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Equilibrium shape analysis of single layer structure by measure potential function

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Abstract. A unified theory is presented for the shape analysis of curved surface with a single layer structure composed by frame, membrane or shell. The shapes produced by the theory have no shear stress in elements, and the stress states in the whole shape are as uniform as possible under an ordinary load. The theory starts from defining an element potential function expressed by the measurement of the element length or the element area. Therefore, the shape analysis can produce various forms according to the definition of the potential function, and each of those form or the cable net form with the potential function of the second power of element length is simply gotten by the linear analysis. The form in tensile stress is mechanically equal to an isotropic tension form.

Key words: formfinding; shape analysis; tangent stiffness method; geometrically nonliner analysis.

1. Introduction

A shape analysis is to find a form in rational stress state suitable for the material composing a structure. In the experimental method, as well known, there are soap film forms for membrane structures (Otto 1960 and Otto, *et al.* 1982). There are forms for RC shell structures in compression only, which are gotten by reversing suspended formes in tension only (Isler 1993). They produced the original and practical forms suitable for the structure materials.

Simultaneously, the computational methods by mechanical models or mathematical models of the experiments are being published. For example, the farmers are Haug (1971), Ishii (1976), Barnes (1976), etc. Haug used the finite element method for a shape analysis of an isotropic tension form, and it seems that the computational method has established. On the other hand, if it is called the mechanical method that the mechanical equilibrium state in shape is clear, Ohmori (1988) is the latter. Ohmori solved the minimal surface area as a variation problem with additional conditions. The study gives only a design surface, but the mechanical equilibrium state is not clear.

The paper proposes a mechanical method freely defining a potential function of element itself by a measure of element. The method can not only treat an isotropic tension form and a compression shell form without bending moment, but also has potential ability to create an original form. In the paper, for example, an element potential function found by the theory

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is interesting for the forms by cable net and/or frames. The element potential is proportional to the second power of the element length. The analysis becomes linear, while many potentials defined for element bring the nonlinear analyses. As far as the stiffness matrix derived from the element potential is nonsingular, the solution form can surely exist and be easily gotten. When axial forces in a structure are tension only or compression only, the stiffness matrix is always nonsingular.

The method does not use any strain of element, which is generally used in the finite element method. Therefore, the paper does not call energy in element strain energy, but potential energy. The strain needs a non-stress shape or a shape of element as some standard, and it is impossible to express exactly the strain by the function of nodal displacment. On the other hand, the isotropic tension element as a mechanical model of soap film has no strain, and even if the area of element becomes zero, the element has surface tension. When an axial element has the axial force proportional the the element length, the non-stress length of element is zero. Nevertheless, using a non-stress shape disturbs forming an ideal stress state.

In the paper, if an element potential function is defined, differentiating the potential by the measurement of element derives the element force. the element force in the paper works at the element edge. A set of the element forces in an element is statically determined. The external force at node and the element force make the equilibrium equation. Differentiating the equilibrium equation by the nodal positions gives the tangent stiffness equation, and clearly separates the geometrical stiffness and the element stiffness. Goto (1983) applied the tangent stiffness method to the real structures, and showed getting exactly the large deformation. In the method, it is important to get the exact equilibrium equation, without using the relation between the strain and the nodal displacement. In the shape anlysis, defining the element potential function deduce to get the exact equilibrium equation.

2. Theory

The element potential function is defined by using a measurement such as the element length or the element area, as follows,

$$P=f(A) \tag{1}$$

where P is the element potential, and A is the element measurement. The element potential must not be dependent on the initially assumed form prior to the calculation, and the sum including the potential of external load must have a extremum.

The derivative of Eq. (1) by the measurements independent each other in determining the element shape is the element force expressed as,

$$\boldsymbol{N}=\frac{\partial P}{\partial \boldsymbol{l}} \tag{2}$$

where $\boldsymbol{N}$ is the vector consisting of the element forces in the measurement directions, and $\boldsymbol{l}$ is the vector consistinng of the element measurements.

Eq. (2) is the element force equation. Without defining the element potential, we can start from defining the element force equation. In the case, though the types of forms increase more widely, the tangent stiffness equation is not always symmetric.

The differential calculus in Eq. (2) is the vector, and expressed as,

$$\frac{\partial}{\partial \boldsymbol{l}}=\left(\frac{\partial}{\partial l_1}, \frac{\partial}{\partial l_2}, \cdots\cdots, \frac{\partial}{\partial l_m}\right)^T \tag{3}$$

where m is the number of measurements. The superscirpt means the transposition of the vector. If the element is axial, $\boldsymbol{l}$ is unique as the element length, and A itself. If the element is triangle, A is the area. $\boldsymbol{l}$ can have some sets of measurements in the triangle shape: the three lengths of sides, the three distances from the orthocentre to the apexes, etc.

Differentiating Eq. (1) by the universal coordinate in consideration of Eq. (2) and the external load gives the equilibrium equation.

$$\boldsymbol{U}=\frac{\partial \boldsymbol{l}^T}{\partial \boldsymbol{u}}\frac{\partial P}{\partial \boldsymbol{l}}=\boldsymbol{a}\boldsymbol{N} \tag{4}$$

where $\boldsymbol{U}$ and $\boldsymbol{u}$ are the nodal force vector and the nodal position vector, respectively. $\boldsymbol{a}$ is the transforming matrix and generally consists of the direction cosines of the measurement lines. When the nodal force vector has the potential, Eq. (4) expresses the state in an extremum of the total potential. If there is no nodal force, that is $\boldsymbol{U}=\boldsymbol{0}$, and the element potential is the element area, Eq. (4) is the equilibrium equation on the surface of the extreme minimum area. Eq. (4) is generally nonlinear with the nodal positions, except the case of the axial element force proportional to the element lenght.

Further, differentiating Eq. (4) by the universal coordinate gives the tangent stiffness equation.

$$d\boldsymbol{U}=d\boldsymbol{a}\boldsymbol{N}+\boldsymbol{a}d\boldsymbol{N} \tag{5}$$

The minute displacement theory uses the second term only.

The first term becomes,

$$d\boldsymbol{a}\boldsymbol{N}=\left\{\frac{\partial}{\partial \boldsymbol{u}}(\boldsymbol{l}^T\boldsymbol{N})\frac{\partial}{\partial \boldsymbol{u}^T}\right\}d\boldsymbol{u}=\boldsymbol{K}_G\, d\boldsymbol{u} \tag{6}$$

In Eq. (6), $\boldsymbol{N}$ is constant in differentiating by the nodal position. Therefore, $\boldsymbol{K}_G$ expresses the geometrical stiffness induced by changing the direction of the element forces in the universal coordinate. $d\boldsymbol{u}$ is the nodal displacement by the increment of the nodal forces.

The increment of the element force becomes,

$$d\boldsymbol{N}=\left\{\frac{\partial}{\partial \boldsymbol{l}}\, P\frac{\partial}{\partial \boldsymbol{l}^T}\right\}\, d\boldsymbol{l}=\boldsymbol{k}d\boldsymbol{l} \tag{7}$$

The increment of the element measurement becomes,

$$d\boldsymbol{l}=\frac{\partial \boldsymbol{l}}{\partial \boldsymbol{u}^T}\, d\boldsymbol{u}=\boldsymbol{a}^T\, d\boldsymbol{u} \tag{8}$$

Substituting Eqs. (7) and (8) into the second term in Eq. (5) gives the stiffness related to the element itself.

$$\boldsymbol{a}d\boldsymbol{N}=\boldsymbol{a}\boldsymbol{k}\boldsymbol{a}^T\, d\boldsymbol{u}=\boldsymbol{K}_o d\boldsymbol{u} \tag{9}$$

Since $\boldsymbol{k}$ is the element stiffness, $\boldsymbol{K}_o$ expresses the stiffness of the element itself.

Substituing Eqs. (6) and (9) into Eq. (5), finally, the tangent stiffness equation is expressed as,

$$dU=(K_G+K_0)\ du \tag{10}$$

When the equilibrium equation is expressed by the element forces without including the nodal displacements, the differential can separate the geometrical stiffness and the stiffness of the element itself. One of the authors has already presented the advantages of this separation in the geometrically nonlinear structure analysis. Namely, the geometrical stiffness is independent of the element stiffness, as shown in Eq. (6). The separation is not clear in the finite element method, so that the geometrical stiffness also is not clearly understood. In the shape analysis, the separation typically shows the advantage, that the change of the element potential or the attachment of a real structural material affects only changing Eq. (2) and Eq. (7).

The solution fulfilling Eq. (4) is got by Newton-Raphson's method with the tangent stiffness equation of Eq. (10). Namely, since an initially assumed shape does not fulfill Eq. (4), the unbalanced nodal force vector in the shape is,

$$\Delta U=U-aN \tag{11}$$

In Eq. (10), changing the increment nodal force to the unbalanced force in Eq. (11), and solving Eq. (10) gives the nodal displacements from the present shape. Adding the nodal displacement to the present nodal position renews the nodal positions with the smaller unbalanced force than the previous shape. If iterating this procedure until the unbalanced forces become less than the allowable value, the solution shape is produced.

3. Example of formulation for 1D element

In one dimensional element, the element potential can be defined by various types of function. As an example, the paper defines the potential function as,

$$P=Cl^n \tag{12}$$

where l is the element length. C and n are the constants.

The axial force and the equilibrium equation are given by Eq. (13) and Eq. (14), respectively.

$$N=nCl^{n-1} \tag{13}$$

$$\begin{Bmatrix} U_i \\ U_j \end{Bmatrix}=\begin{Bmatrix} -a \\ a \end{Bmatrix} N \tag{14}$$

where U_i and U_j are the nodal force at both ends of the element, and N is the axial force. In Eq. (14), a is the direction cosiune vector of the axial element, as following,

$$a=(u_j-u_i)/l \tag{15}$$

where u_i and u_j are the nodal position of both ends of the element.

Eq. (13) shows that if $n=1$, the axial force is constant, and the element has no rigidity. Therefore, when the axial force is fixed, the element potential is proportional to the element length, and the equilibrium state is the minimum sum of element lengths in the same constant of the all elements. If $n>1$, the nonstress length of element is zero. Therefore, the equilibrium shape composed by the elements is independent of initially assumed shapes.

The sign of C distinguishes tension or compression from the axial forces of elements. However, the sign has no influence on the convergent process of the solution in the shape analysis. Namely, when the all elements are compression, the sum of the element potential must be the maximum. The shape is in unstable equilibrium, but the solution is surely obtained.

From Eq. (14), the tangent stiffness equation is,

$$\begin{Bmatrix} \delta U_i \\ \delta U_j \end{Bmatrix} = \left\{ \frac{N}{l} \begin{bmatrix} e - aa^T & -e + aa^T \\ -e + aa^T & e - aa^T \end{bmatrix} + n(n-1)Cl^{n-2} \begin{bmatrix} aa^T & -aa^T \\ -aa^T & aa^T \end{bmatrix} \right\} \begin{Bmatrix} \delta u_i \\ \delta u_j \end{Bmatrix} \tag{16}$$

$$= nCl^{n-2} \begin{bmatrix} e + (n-2)\, aa^T & -e - (n-2)\, aa^T \\ -e - (n-2)\, aa^T & e + (n-2)\, aa^T \end{bmatrix} \begin{Bmatrix} \delta u_i \\ \delta u_j \end{Bmatrix} \tag{17}$$

where $\boldsymbol{e}$ is the unit matrix of 3×3.

In Eq. (16), the first term is the geometrical stiffness, and the second term is the stiffness of the element itself. In order to reshape the form by a real material, the second term in the equation changes the real stiffness. Furthermore, Eq. (16) clearly shows the effective direction of the geometrical stiffness. In the universal coordinate, taking the component in the direction of $\boldsymbol{a}$ from the nodal force of $\boldsymbol{U}$, the remainder is $(\boldsymbol{e} - \boldsymbol{a}\boldsymbol{a}^T)\boldsymbol{U}$. Therefore, the geometrical stiffness of N/l works in only the plane perpendicular to the element direction of $\boldsymbol{a}$.

The tangent stiffness in Eq. (17) shows that, if $n=2$, the tangent stiffness equation becomes linear. The analysis is linear, while the nodal displacement from the initially assumed shape is really large. The axial force is proportional to the element length. In the shape determined by only the boundary condition without external load, the sum of the second power of each element length is minimized. Therefore, the lengths in the shape are more uniform than the shape having the minimum sum of element lengths.

Furthermore, in the case of $n=2$, if C in all element is the same value and each node joins the same number of elements, such as a honeycombed net or a latticed net, the geometrical stiffness at each node is constant. This is because $N/l = C$ in Eq. (16). Particularly, in the honeycombed net, since each node connects three elements, and the three elements in equilibrium make the plane, the geometrical stiffness at each node is $3C$ in the perpendicular to the plane. The geometrical stiffness of an isotropic tension form has the identical value on all over the surface. Therefore, the mechanical property is the same between the net form with the potential of $n=2$ and the isotropic tension form.

4. Example of formulation for 2D element

In a triangle plane element as a two dimensional element, the paper defines the potential function, as follows,

$$P = CA^n \tag{18}$$

where A is the area of the triangle element, and C and n are the constants.

There are some sets of measurements that determine the element area. The paper adopts the three lengths of sides, as shown in Fig. 1. Differentiating the element area by the side length, l_j, gives,

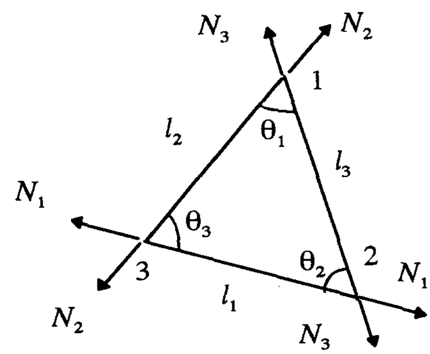

Fig. 1 Element force along side.

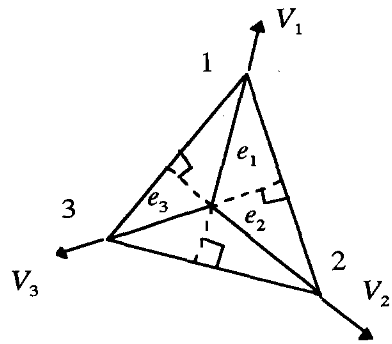

Fig. 2 Element force in perpendicular.

$$\frac{\partial A}{\partial l_j} = R\cos\theta_j = \frac{e_j}{2} \tag{19}$$

where R is the radius of the circumscribed circle of the triangle element. θ_j is the angle at the apex, and e_j is the distance from the orthocenter to the apex.

Differentiating Eq. (18) by the side length gives the element force in the side direction.

$$N_j = \frac{\partial P}{\partial l_j} = nCA^{n-1}R\cos\theta_j = nCA^{n-1}\frac{e_j}{2} \tag{20}$$

Incidentally, differentiating Eq. (18) by the distance of e_j gives the element force of V_j in Fig. 2.

$$V_j = \frac{\partial P}{\partial e_j} = nCA^{n-1}\frac{\partial A}{\partial e_j} = nCA^{n-1}\frac{l_j}{2} \tag{21}$$

Since V_j is proportional to the opposite side length, as shown Fig. 2, V_1, V_2, and V_3 make the force triangle. Therefore, the three element forces are in equilibrium at the orthocenter. In the theory, it is important that the selected measurements must be independent each other and determinative for the potential function. Therefore, the perpendicular distance is not adequate as the measurements adopted, but the distance of e_j is reasonable. It is interesting that the element measurements deducing the element forces in equilibrium are determinative on the element

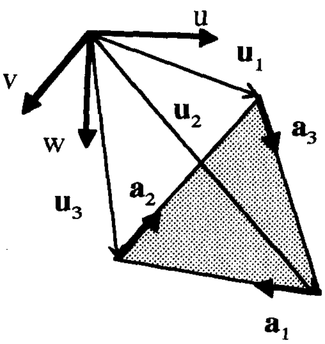

Fig. 3 Element position in universal coordinate.

area.

As well known, in an isotropic tension form, the equilibrium state is in the minimum surface area. It corresponds to $n=1$ in Eq. (18) in the theory. In the triangle element with isotropic tension, the element force in the side direceion is,

$$N_j=\tau\frac{e_j}{2} \tag{22}$$

where τ is the tension density. When C is τ, the differential of the element potential by the measurements expresses the element force.

If $n>1$ in the element potential function, the element in the nonstress state is the point, and the form is independent of initially assumed shapes prior to the calculation.

From Eq. (4), the equilibrium equation of the element in the universal coordinate is,

$$\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix}=\begin{bmatrix} \frac{\partial l_1}{\partial \boldsymbol{u}_1} & \frac{\partial l_2}{\partial \boldsymbol{u}_1} & \frac{\partial l_3}{\partial \boldsymbol{u}_1} \\ \frac{\partial l_1}{\partial \boldsymbol{u}_2} & \frac{\partial l_2}{\partial \boldsymbol{u}_2} & \frac{\partial l_3}{\partial \boldsymbol{u}_2} \\ \frac{\partial l_1}{\partial \boldsymbol{u}_3} & \frac{\partial l_2}{\partial \boldsymbol{u}_3} & \frac{\partial l_3}{\partial \boldsymbol{u}_3} \end{bmatrix}\begin{Bmatrix} \frac{\partial P}{\partial l_1} \\ \frac{\partial P}{\partial l_2} \\ \frac{\partial P}{\partial l_3} \end{Bmatrix}=\boldsymbol{a}N=\begin{bmatrix} \boldsymbol{0} & \boldsymbol{a}_2 & -\boldsymbol{a}_3 \\ -\boldsymbol{a}_1 & \boldsymbol{0} & \boldsymbol{a}_3 \\ \boldsymbol{a}_1 & -\boldsymbol{a}_2 & \boldsymbol{0} \end{bmatrix}\begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} \tag{23}$$

where $\boldsymbol{U}_j$ is the nodal force at the node connected with the apex and $\boldsymbol{a}_j$ is the direction cosine vector of the side, as shown in Fig. 3.

The tangent stiffness equation is obtained by differentiating Eq. (23) by the nodal position. The geometrical stiffness, $\boldsymbol{K}_G$, becomes,

$$\boldsymbol{K}_G d\boldsymbol{u}=\begin{bmatrix} \boldsymbol{k}_2+\boldsymbol{k}_3 & -\boldsymbol{k}_3 & -\boldsymbol{k}_2 \\ -\boldsymbol{k}_3 & \boldsymbol{k}_3+\boldsymbol{k}_1 & -\boldsymbol{k}_1 \\ -\boldsymbol{k}_2 & -\boldsymbol{k}_1 & \boldsymbol{k}_1+\boldsymbol{k}_2 \end{bmatrix}\begin{Bmatrix} d\boldsymbol{u}_1 \\ d\boldsymbol{u}_2 \\ d\boldsymbol{u}_3 \end{Bmatrix} \tag{24}$$

where

$$\boldsymbol{k}_j=\frac{N_j}{l_j}(\boldsymbol{e}-\boldsymbol{a}_j\boldsymbol{a}_j^T) \tag{25}$$

The stiffness of element itself becomes,

$$K_o du = a k_o a^T du \tag{26}$$

$$k_o = \begin{bmatrix} \frac{\partial^2 P}{\partial l_1^2} & \frac{\partial^2 P}{\partial l_1 \partial l_2} & \frac{\partial^2 P}{\partial l_3 \partial l_1} \\ \frac{\partial^2 P}{\partial l_1 \partial l_2} & \frac{\partial^2 P}{\partial l_2^2} & \frac{\partial^2 P}{\partial l_2 \partial l_3} \\ \frac{\partial^2 P}{\partial l_3 \partial l_1} & \frac{\partial^2 P}{\partial l_2 \partial l_3} & \frac{\partial^2 P}{\partial l_3^2} \end{bmatrix}$$

$$= nCA^{n-2}\ R\begin{bmatrix} n\beta_1^2+\beta_1\beta_2\beta_3-1 & (n-1)\beta_1\beta_2+\beta_3 & (n-1)\beta_3\beta_1+\beta_2 \\ (n-1)\beta_1\beta_2+\beta_3 & n\beta_2^2+\beta_1\beta_2\beta_3-1 & (n-1)\beta_2\beta_3+\beta_1 \\ (n-1)\beta_3\beta_1+\beta_2 & (n-1)\beta_2\beta_3+\beta_1 & n\beta_3^2+\beta_1\beta_2\beta_3-1 \end{bmatrix} \tag{27}$$

where $\beta_j = \cos\theta_j$.

If an element is triangle and the three element forces are in the side directions, Eq. (24) does not change, no matter what is the element stiffness. For example, even if a structure is composed by truss units of three axial elements, Eq. (24) can be used in the geometrically nonlinear analysis.

5. Computational examples

The element potential function of Eq. (12) in an axial element can produce various shapes of structure such as cable nets and latticed frames, by changing not only boundary and load conditions, but also the constants of C and n. Some interesting examples in those are shown in Fig. 4 to Fig. 8.

Figs. 4 and 5 are the equilibrium shapes by the vertical load of W=0.2N at the all of nodes. The boundary with rectangle perimeter of 8×12m is fixed. In Fig. 4, the element potential function of the axial element is proportional to the element length, and $P=-l$ Nm. On the other hand, in Fig. 5, the element potential function is proportional to the second power of the element length, and $P=-0.423\,l^2$ Nm. The two shapes are the same height at the crown.

Fig. 4 indicates more remarkable difference between element lengths composing the latticed frames than Fig. 5. The all of axial element are in compression, so that the total potential including the external load is the maximum in the equilibrium state. If the element forces are in tension, it is the minimum. However, in either element force, the sum of the absolute value of the element potential becomes as less as possible under the given external load. Therefore, Fig. 5, in which the sum of the second power of element length is minimum, consists of more

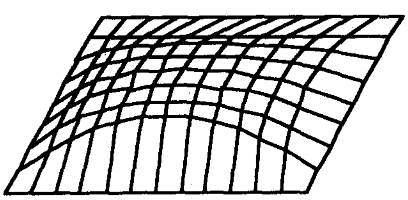

Fig. 4 Form with $P=-l$.

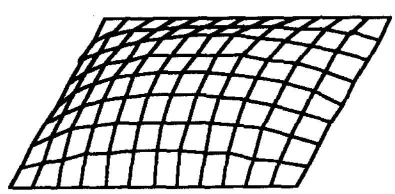

Fig. 5 Form with $P=-0.423l^2$.

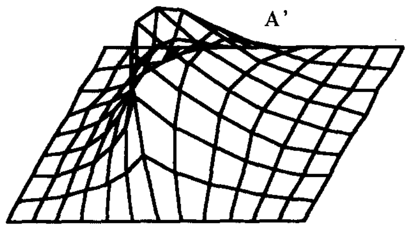

Fig. 6 Form by cable net and arch.

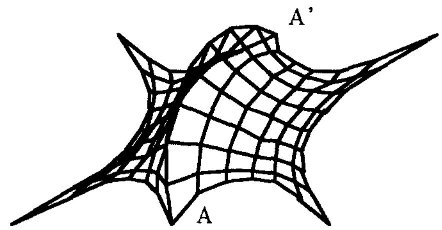

Fig. 7 Form by cable net and arch.

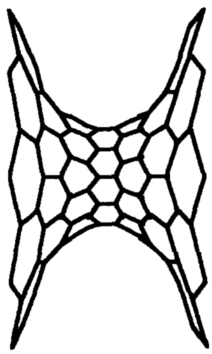

Fig. 8 Catenoid by cable net.

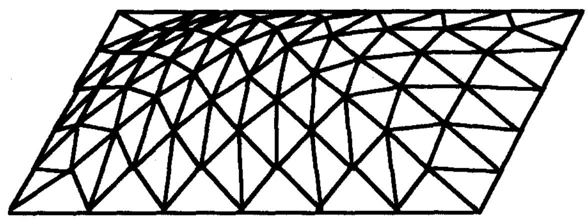

Fig. 9 Shell form in compression.

even element than Fig. 4 in which the sum of element length is minimum. If the power number is greater than 2, each element length is more uniform, but each axial force becomes more uneven.

The shapes of Fig. 5 to Fig. 8 consist of the axial element with the potential proportional to the second power of element length. If boundary conditions are given, each shape is derived by the linear analysis, no matter what initial shape is assumed. Therefore, there is no need of iterative calculation to get these forms. Cable nets modeled with extensivle rubber string are similar to these forms, because the element with the potential proportional to the second power of element length has no length in the non-stress state. The analysis can more easily produce the forms than modeling with rubber string.

Fig. 6 and Fig. 7 consist of the cable net and the arch. The axial elements from A to A' in the figures possess only compression with no bending moment. The perimeter in Fig. 6 is fixed. The element potential functions are $P=-3l^2$ and $P=l^2$. The vertically downward loads work at the all of nodes. In Fig. 7, the six points are fixed, and the element potential functions are $P=-4l^2$ and $P=l^2$. The vertically downward loads work the nodes except the part of arch. Considering the self-weight of arch member, in order to make the form of Fig. 7, the nodes in arch part have to resist bending moment.

The cable net form in Fig. 8 is produced by fixing the boundaries along the two circles. If the curved surface consists of isotropic tension element such as soap film, the form is called catenoid. The form has restriction of the boundary condition. The axial element with the potential of the second power of element length can make the similar form of the catenoid without the restriction. In Fig. 8, all of the elements have the same value of $N/l=1.0$. Therefore, the all of nodes have the same geometrical stiffness in the direction perpendicular to the plane consisting of three elements at each node. This mechanical property is equal to one of the isotropic tension form.

Fig. 9 consists of the triangle elements with the potential of $P=A^2$ Nm on the condition of the element weight of 0.4 N/m^2. The isotropic stress form can not be produced by the condition.

6. Conclusions

By defining freely the potential of element, various shapes with single layer structure were gotten. The paper presented the power function of the element length or the element area, as the examples of formulation. The shapes have the mechanical and geometrical property according to the definition of the potential. Particularly, the second power of element length as the potential can produce the shape mechanically equal to the isotropic tension form. In membrane or shell structures, the potential with more than the second power of element area can produce the the shapes unrestricted by boundary and load conditions.

If defining more various types of potential function of element, rational and practical shapes can be produced.

The proposed method is based on that the element potential is a scalar invariable in any coordinate transformation, as well as the element measurments. On the other hand, many geometrically nonlinear analysis of structures depends on the defined coordinate. Therefore, in many cases, the solutions may not strictly fulfill the equilibrium equation. These problems are not significant in the presented method at all.

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