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Determination of cable force based on the corrected numerical solution of cable vibration frequency equations

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Abstract. The accurate determination of cable tension is important to the monitoring of the condition of a cable-stayed bridge. When applying a vibration-based formula to identify the tension of a real cable under sag, stiffness and boundary conditions, the resulting error must not be overlooked. In this work, by resolving the implicit frequency function of a real cable under the above conditions numerically, indirect methods of determining the cable force and a method to calculate the corresponding cable mode frequency are investigated. The error in the tension is studied by numerical simulation, and an empirical error correction formula is presented by fitting the relationship between the cable force error and cable parameters λ^2 and ξ. A case study on two real cables of the Shanghai Changjiang Bridge shows that employing the method proposed in this paper can increase the accuracy of the determined cable force and reduce the computing time relative to the time required for the finite element model.

Keywords: cable; cable force; implicit frequency function; numerical solution

1. Introduction

The cable force is an important aspect of the working condition of a cable-stayed bridge because it makes important contributions to the load bearing and deformation bearing of the overall structure of the bridge (Irvine 1991). The force thus needs to be measured accurately. Presently, the frequency formula for ambient vibration is commonly used for measurements of a cable-stayed bridge (Kim *et al*. 2007, Nam *et al*. 2011). When actual factors such as sag, flexural stiffness, and boundary conditions are considered, the explicit relationship between the cable force and cable vibration frequency cannot be presented; i.e., it is inaccurate to explore their relationship employing a formula based on the taut string assumption (Geier *et al*. 2006, Marcelo *et al*. 2008, Choi and Park 2011). Therefore, a number of methods that aim to increase the accuracy of determining the cable force have been proposed. These methods focus on the technical resolution to establish an explicit relationship between the cable force and cable modal frequency. Scholars adopted the finite difference method (Mehrabi *et al*. 1998), finit element method (Wang *et al*.

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2011), and spline fitting to fit the completely empirical and explicit relationship between the cable vibration frequency and force (Zui *et al*. 1996). Others such as Ren *et al*. (2005) and Fang *et al*. (2011) proposed a practical formula that takes cable sag and flexural stiffness into account separately and calculates the cable force from the fundamental frequency employing energy considerations and curve fitting, and this explicit relationship is partially empirical and partially theoretical. Chen *et al.* (2013) make revision on effective cable length and used multiple measurements to increase the accuracy of identified cable force. Although these methods are suitable for obtaining approximate values of the cable force, in certain circumstances, especially when sag, flexural stiffness and boundary conditions need to be considered simultaneously, the accuracy is unsatisfactory. Another method is model updating, which determines the cable force by modifying the parameters of mechanical models (including differential equation models, finite difference models and finite element models). Employing an optimization method, Kim *et al*. (2007) modified the differential analysis model to take flexural stiffness into account when obtaining the cable force. Zhang (2010) modified the finite element model of the cable using an intelligent algorithm. The former of these methods encounters difficulties in terms of correcting systematic errors through parameter modification, whereas the latter can overcome model errors but is not suitable for immediate and online determination of the cable force owing to its time-consuming finite element calculation. Anyway, the frequency formula based approaches often need to be verified with other direct cable force measurements like pressure ring method and elasto-magnetic sensors approach (Yim *et al*. 2013).

To achieve a measurement that is accurate and has low computational cost, this paper investigates the implicit frequency equation for the cable without a damper that is presented in the literature and considers sag, flexural stiffness and clamped boundary conditions, and establishes a relationship between the cable force and frequency with an implicit frequency equation. Therefore, on the one hand, the cable force can be determined easily, with the accuracy improved by empirical error modification, and on the other hand, a computationally inexpensive, accurate and online cable model modification can be easily achieved through the accurate frequency calculation for a given cable force.

2. Cable frequency equatıon

The cable frequency equation describes the relationship between the cable vibration frequency and cable force. If the cable parameters are given, mutual computing can be achieved using this frequency equation. Under certain assumed conditions, the frequency equation can be presented explicitly. For example, when a pinned–pinned taut string is assumed, the cable frequency equation is explicit and is used as the formula of the vibration frequency method commonly used in engineering. In addition, under the conditions assumed in the literature (Ricciardi *et al*. 2008), the cable frequency equation cannot be presented explicitly but can be presented according to its implicit function relationship, namely the implicit frequency equation, and mutual computing between cable tension and modal frequencies can be realized employing the numerical method. The following are frequency equations for these two situations.

2.1 Explicit frequency equation for the taut string

The equation for free vibration of the horizontal taut string is

$$EI\frac{\partial^4 v}{\partial x^4} - H\frac{\partial^2 v}{\partial x^2} + \rho\frac{\partial^2 v}{\partial t^2} = 0 \tag{1}$$

where x is the coordinate along the cable length and $v(x,t)$ is the transverse deflection of the cable location x at time t, EI is the flexural stiffness, H is the cable force, and ρ is the mass per unit length.

Considering the pins of the two cable ends, the explicit frequency equation for vibration and the corresponding cable-tension equation in the cable plane are

$$f_n = \frac{n}{L}\sqrt{\frac{TL^2 + n^2\pi^2 EI}{4\rho L^2}} \tag{2a}$$

$$T = \frac{4\rho L^2 f_n^2}{n^2} - \frac{n^2\pi^2 EI}{L^2} \tag{2b}$$

where n is the order of the cable vibration frequency, f_n is the frequency of vibration of order n, and L is the cable length.

2.2 Implicit frequency equation for the cable

Considering sag and flexural stiffness conditions, the free-vibration equation within the cable plane is

$$EI\frac{\partial^4 v}{\partial x^4} - H\frac{\partial^2 v}{\partial x^2} - h(t)\frac{d^2 y}{dx^2} + \rho\frac{\partial^2 v}{\partial t^2} = 0 \tag{3}$$

where $h(t)$ is the additional cable force due to the cable vibration. In the case of the anti-symmetric mode, $h(t)$=0, Eq. (3) has the same form as Eq. (1). By applying a doubly clamped boundary to Eq. (3), two implicit frequency equations are obtained as follows (Ricciardi *et al*. 2008).

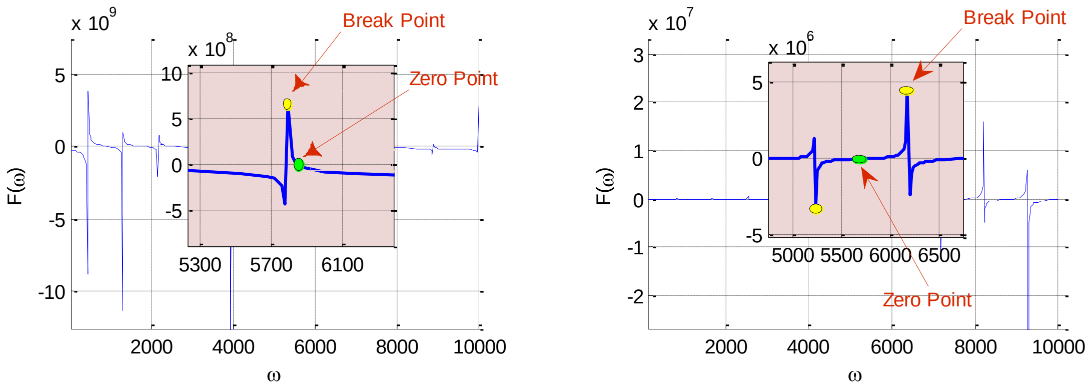
Symmetric mode

$$\frac{\hat{\omega}^2}{\lambda^2} = \xi^2\{1 - \frac{2(\hat{\beta}/\hat{\alpha} + \hat{\alpha}/\hat{\beta})}{\hat{\alpha}\coth(\hat{\beta}/2) + \hat{\beta}\coth(\hat{\alpha}/2)}\} \tag{4a}$$

Anti-symmetric mode

$$\frac{\hat{\alpha}(\hat{\omega})}{\hat{\beta}(\hat{\omega})} = \frac{\tan[\hat{\alpha}(\hat{\omega})\ /2]}{\tanh[\hat{\beta}(\hat{\omega})\ /2]} \tag{4b}$$

Here, $\hat{\alpha}(\hat{\omega}) = \sqrt{\sqrt{(\frac{\xi^2}{2})^2 + \hat{\omega}^2} - \frac{\xi^2}{2}}$, $\hat{\beta}(\hat{\omega}) = \sqrt{\sqrt{(\frac{\xi^2}{2})^2 + \hat{\omega}^2} + \frac{\xi^2}{2}}$, $\hat{\omega}=\omega\frac{L^2}{\sqrt{EI/\rho}}$, $\xi=L\sqrt{\frac{H}{EI}}$, $\lambda^2=\frac{(\rho g L)^2 EAL}{L_e H^3}$, and $L_e \cong L[1+(\rho g L)^2/(8H^2)]$, ω is the angular frequency.



(a) $F(\hat{\omega})$ curve of formula 5a (b) $F(\hat{\omega})$ image curve of formula 5b

Fig. 1 Mesh grid of topographic model

3. Numerical solution to the implicit frequency equation

When the cable parameters $\theta(\theta_1,\theta_2,\theta_3\ldots)$ are given, Eq. (4a) and (4b) provides the implicit equation relationship between the frequency and cable force through the operations of transposition and simplification. This equation is a complex transcendental equation, and it cannot be used to resolve the cable force and other parameters directly.

Eqs. (2a) and (2b) presents the explicit relationship between the cable vibration frequency and cable force, and it can easily be applied to the computation of a cable force. However, it does not take the sag, flexural stiffness, or boundary hinge condition into account, and it thus differs from the relationship for a real bridge cable. Therefore, the cable force obtained using Eq. (2b) may be inaccurate. Even in the case of large sag and long cable length, the result obtained with Eq. (2b) is not applicable because of the generation of unacceptable errors. Eqs. (4a) and (4b) is based on all above-mentioned factors simultaneously, and is closer to the relationship for a real cable; therefore, the described force–frequency relationship for the cable is more accurate than the explicit relationship relating to a taut string but applied to real cables. Although it is an implicit relationship, a numerical solution to Eq. (4a) and (4b) can be used to obtain the cable vibration frequency and cable force.

3.1 Root-seeking mode selection for the implicit frequency equation

Solving Eq. (4a) and (4b) is a process of finding the zero root of the equation $F(\omega,H,\theta)=0$. Obviously, any primary operations acting on the left term of the expression $F(\omega,H,\theta)$ will not affect the zero root value. Therefore, we make suitable changes to Eq. (4a) and select the most suitable form to obtain the numerical solution. The target zero roots of the frequency equation can be limited to a continuous interval of the argument and set as far from the break point as possible, which will help with the stability and convergence of numerical algorithms.

When solving for the symmetric mode, we can rearrange Eq. (4a) as

$$\xi^2\{1-\frac{2(\hat{\beta}/\hat{\alpha}+\hat{\alpha}/\hat{\beta})}{\hat{\alpha}\coth(\hat{\beta}/2)+\hat{\beta}\coth(\hat{\alpha}/2)}\}-\frac{\hat{\omega}^2}{\lambda^2}=0 \tag{5a}$$

Adopting Eq. (5a), it is found that most roots of the equation approach the break points of intervals, which leads to difficulties in finding the zero roots and solving the equation numerically (see Fig. 1(a)). However, the following mode can be used to locate the zero roots at nearly the mid-point between two break points (Fig. 1(b)). As a result, the solution is easy and concentrated and is unlikely to be confined to infinite recursion.

$$\frac{(\xi^2\lambda^2 - \hat{\omega}^2)\left[\dfrac{\hat{\alpha}}{\hat{\beta}}\coth(\hat{\beta}/2) + \coth(\hat{\alpha}/2)\right]}{2\lambda^2\xi^2\dfrac{\hat{\beta}^2+\hat{\alpha}^2}{\hat{\alpha}\hat{\beta}^2}} - 1 = 0 \tag{5b}$$

To solve for the anti-symmetric mode, we apply the following expression to the frequency equation

$$F(\omega, H, \theta) = \frac{\hat{\alpha}}{\hat{\beta}} - \frac{\tan(\hat{\alpha}/2)}{\tanh(\hat{\beta}/2)} = 0 \tag{6}$$

3.2 Determination of the equation solution interval

Owing to the presence of periodic terms ‘tan’ on the left side of the cable frequency equation, the domain of the definition of the functions determined by Eqs. (5a), (5b) and (6) is periodically separated as numerous continuous intervals with discontinuous break points, with each period forming a resolving interval, within which the function has at least one zero root corresponding to one order mode of the cable. Thus, the first step to solving this type of periodic implicit function is to determine the resolving intervals throughout the domain of definition of the functions.

Anti-symmetric mode:

According to the parameter values of Eqs. (4b) and (6), break points arise from the periodic discontinuous point of the term tan($\hat{\alpha}$/2) in the left expression $F(\omega,H,\theta)$, and the resolving intervals can be determined by these periodic break points.

Let

$$n\pi - \pi/2 < \hat{\alpha}/2 < n\pi + \pi/2 \tag{7}$$

where n=0,1,2,... Solving the above inequality, the argument ranges of ω and ξ^2 can be determined as

$$\frac{2n\pi\sqrt{\dfrac{(2n)^2\pi^2 EI}{m}}}{L^2} < \omega < \frac{(2n+1)\pi\sqrt{\dfrac{(2n+1)^2\pi^2 EI + HL^2}{m}}}{L^2} \tag{8}$$

$$\frac{\hat{\omega}^2 - (2(n+1)-1)^4\pi^4}{(2(n+1)-1)^2\pi^2} < \xi^2 < \frac{\hat{\omega}^2 - (2n-1)^4\pi^4}{(2n-1)^2\pi^2} \tag{9}$$

When the cable force is known and the frequency is to be resolved, Eq. (8) can be used to define the resolving interval, where ω is the circular frequency of the structure. To solve the resolving interval for the cable force, Eq. (9) is used to first obtain the intervals of ξ^2, and the resolving intervals of the cable force are then obtained according to

$$H=\xi^2\frac{EI}{L^2} \tag{10}$$

It should be mentioned that in Eq. (6), if the zero root exists and $\hat{\alpha}>0$, $\hat{\beta}>0$ and $\tanh(\hat{\beta}/2)>0$ hold throughout the definition domain, the resolving interval can be shortened as

$$n\pi<\hat{\alpha}/2<n\pi+\pi/2 \tag{11}$$

As a result, the narrowed interval of ω and H can be determined in the same way using the above-mentioned operations.

Symmetric mode:

In Eq. (5b), the $\cot(\hat{\alpha}/2)$ term has periodic break points, and the resolving interval of ω and ξ^2 can be defined by these break points; i.e., substituting the definition expression of $\hat{\alpha}$ into the inequality $n\pi<\hat{\alpha}/2<(n+1)\pi$, n=0, 1, 2, …, we get

$$\frac{2n\pi\sqrt{\dfrac{(2n)^2\pi^2EI+HL^2}{m}}}{L^2}<\omega<\frac{2(n+1)\pi\sqrt{\dfrac{(2(n+1))^2\pi^2EI+HL^2}{m}}}{L^2} \tag{12}$$

$$\frac{\hat{\omega}^2-(2(n+1))^4\pi^4}{(2(n+1))^2\pi^2}<\xi^2<\frac{\hat{\omega}^2-(2n)^4\pi^4}{(2n)^2\pi^2} \tag{13}$$

Similarly, the resolving interval of H can also be deduced from Eq. (13).

3.3 Selection of method for resolving numerical rooting

Commonly used numerical solutions for nonlinear equations include the procedures of dichotomy, the rule of thirds, the bubble method, simple iteration, the golden section method and Newton-Raphson (NR) iteration. NR iteration provides a quick searching rate but cannot be applied in solution crossing intervals, as it can give rise to the interference of intervals and the acquired root will not be within the corresponding interval. Additionally, NR iteration requires convergence conditions to be met to avoid any departure from the target root or falling into infinite iterations. Interval solution methods including dichotomy procedures, the rule of thirds, the golden section method, and the bubble method can provide at least one root in a desired resolving interval of the equation if the root exists. The only restriction is that the function values have different signs at the two ends of the resolving intervals. The searching of the dichotomy method is slow, whereas that of the golden section method is quicker, but the former method has a higher convergence rate.

According to the features of the real-cable implicit frequency equation, a method with a quick convergence rate should be adopted. Eqs. (5a), (5b) and (6) contain periodic function terms, and the left function value of the two equations thus crosses the zero axis periodically and there is more than one root. Generally, there is only one zero root in the resolving interval. In particular cases, there can be two or more zero roots, and this paper therefore adopts a combination of the interval solution method (implemented by the *fzero* function in Matlab) and NR method to solve the cable vibration frequency equation.

In the event that there are many zero points in one interval, further discrimination is needed to gain the actual root. As for the cable force, whichever order of frequency is adopted, the acquired cable force should be the same. On this basis, the root closest to the single zero root in another resolving interval can be selected.

4. Identification error and empirical modification of the cable force and frequency

Employing the numerical method of the previous frequency equation, we can easily determine the cable force when the cable vibration frequency is provided and determine the frequency when the cable force is provided. The cable force and vibration frequency obtained employing the methods proposed in this paper are closer to the actual quantities than those obtained using the explicit frequency equation for a taut string. However, the error in the proposed method cannot be neglected. For comparison, if the cable is divided into enough elements, the cable force and vibration frequency obtained using the FEM are close to the real values. Therefore, the vibration frequency obtained using the FEM can be assumed as the frequency of the real structure and is used to study the error tendencies of the methods proposed in this paper.

The frequency error is defined as relative error between the frequencies gained using the examined method and the finite element procedure

$$Err(f_n) = \frac{f_n - f_n^{\mathrm{FE}}}{f_n^{\mathrm{FE}}} \tag{14}$$

The basis frequency f_n^{FE} is the n^{th} order mode frequency without an additional damper and with a clamped boundary condition, and f_n is the n^{th}-order frequency gained employing the examined method.

The error in the cable force is defined as the relative error between the determined cable force $T_{identified}$ and the designated cable force $T_{designated}$ of the FEM

$$Err(T) = \frac{T_{identified} - T_{designated}}{T_{designated}} \tag{15}$$

Carrying out numerical simulation to numerically determine the cable force using the implicit frequency equation with different cable technique parameters, we can investigate the relationship between the errors and synthetic cable parameters such as λ^2 and ξ (see Fig. 2 and Fig. 3). During the simulation, to obtain reasonable values of λ^2 and ξ, we take the upper margin of cable parameters in an engineering scope (including $T_{designated}$) as the boundary, traverse the values within this range, and maintain these parameter values independently. Each step of the simulation matches a set of parameters, and we first accurately calculate the first seven orders of cable vibration frequencies employing the FEM, and then take these frequencies as the input and determine the cable force $T_{identified}$ numerically. Eqs. (14) and (15) can be used to calculate the errors in the cable vibration frequency and cable force. The relationship between the errors and synthetic parameters λ^2 and ξ can then is investigated.

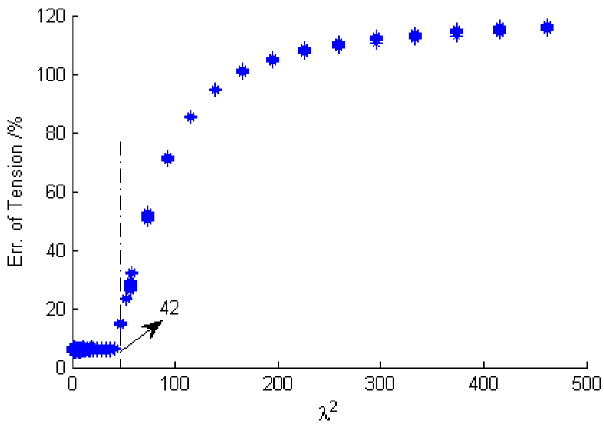


Fig. 2 Relationship between error in the cable force and λ^2

Table 1 Some Irvine parameters λ^2 from 3 cable stayed bridges in shanghai

bridge	Yangpu			Xupu			Donghai		
Cables	Mid23	Mid11	Mid1	S29	S15	S2	S32	S16	S3
λ^2	0.467	0.304	0.024	0.769	0.432	0.358	0.674	0.532	0.293

4.1 Relationship between cable-force error and λ^2

Fig. 2 shows that λ^2 has a marginal value of 42, which defines the applicable scope of the numerical method of the frequency equation. When λ^2<42, the error in the determined cable force is 5%–7% (the determined cable force is larger than the real cable force), and the error has no obvious relationship with λ^2; when λ^2>42, the error increases sharply. The explanation is that, for pure cable behavior, if $\lambda^2<4\pi^2$, the frequency of first symmetric mode is less than the frequency of first anti-symmetric mode, and when $\lambda^2=4\pi^2$, the two frequencies are equal; if $\lambda^2>4\pi^2$, the frequency of first symmetric mode is greater than the frequency of first anti-symmetric mode. In this study, considering other factor, this critical value come to 42, a value approached to $4\pi^2$. Fortunately, in the scope of engineering, λ^2 is usually less than the marginal value of 42, and the method presented in this paper can thus be applied to the determination of the cable force of a cable-stayed bridge. Caetano, E. D. S. have investigated many real cables and have proved that is true. Also we can verify it by the data in following Table 1, which are collected from three cable stayed bridge in Shanghai area.

4.2 Relationship between cable-force error and ξ

We now discuss the relationship between cable-force error and ξ in the scope of engineering (λ^2<42).

Fig. 3 shows the relationship between the error in determining the cable force using the numerical method of the frequency equation for the first seven orders of vibration and cable parameter ξ, and its fitting curve. The figure shows that no matter whether the fundamental frequency or frequencies from second order to seventh order is chosen to determine the cable

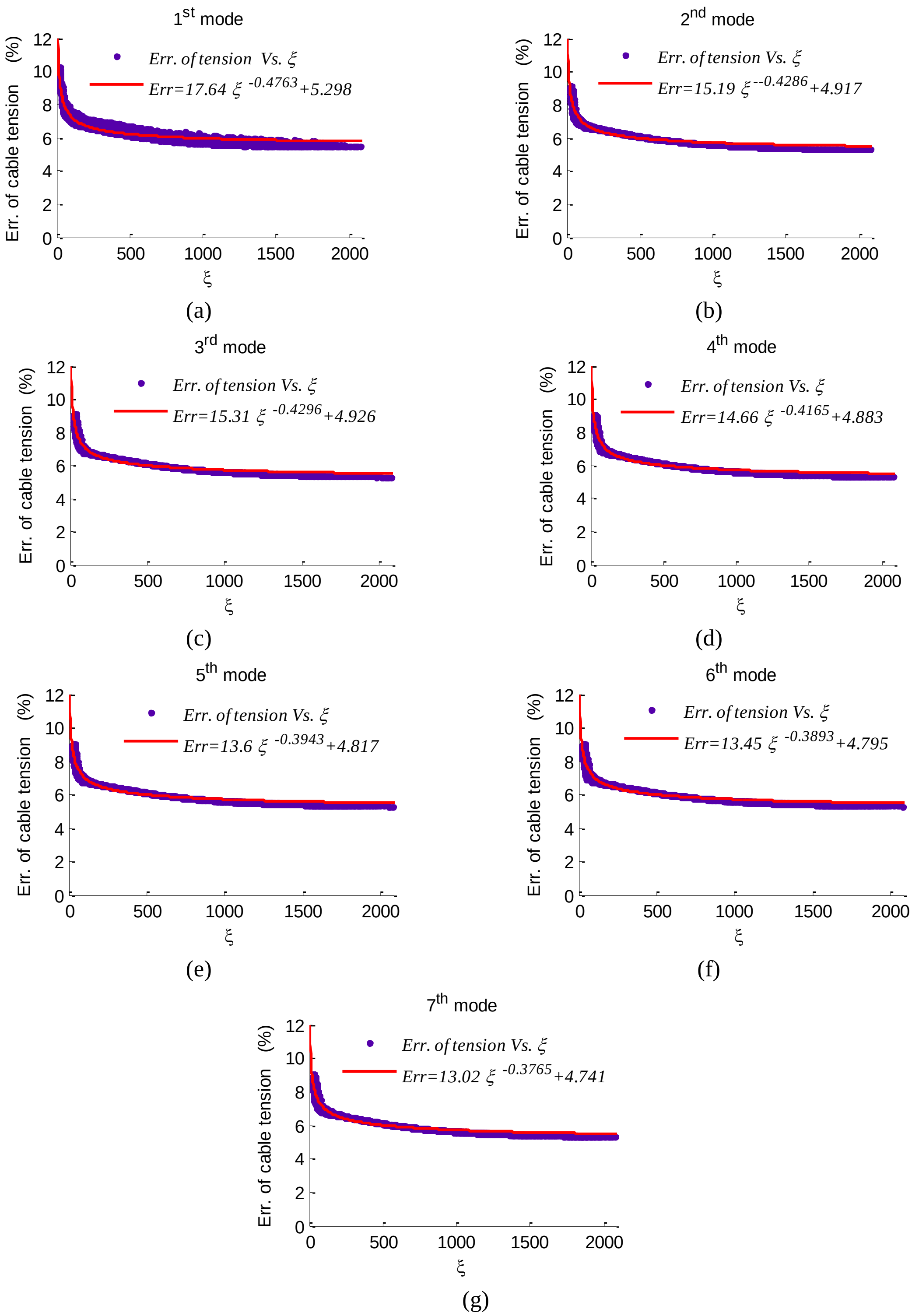


Fig. 3 Cable force determined using the frequencies of the first to seventh modes of vibration

Table 2 Fitting coefficients for the error-ξ relationship in Eq. (16)

	Baseband	2	3	4	5	6	7	2~6 composite
a_i	17.64	15.19	15.31	14.66	13.6	13.45	13.02	14.205
b_i	-0.4763	-0.4286	-0.4296	-0.4165	-0.3943	-0.3893	-0.3765	-0.4058
c_i	5.298	4.917	4.926	4.883	4.817	4.795	4.741	4.8465

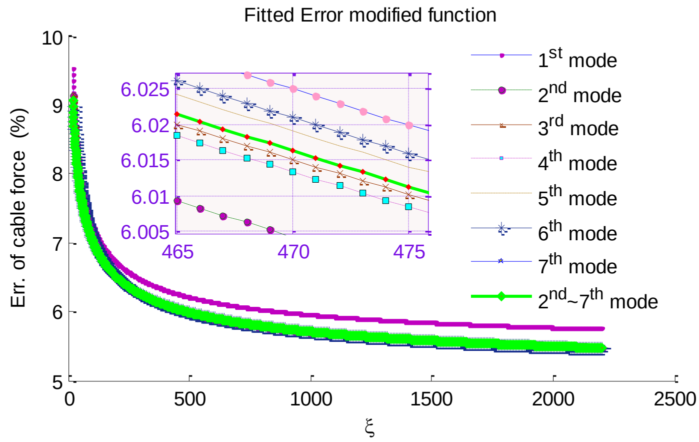


Fig. 4 Relationship between ξ and the error in the cable force determined using the first seven orders of vibration

force, the cable force error and ξ have an approximately exponential relationship. When 0<ξ <200, as the ξ value of the cable increases, and cable-force error decreases from 14% to 7%. When ξ>300, the decrease in error becomes more gradual and the error value approaches 5%.

To fit the relationship between the cable-force error and ξ using

$$Err_i = a_i \xi^{b_i} + c_i \tag{16}$$

Where the subscript i represents the order of vibration adopted, and a_i, b_i and c_i are the coefficients of the fitting expression for the relationship between the cable-force error calculated for the i^{th} order of frequency and ξ . The values of the fitting coefficients are listed in the table. The fitting curves for all orders of vibration are shown in Fig. 2. It is seen that when the orders two to seven are adopted to determine the cable force, the fitting expressions for the error and ξ almost superpose. However, when the base frequency is adopted, the fitting curve is above that for orders two to seven. This means that, under the same cable conditions, the error is slightly larger when the base frequency is used to determine the cable force, and when frequencies of second to seventh order are applied, the errors are almost equal. Therefore, the cable force can be corrected by one of two fitting formulas. The last column in Table 2 gives the coefficients for the fitting of the error-ξ relationship that are suitable for the second- to seventh-order vibrations, and Fig. 4 presents the integrated modification curve (in green) for cable-force error that is suitable for second- to seventh-order vibrations.

Because all the determined results are monotonously larger than the actual cable force, the previous fitting formula can be applied to modify the numerical results of the implicit frequency equation. After modification, the cable force is calculated as

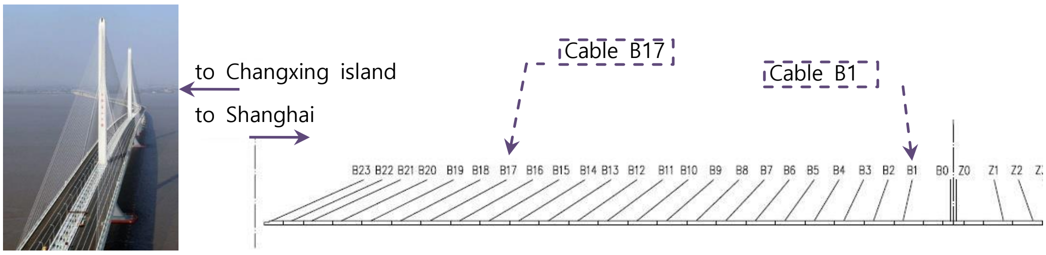
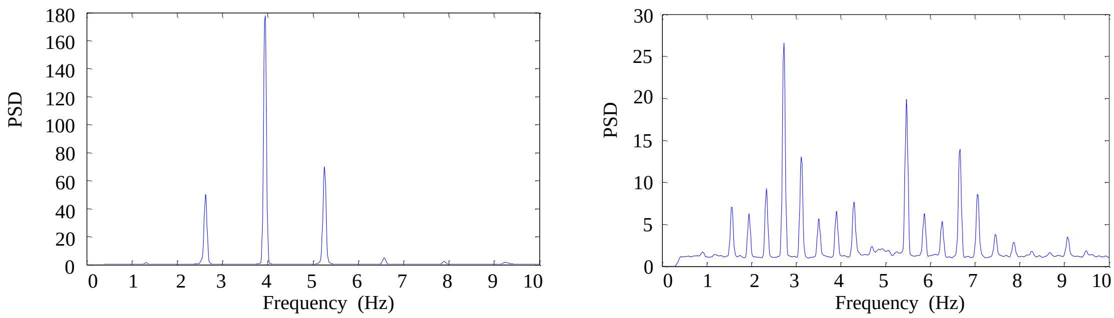


Fig. 5 Main bridge of Shanghai Changjiang Bridge and the cable layout



(a) B01 (b) B17

Fig. 6 Power spectrum density of cables B01 and B17

$$H_{\_\mathrm{mod}ified} = H_{\_identified} * (1 - Err_i) \tag{17}$$

4.3 The proposed modified cable force determination procedure

The following flowchart is proposed to determine the cable force of any given cables,

Step 1, to identify the mode frequencies from the vibration data;

Step 2, to determine the solution interval according to the frequency mode order.

Step 3, to do root seeking of frequency equations according to the proposed method. If the symmetric mode frequency is used, the equation is (5b), and if anti-symmetric frequencies are chosen, the rooting seeking in done on equation (6) ;

Step 4, to estimate the Err according to the experience formula (16);

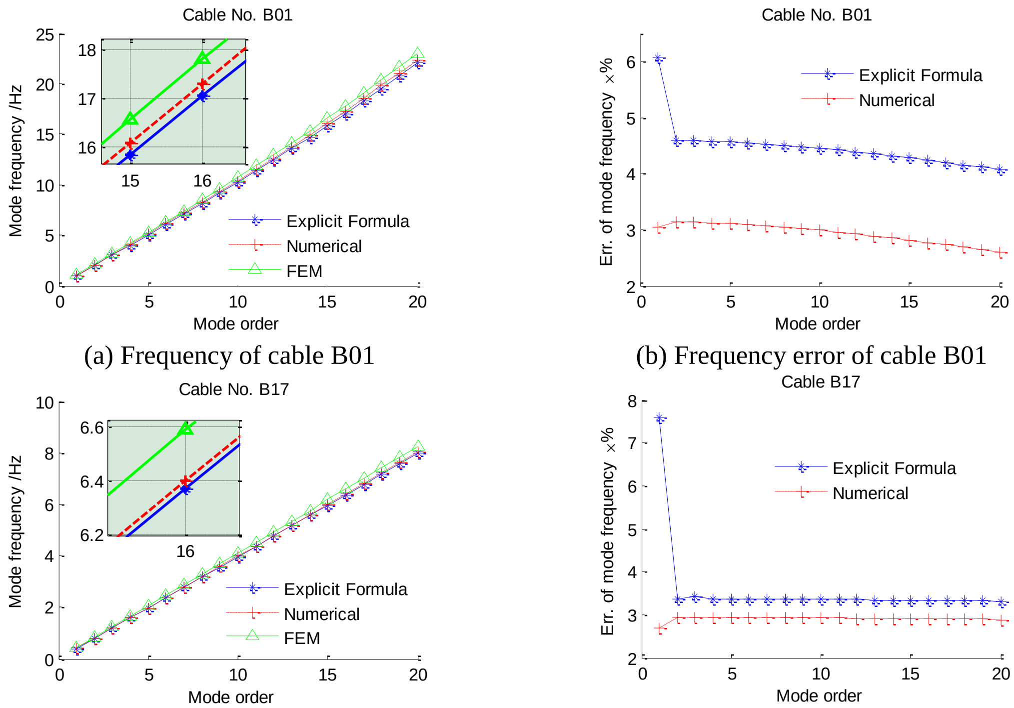
Step 5, to modify the initial Cable force identified in step 3, according to expression (17).

5. Applications

To demonstrate the effectiveness of the methods proposed in this paper, employing the numerical method of the implicit frequency equation (simply referred to as the numerical method hereafter), the cable forces are determined from the vibration frequency for the B01 and B17 cables of the Shanghai Changjiang Bridge (see Fig. 3). For comparison, the FEM and taut-string explicit vibration frequency formula (simply referred to as the explicit formula hereafter) are used. Before the investigation of these three methods, the FEM model is updated for both cables to

Table 3 Cable parameters for Shanghai Changjiang Bridge undergo process of model updating

Cable	Cable length (m)	Elastic modulus (Pa)	Section area (m^2)	Second moment of area (m^4)	Linear density (kg/m)	Cable force (KN)
B01	97.6	2e11	1.0029e-2	8.0036e-6	79.15	3.01e3 (dip70°)
B17	300	2e11	1.2272e-2	1.1984e-5	96.85 kg/m	5.46e3 (dip28°)



(a) Frequency of cable B01 (b) Frequency error of cable B01

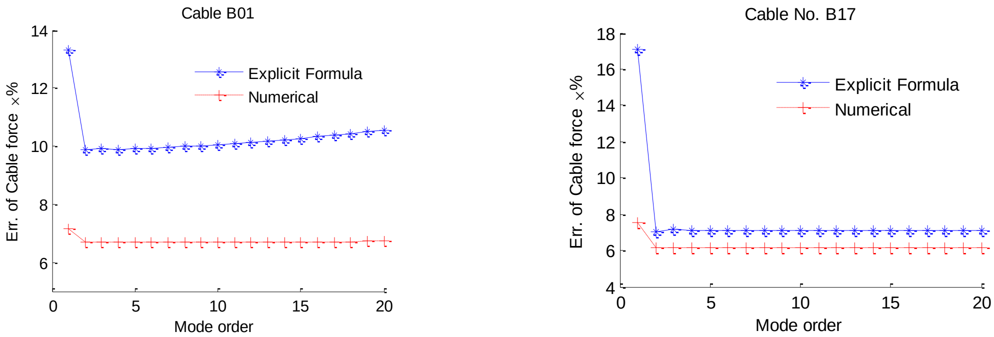
(c) Frequency of cable B17 (d) Frequency error of cable B17

Fig. 7 Frequency results and their errors for cables B01 and B17

obtain a set of reasonable physical constants, which are presented in the table. The optimum target functions of the updating process are the root-mean-square of the difference vector between the measured frequencies and the corresponding calculated frequencies. The measured frequencies are obtained from power spectrum analysis of the measured raw acceleration shown in Fig. 4. Three-dimensional beam elements are used to establish finite element models of the two cables, and each cable is divided into 100 elements. In the modal analysis of the parameters for the first 20 orders of frequency, this finite element number provides very accurate modal frequencies, which can be regarded as good approximations of the real modal frequencies of the cable.

5.1 Cable force is provided to calculate frequency

First, the designated cable force of the two cables in the table is taken as the input for the above three methods, and the frequencies for the first 20 orders of vibration are obtained (see Fig. 5(a) and (c)). According to the definition of frequency error in Eq. (14), the error in the frequencies can also be calculated (Fig. 5(b) and (d)).



(a) cable B17 (b) cable B17

Fig. 8 Error in determining the cable force for cables B01 and B17

Fig. 5(a) and (c) show that the solution of the cable vibration frequency obtained by adopting the numerical method is closer than the solution obtained from the vibration frequency formula to the FEM solution. The frequencies obtained using the numerical method discussed in this paper are lower than the FEM frequencies. Therefore, if these errors are estimated accurately, the frequency obtained employing the numerical method can be simply modified through subtraction. Fig. 7(b) and (d) show that the accuracy of the numerical method is better than that of the explicit formula. In the case of the B01 cable, the frequency errors for 2nd- to 20th-order vibrations resulting from the explicit formula vary from 4.2% to 4.6%, while the error arising from using the numerical method only fluctuates from 2.6% to 3.1%; thus, the maximum error difference between the two methods is 1.5%. In the case of the B17 cable, the maximum error difference is only 0.8%. The error in the fundamental frequency can be twice that in the frequencies of other orders when using the explicit formula, but the error in the fundamental frequency obtained using the numerical method is only 2%-3%, much smaller than that for the explicit formula method.

5.2 Frequency is provided to calculate the cable force

In the reverse process, when the measured vibration frequencies of the two cables are the input for the explicit formula and the numerical method, the cable force can be determined. According to the definition of cable force error in Eq. (15), the errors in cable force are also calculated and shown in Fig. 6.

Fig. 6(a) and (b) show that the measurement accuracy of the numerical method is obviously higher than that of the explicit formula. When adopting frequencies of vibrations other than the fundamental frequency to estimate the cable force, for the short cable that is hardly affected by sag, the measurement accuracy of the numerical method is 3%-5%, better than that of the explicit formula. For a cable of length 300 m, the numerical method provides about 1% more accuracy than does the explicit formula. When adopting the fundamental frequency, no matter for a long or short cable, the numerical method acquires impressively better accuracy for the cable force. Compared with using the explicit formula, the numerical method decreases the cable force error by 5%-10%. This can be interpreted as the numerical method taking the effects of sag, stiffness and the clamped support condition into account, and thus gaining higher cable-force accuracy.

It is found that all cable forces determined by the numerical method are higher than the real values. This one-sided-biased estimation can be corrected if the a priori error mechanism is

Table 4 Cable force modified by using the correction relationship

	1	2	3	4	5	6	7	2~6composite
B01 designated cable force (N)					3.01e6			
B01 determined cable force (N)	3.004	2.998	2.998	2.998	2.998	2.997	2.997	2.998
B01Error of cable (%)	0.14	0.08	-0.08	-0.08	-0.08	-0.09	-0.09	-0.09
B17 designated cable force (N)					5.46e6			
B17 determined cable force (N)	5.532	5.468	5.469	5.469	5.469	5.468	5.468	5.468
B17 Error of cable (%)	1.31	0.14	0.17	0.16	0.16	0.15	0.16	0.13

known. Using the empirical formulae (16) and (17), the modified cable forces and their errors for the two cables are presented in the table.

As shown in the table, after the modification of Eqs. (16) and (17), the cable identification results obtained with the numerical method are improved, and the error can be controlled to within 1.5%.

6. Conclusions

When cable sag, flexural stiffness and boundary conditions are considered, the frequency equation for a cable is a complex transcendental equation and cannot be solved explicitly. This paper showed that the implicit equation for the cable vibration frequency can be made explicit employing a numerical method and the cable vibration frequency and cable force can then be determined easily. Compared with the explicit frequency formula applicable to a taut string, the numerical method based on cable's implicit frequency can estimate the cable vibration frequency and cable force with greater accuracy. After further modification of the numerical method, the estimation accuracy becomes satisfactory.

Engineering applications showed that in the calculation of cable vibration frequency, the solution obtained by the numerical method is closer than that obtained by the explicit formula to that obtained by the FEM. The frequency obtained for each order of vibration is 1.5% higher than the result obtained using the explicit formula method, and the accuracy improvement for the first-order vibration is the most obvious. In the determination of a cable force, the proposed method produces errors for a short cable are 3% less than those produced by the explicit formula. In the case of a long cable, when frequencies of second- to seventh-order vibrations are adopted, the calculation accuracy of the cable force is about 1% better. The sag has a greater effect on the accuracy of cable-force determination when the fundamental frequency is used.

The proposed numerical method for determining the cable force is applicable when λ^2 is less than 42. This is the case for ordinary engineering applications because the cable commonly used for a cable-stayed bridge meets this limitation. Reasonable calculation accuracy for the cable force can then be achieved using the frequencies of vibrations of first to seventh order. In the case of cables for which λ^2 is greater than 42, it is better to determine a cable force according to the fundamental frequency.

FEM model updating method can be used to determine the cable force. In that situation, the parameters plus cable force are underdetermined and the difference between the measured mode frequencies and their corresponding values calculated by FEM model are reduced step by step until a convergence value is reached. Then the final updated cable force is the desired solution. This approach is time-consuming processes because of many iteration computing are needed. The proposed numerical method of the implicit frequency equation, which is used to compute the cable force, can effectively reduce the computational cost relative to FEM updating, and can be applied to the real-time monitoring of the cable force.

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References

Irvine, H.M. (1991), *Cable Structures*, Dover Publications, New York

Kim, B.H., Park, T., Shin, H.Y. and Yoon, T.Y. (2007), “A comparative study of the tension estimation methods for cable supported bridges”, *Int. J. Steel Struct.*, **7**(1), 77-84.

Nam, H. and Nghia, N.T. (2011), “Estimation of cable tension using measured natural frequencies”, *Proceedings of the 12th East Asia-Pacific Conference on Structural Engineering and Construction (EASEC)*, Hong Kong.

Geier, R.G., Roeck, D. and Flesch, R. (2006) “Accurate cable force determination using ambient vibration measurements”, *Struct. Infrastr. Eng.*, **2**(1), 43-52.

Marcelo, A.C. and Carlos, A.P. (2008), “Determination of the axial force on stay cables accounting for their bending stiffness and rotational end restraints by free vibration tests”, *J. Sound Vib.*, **317**(1-2), 127-141.

Choi, D.H. and Park, W.S. (2011), “Tension force estimation of extradosed bridge cables oscillating nonlinearly under gravity effects”, *Int. J. Steel Struct.*, **11**(3), 383-394.

Mehrabi, A.B. and Tabatabai, H. (1998), “Unified finite difference formulation for free vibration of cables”, *J. Struct. Eng., ASCE*, **124**(11), 1313-1322.

Wang, R., Gan, Q., Huang, Y. and Ma, H. (2011), “Estimation of tension in cables with intermediate elastic supports using finite-element method”, *J. Bridge Eng.*, **16**(5), 675-678.

Zui, H., Shinke, T. and Namita, Y.H. (1996), “Practical formulas for estimation of cable tension by vibration method”, *J. Struct. Eng., ASCE*, **122**(6), 651-656.

Ren, W.X., Chen, G. and Hu, W.H. (2005), “Empirical formulas to estimate cable tension by cable fundamental frequency”, *Struct. Eng. Mech.*, **20**(3), 363-380.

Fang, Z. and Wang, J. (2012), “Practical formula for cable tension estimation by vibration method”, *J. Bridge Eng.*, **17**(1), 161-164.

Chen, C.C., Wu, W.H., Huang, C.H. and Lai, G.> (2013), “Determination of stay CABLE FORCE based on effective vibration length accurately estimated from multiple measurements”, *Smart Struct. Syst.*, **11**(4), 411-433.

Kim, B.H. and Park, T. (2007). “Estimation of cable tension force using the frequency-based system identification method”, *J. Sound Vib.*, **304**(3-5), 660-676.

Zhang, P. (2010), “Cable force identification based on corrected numerical resolution of cable frequency functions”, Tongji University Master's Degree Thesis.,12-14. (in Chinese)

Ricciardi, G. and Saitta, F. (2008), “A continuous vibration analysis model for cables with sag and bending stiffness”, *Eng. Struct.*, **30**(5), 1459-1472.

Caetano, E.D.S. (2007). "Cable vibrations in cable-stayed bridges", Vol. 9, IABSE.

Yim, J.S., Wang, M.L., Shin, S.W., Yun, C.B., Jung, H.J., Kim, J.T. and Eem, S.H. (2013), "Field application of elasto-magnetic stress sensors for monitoring of CABLE tension FORCE in CABLE-stayed bridges", *Smart Struct. Syst.*, **12**(3), 465-482.