the PML domain can be reduced by opting for stretching functions which establish a more attenuative medium. However, to do so, one has to employ smaller elements to discretize the PML area. Here, the size of elements is chosen as if the whole foundation is modeled by solid elements and stretching functions and the PML domain size are selected accordingly. The function parameters ($f^e$, $f^p$ and $m$) are presented in Table 2. The term $a_0$ in the table is a non-dimensional frequency defined as $a_0 = \omega B/2c_s$. The functions are defined in such a way that the equation governing PML medium can be transformed into time domain.

In Table 2, $E_s$ and $E_f$ are the elastic modulus of the structure and the foundation rock, respectively. Note that two different functions have been suggested for two ranges of the elastic modulus ratio ($E_s/E_f$).




(a)

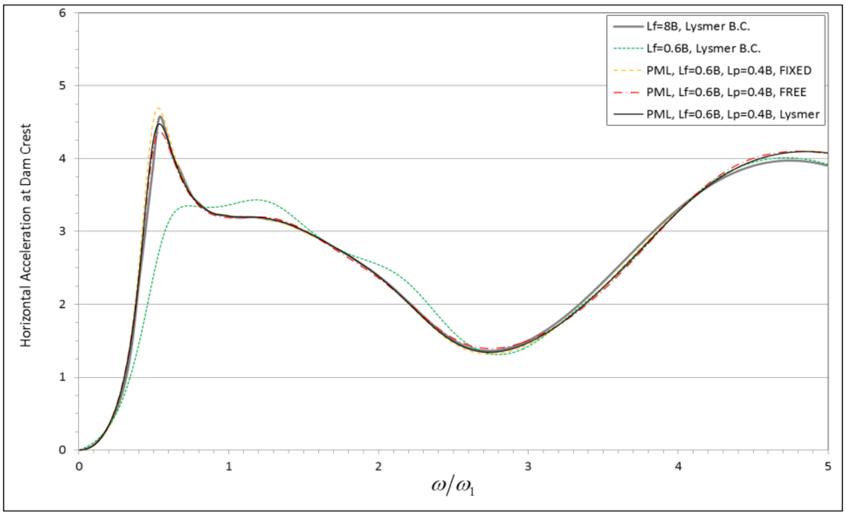



(b)

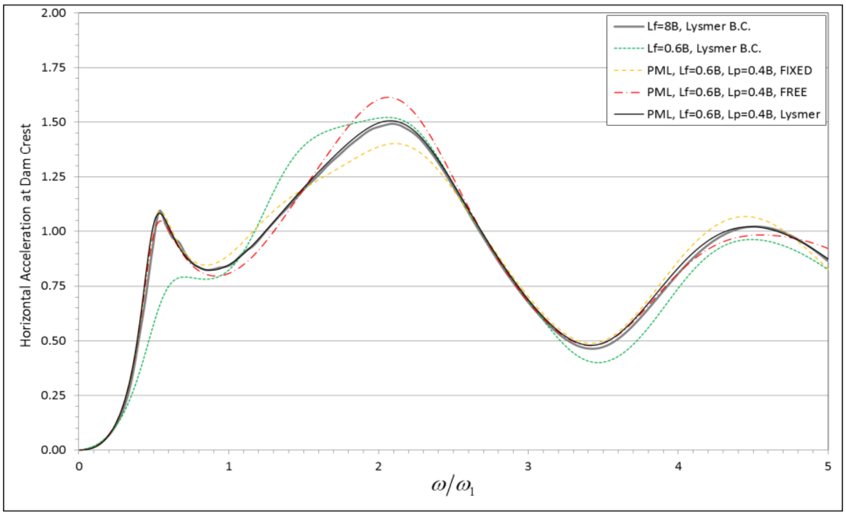
Fig. 6 Horizontal acceleration at dam crest due to harmonic ground motions, for the case $E_f/E_s$=0.50 with different boundary conditions of PML region, (a) Horizontal ground motion, (b) Vertical ground motion

Figs. 4-7 present transfer functions of the dam-foundation rock system for a model in which domain sizes are chosen to be $L_f$=0.6*B* and $L_p$=0.4*B* (Fig. 3). Both horizontal and vertical ground motions are considered. By applying local boundary conditions at the truncation boundary of the PML medium, reasonably accurate results have been obtained for four different elastic modulus ratios ($E_s/E_f$) for both horizontal and vertical ground motions even though the computational domain is relatively small. The inaccuracy of results from models using the Lysmer boundary condition (with the same size of the foundation rock extension ($L_f$=0.6*B*) is a proof for the smallness of the domain.

The results obtained by adopting local boundary conditions at the truncation boundary of the PML medium are also compared with the cases in which no boundary conditions are applied at the




(a)




(b)

Fig. 7 Horizontal acceleration at dam crest due to harmonic ground motions, for the case $E_f/E_s$=0.25 with different boundary conditions of PML region, (a) Horizontal ground motion, (b) Vertical ground motion

outer boundaries of the PML area. The cases named “Fixed” and “Free” are the ones with no boundary conditions at the truncation boundary and the external nodes of their corresponding models are fixed and free, respectively.

As it is shown in Figs. 4-7, applying the Lysmer boundary condition at the truncation boundary of the PML area improves the accuracy of results. Under horizontal excitations, the difference between the results corresponding to different boundary conditions decreases as the elastic modulus ratio ($E_f/E_s$) decreases, however, one could use a smaller PML domain to gain results with the same accuracy by applying the Lysmer relation at the truncation boundary of the PML region even for smaller elastic modulus ratios.

## 5. Conclusions

Perfectly matched layers have been employed in the time harmonic analysis of dam-foundation rock systems. The Lysmer boundary condition at the truncation boundary of the PML region has been incorporated in the formulation of the dam-foundation rock FE model. Several numerical experiments have been carried out on an ideal dam-foundation system. The transfer function of the horizontal acceleration of the dam crest has been determined using different truncation boundary conditions and various numerical model parameters. Results show that:

• If the Lysmer boundary condition is applied at the truncation boundary of the PML area, highly accurate results can be obtained at a relatively small computational cost. Choosing the foundation domain size equal to 0.60B (B is the dam width) will yield reasonably accurate results. It is noteworthy that the stretching function used for defining PML has been selected in such a way that the governing equation of PML can be transformed into time domain.

• Applying the Lysmer boundary condition at the truncation boundary of the PML region improves the accuracy of the results for a specific size of the foundation rock which has to be discretized by finite elements. Hence, it reduces the computational cost and makes the PML approach more efficient for the harmonic analysis of dam-foundation systems.

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