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Bypass, homotopy path and local iteration to compute the stability point

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Abstract. In nonlinear finite element stability analysis of structures, the foremost necessary procedure is the computation to precisely locate a singular equilibrium point, at which the instability occurs. The present study describes global and local procedures for the computation of stability points including bifurcation points and limit points. The starting point, at which the procedure will be initiated, may be close to or arbitrarily far away from the target point. It may also be an equilibrium point or non-equilibrium point. Apart from the usual equilibrium path, bypass and homotopy path are proposed as the global path to the stability point. A local iterative method is necessary, when it is inspected that the computed path point is sufficiently close to the stability point.

Key words: stability; buckling; bifurcation; path-tracing; pinpointing; path-switching; nonlinear solution methods; local iteration

1. Introduction

Path-tracing, pinpointing and path-switching are indispensable in computational stability theory (Fujii and Choong 1992, Fujii and Okazawa 1997, Fujii and Ramm 1995). Path-tracing procedures for the equilibrium path, as represented by arc-length control, are well established. The equilibrium path is, however, not only the global path, which leads to the stability point. It is also possible to introduce a bypass and homotopy path to guide the solution from an arbitrary point to the target. In stability analysis, the precise computation of buckling points is a primary concern too. Bisection or bracketing technique requires interactive operation and is not efficient in finite element codes. The pinpointing technique based upon Newton-Raphson iteration seems to be a more sophisticated and innovative procedure in computational practice. For bifurcation, the branching predictors are briefly described for path-switchig. A simple two-dimensional toggle frame with 8 singular equilibrium points is presented to test the proposed global and local procedures.

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2. Stiffness matrix and eigenpairs

In finite displacement theory of nonlinear elastic structures subject to conservative loads p $\boldsymbol{e}$, the nonlinear equilibrium equations may be written in the form

$$\boldsymbol{E}(\boldsymbol{u},\ p)=\boldsymbol{0} \tag{1}$$

with

$$\boldsymbol{E}(\boldsymbol{u},\ p)=\boldsymbol{R}(\boldsymbol{u})-p\boldsymbol{e} \tag{2}$$

$\boldsymbol{E}$, $\boldsymbol{R}$, $\boldsymbol{u}$, p and $\boldsymbol{e}$ represent out-of-balance load vector, inner resistance, nodal displacements (N degrees of freedom), load parameter and reference load vector respectively. The tangent stiffness matrix $\boldsymbol{K}$ defined by

$$\boldsymbol{K}=\frac{d\boldsymbol{R}}{d\boldsymbol{u}} \tag{3}$$

is dependent only upon the nodal displacements $\boldsymbol{u}$. The eigenpair $(\lambda_j,\ \theta_j)$ such that

$$\boldsymbol{K}\theta_j=\lambda_j\theta_j \qquad (j=1,\ 2,\ \ldots,\ N) \tag{4}$$

and

$$|\theta_j|=1 \qquad (j=1,\ 2,\ \ldots,\ N) \tag{5}$$

will provide the most useful information on the stability behavior of the nonlinear structure. The spectral decomposition of $\boldsymbol{K}$ is

$$\boldsymbol{K}=\sum_{j=1}^{N}\lambda_j\theta_j\theta_j^T \tag{6}$$

and when $\boldsymbol{K}$ is invertible, it holds that

$$\boldsymbol{K}^{-1}=\sum_{j=1}^{N}\frac{1}{\lambda_j}\theta_j\theta_j^T \tag{7}$$

In the vicinity of a singular point, at which the critical eigenpair $(\lambda_s,\ \theta_s)$ will be such that,

$$\lambda_s=0 \tag{8}$$

$$\boldsymbol{K}\theta_s=\boldsymbol{0} \tag{9}$$

Eq. (7) become approximately,

$$\boldsymbol{K}^{-1}\cong\frac{1}{\lambda_s}\theta_s\theta_s^T \tag{10}$$

so that in general all columns of $\boldsymbol{K}^{-1}$ is approximately parallel to the critical eigenvector θ_s. For computation of the critical eigenvector θ_s, therefore, it is only necessary to multiply both sides of Eq. (10) from the right with an arbitrary vector which is not orthogonal to θ_s. Noguchi and Hisada (1994) discovered that the displacement corrector $\delta\mathbf{u}_0$ such that

$$\boldsymbol{K}\delta\boldsymbol{u}_0=-\boldsymbol{E} \tag{11}$$

at the last iterative step ($E \rightarrow 0$) to a regular equilibrium point close to a bifurcation point is nearly in the direction of the critical eigenvector θ_s. The displacement corrector δu_0 was then scaled ("scaled corrector") and successfully used as the branching predictor to compute an equilibrium point on the bifurcation path. In any way, it requires less computiational effort to evaluate the critical eigenvector θ_s near the singular equilibrium point than at a regular equilibrium point.

3. Global paths to the stability point

The key idea in the background of the globally nonlinear solution procedure is to define a path connecting the starting point and the target (Fujii and Okazawa 1997, Fujii and Ramm 1995). The successive computation of path points will guide the solution to the target and provide a reliable initial solution required for the succeeding pinpointing iteration.

The simplest idea to come close to the stability point is to track the equilibrium path (Fig. 1) and this has been most frequently done in stability analysis. Two more optional global paths, namely bypass and homotopy path, will be proposed in the present study.

The bypass starting from a regular equilibrium point to the target singular equilibrium point is defined by

$$\boldsymbol{E}(\boldsymbol{u},\ p)-(q-q^2)\boldsymbol{f}=\boldsymbol{0} \tag{12}$$

and

$$\lambda-(1-q^2)\ \lambda_A=0 \tag{13}$$

with a disturbance parameter q associated with a load vector $\boldsymbol{f}$. λ is the observed eigenvalue, which will be critical at the target. λ_A is the initial value of λ at the starting point A with $q=0$ and

$$\boldsymbol{K}_A\theta_A=\lambda_A\theta_A \tag{14}$$

The corresponding eigenvector θ_A may be substituted into the load vector $\boldsymbol{f}$ at the starting point. While tracing the bypass, the change in the magnitude of q should be examined. When $q=1$,

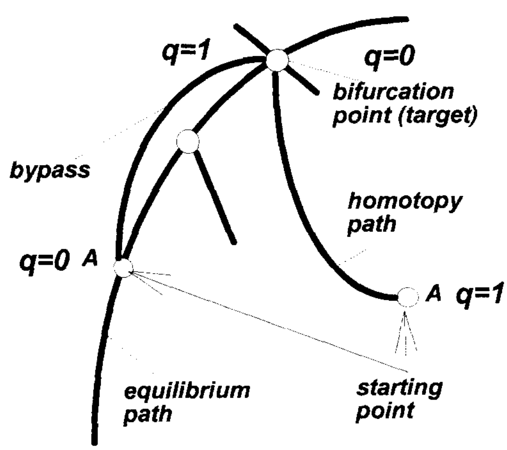


Fig. 1 Paths to the stability point.

the singular equilibrium point is reached, at which the observed eigenpair will be critical (Fig. 1) and local Newton-Raphson iteration must be initiated.

To trace the bypass, the predictor ($\boldsymbol{du}$, dp, dq) may be computed from the linearized path equations,

$$\boldsymbol{Kdu} - dp\,\boldsymbol{e} - dq(1-2q)\,\boldsymbol{f} = \boldsymbol{0} \tag{15}$$

$$d\lambda - dq(-2q)\ \lambda_A = 0 \tag{16}$$

$$\boldsymbol{du}^T\ \boldsymbol{du} + dp^2 + dq^2 = \Delta^2 \tag{17}$$

with a step-size constraint Δ. In the corrector steps,

$$\boldsymbol{K\delta u} - \delta p\boldsymbol{e} - \delta q\ (1-2q)\boldsymbol{f} = -\{\boldsymbol{E} - (q-q^2)\boldsymbol{f}\} \tag{18}$$

$$\delta\lambda - \delta\boldsymbol{q}(-2\boldsymbol{q})\lambda_A = -\{\lambda - (\boldsymbol{1}-\boldsymbol{q}^2)\lambda_A\} \tag{19}$$

$$\boldsymbol{du}^T\ \boldsymbol{\delta u} + \boldsymbol{dp\delta p} + \boldsymbol{dq\delta q} = 0 \tag{20}$$

will be solved for ($\boldsymbol{\delta u}$, δp, δq). For each predictor/corrector step to trace the bypass, the eigenpair (λ, θ) may be updated by Rayleigh quotient iteration and eigenvalue sensitivity $\delta\lambda$ computed from Eq. (40).

One more optional path to the stability point is a homotopy path defined by

$$\boldsymbol{E}(\boldsymbol{u},\ p) - q\boldsymbol{E}_A = \boldsymbol{0} \tag{21}$$

and

$$\lambda - \boldsymbol{q}\lambda_A = \boldsymbol{0} \tag{22}$$

where $\boldsymbol{E}_A$ is the out-of-balance load vector at the starting A (non-equilibrium point) (Fig. 1).

$$\boldsymbol{E}_A = \boldsymbol{E}(\boldsymbol{u}_A,\ p_A) \tag{23}$$

Both starting point A with $q=1$ and target with $q=0$ are on the defined homotopy path, so that by tracing the homotopy path the stability point with $\lambda=0$ is attainable. The predictor ($\boldsymbol{du}$, dp, dq) may be determined from

$$\boldsymbol{Kdu} - dp\boldsymbol{e} - dq\boldsymbol{E}_A = 0 \tag{24}$$

$$d\lambda - \boldsymbol{dq}\lambda_A = 0 \tag{25}$$

$$\boldsymbol{du}^T\ \boldsymbol{du} + dp^2 + dq^2 = \Delta^2 \tag{26}$$

and the corrector ($\boldsymbol{\delta u}$, δp, δq) from

$$\boldsymbol{K\delta u} - \delta p\boldsymbol{e} - \delta q\boldsymbol{E}_A = -(\boldsymbol{E} - q\boldsymbol{E}_A) \tag{27}$$

$$\delta\lambda - \delta q\lambda_A = -(\lambda - q\lambda_A) \tag{28}$$

$$du^T\ \boldsymbol{\delta u} + \boldsymbol{dp\delta p} + \boldsymbol{dq\delta q} = \boldsymbol{0} \tag{29}$$

When the parameter q changes its sign on the homotopy path in the ($\boldsymbol{u}$, p, q)-space, the local pinpointing scheme will be applied to precisely compute the stability point, as described below.

The path-tracing scheme for bypass and homotopy path is almost the same as the arc-length control for the equilibrium path, except that there are two load parameters (p, q) and the eigenvalue constraint is included in the path equations.

4. Local iteration to compute the stability point

When the target is detected on the defined path, the local pinpointing procedure is necessary to solve

$$\boldsymbol{E}(\boldsymbol{u}, p)=\boldsymbol{0} \tag{30}$$

and

$$\lambda=0 \tag{31}$$

by Newton-Raphson iteration. The linearized stability equations are

$$\boldsymbol{K}\delta\boldsymbol{u}-\delta p\boldsymbol{e}=-\boldsymbol{E} \tag{32}$$

and

$$\delta\lambda=-\lambda \tag{33}$$

From Eq. (32)

$$\delta\boldsymbol{u}=\delta\boldsymbol{u}_E+\delta p\,\delta\boldsymbol{u}_e \tag{34}$$

with

$$\delta\boldsymbol{u}_E=-\boldsymbol{K}^{-1}\boldsymbol{E} \tag{35}$$

and

$$\delta\boldsymbol{u}_e=+\boldsymbol{K}^{-1}\boldsymbol{e} \tag{36}$$

the change $\Delta\boldsymbol{K}$ in $\boldsymbol{K}$ is

$$\Delta\boldsymbol{K}=\Delta\boldsymbol{K}_E+\delta p\,\Delta\boldsymbol{K}_e \tag{37}$$

with the finite difference appoximation

$$\Delta\boldsymbol{K}_E=\left(\frac{1}{\varepsilon_E}\right)\quad\{\boldsymbol{K}(\boldsymbol{u}+\varepsilon_E\delta\boldsymbol{u}_E)-\boldsymbol{K}(\boldsymbol{u})\} \tag{38}$$

and

$$\Delta\boldsymbol{K}_e=\left(\frac{1}{\varepsilon_e}\right)\quad\{\boldsymbol{K}(\boldsymbol{u}+\varepsilon_e\delta\boldsymbol{u}_e)-\boldsymbol{K}(\boldsymbol{u})\} \tag{39}$$

in which (ε_E, ε_e) are intervals in finite difference. A higher-order finite difference approximation may be used to improve the convergence behavior.

The eigenvalue will change due to $\Delta\boldsymbol{K}$ by

$$\delta\lambda=\delta\lambda_E+\delta p\,\delta\lambda \tag{40}$$

with

$$\delta\lambda_E = \theta^T \Delta \boldsymbol{K}_E \theta \tag{41}$$

$$\delta\lambda_e = \theta^T \Delta \boldsymbol{K}_e \theta \tag{42}$$

Substituting Eq. (40) into Eq. (33) and solving for δp, we obtain the following iterative solution.

$$\delta p = -\frac{\lambda + \delta\lambda_E}{\delta\lambda_e} \tag{43}$$

and

$$\delta \boldsymbol{u} = \delta \boldsymbol{u}_E + \delta p \delta \boldsymbol{u}_e \tag{44}$$

The iteration will be terminated, when the out-of-balance load vector $\boldsymbol{E}$ and the eigenvalue λ become practically zeros.

5. Branching predictor

At the bifurcation point $(\boldsymbol{u}_B, p_B)$ with the critical eigenpair (λ_s, θ_s), let $(d\boldsymbol{u}_I, dp_I)$ and $(d\boldsymbol{u}_{II}, dp_{II})$ be the tangent direction of the primary path (path I) and that of the bifurcation path (path II) respectively. The branching predictors (Fujii and Choong 1992, Fujii and Okazawa 1997, Fujii and Ramm 1995) are summarized in Fig. 2 and Table 1.

For symmetric bifurcation, the critical eigenvector exactly equals to the tangent vector of the bifurcation path and the determination of the indefinite magnitude c is reduced to a step size problem. This is true, however, only when the current point exactly coincides with the bifurcation point. The particular solution $d\boldsymbol{u}_p$ of the singular stiffness equations need not be computed for both symmetric and asymmetric bifurcation. This is due to the face that it can be approximately replaced by the incremental displacements δ such that

$$\boldsymbol{K}\delta = \boldsymbol{e} \tag{45}$$

for unit load increment $dp=1$ at a regular equilibrium point on the primary path I close to the bifurcation point (Fig. 2). For hill-top branching, $d\boldsymbol{u}_I$, δ and θ_s are all in a same direction and we have to compute the particular solution $d\boldsymbol{u}_p$ from

$$\widetilde{\boldsymbol{K}} d\boldsymbol{u}_p = dp\boldsymbol{e} \tag{46}$$

with the stabilized stiffness marix

$$\widetilde{\boldsymbol{K}} = \boldsymbol{K} + \lambda_s^* \ \theta_s \theta_s^T \tag{47}$$

The specified eigenvalue λ_s^* will not affect the solution vector $d\boldsymbol{u}_p$ due to

$$dp\boldsymbol{e}^T \ \theta_s = 0 \tag{48}$$

For hill-top branching in the direction orthogonal to the primary path direction, it holds that

$$d\boldsymbol{u}_I^T d\boldsymbol{u}_{II} + 0 \times dp_{II} = 0 \tag{49}$$

so that only the particular solution $d\boldsymbol{u}_p$ is necessary for path-switching.

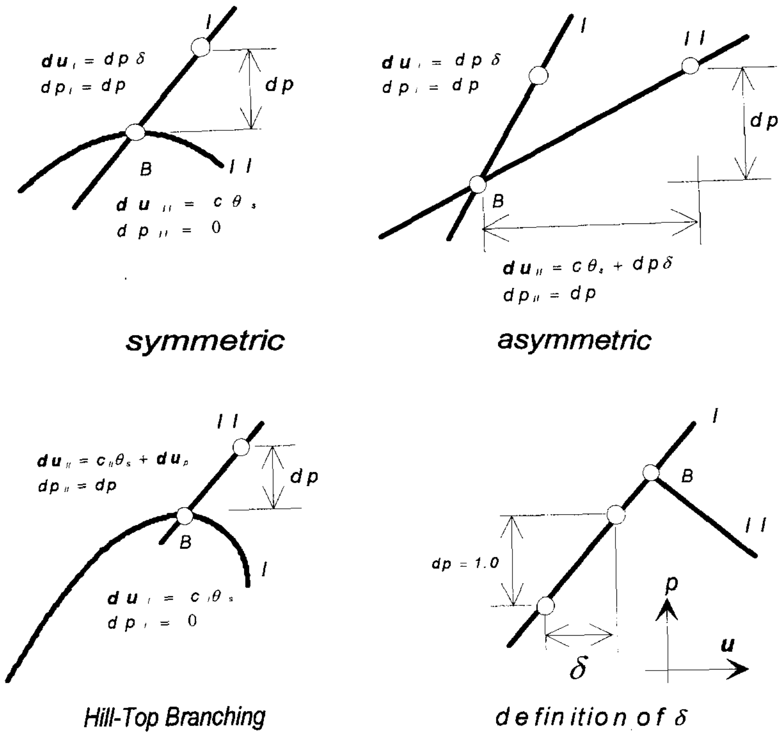


Fig. 2 Branching predictor.

Table 1 Tangent vector for the equilibrium branches at the bifurcation point

Type of bifurcation		Symmetric	Asymmetric	Hill-Top general	Hill-Top orthogonal
Primary path I	stiffness equations & solutions	$K\ du_I = dpe$ $du_I = dp\delta$ $dp_I = dp$	$K\ du_I = dpe$ $du_I = dp\delta$ $dp_I = dp$	$K\ du_I = 0$ $du_I = c_I\theta_s$ $dp_I = 0$	$K\ du_I = 0$ $du_I = c\theta_s$ $dp_I = 0$
Bifurcation path II	stiffness equations & solutions	$K\ du_{II} = 0$ $du_{II} = c\theta_s$ $dp_{II} = 0$	$K\ du_{II} = dpe$ $du_{II} = c\theta_s + dp\delta$ $dp_{II} = dp$	$K\ du_{II} = dpe$ $du_{II} = c_{II}\theta_s + du_p$ $dp_{II} = dp$	$K\ du_{II} = dpe$ $du_{II} = du_p$ $dp_{II} = dp$

6. Example

A series of bifurcation examples including multiple bifurcation and hill-top branching have been computed to test the procedures for the compution of stability points and path-switching (Fujii and Okazawa 1997, Fujii and Ramm 1995). Only one representative example is presented in this paper. The toggle frame in Fig. 3 serves as a bench model with six bifurcation points (BP1 through BP6) and two limit points (LP1 and LP2). The geometry and cross sectional data are as follows: L (half span)$=328.56$, H (height)$=38.6$, EA (axial stiffness)$=100\times84.16$, EI (flexural stiffness)$=100\times265.74$.

Ten two-dimensional beam elements were used in discretization (27 degrees-of-freeddom). Local pinpointing of all 8 singular points on the snapping-through primary path was successful except

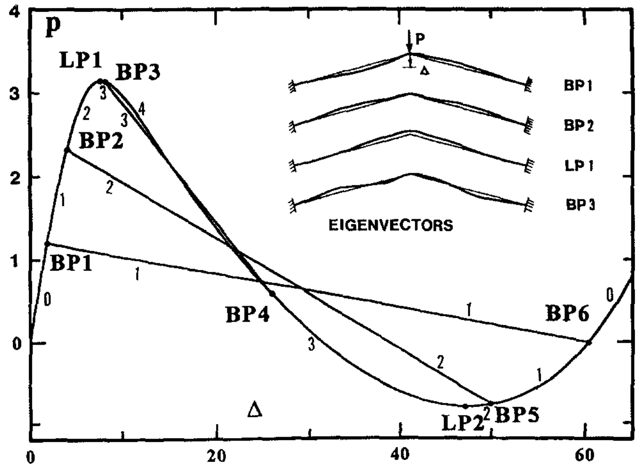


Fig. 3 Toggle frame and equilibrium path.

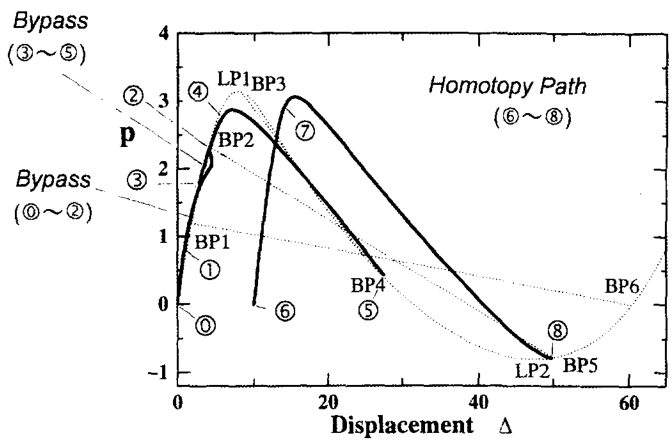
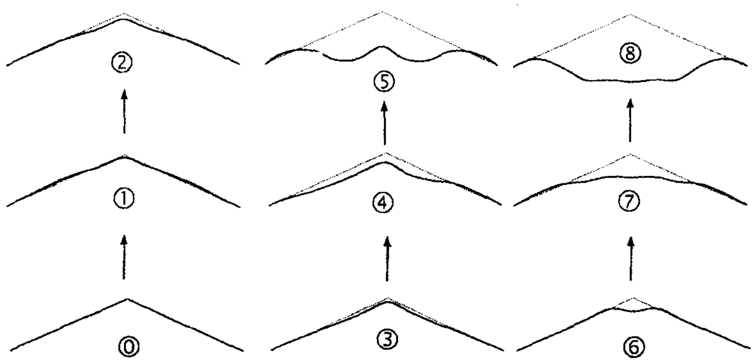


Fig. 4 Global path to the stability point.

that for the limit points the convergence was slow. Path-switching for symmetric bifurcation was nothing but a step-size determination, as long as the bifurcation points were precisely located. The proposed global paths, bypass and homotopy path, were then tested (Fig. 4) and the deformed frame configurations on these paths were depicted in Fig. 5. A total of three attempts were made:

The first attempt was to define a bypass at the unloaded point ◎ to move directly to ② (=BP2) (◎→①→②). The secone eigenvector θ_2 computed at ◎ was substituted into vector f in Eqs. (12). BP1 was not on this defined bypass. One more bypass (③→④→⑤) was defined in the second attempt with the eigenvector θ_4 (non-symmetric mode) computed at a primary path point ③ between BP1 and BP2 to attain ⑤ (=BP4). BP3 was not attainable before reaching BP4. Fig. 5 shows that the deformation at the bypass point ④ is not symmetric due to the non-symmetric mode of $f=\theta_4$. The frame deformation was, however, symmetric at ③ and ⑤ at which the effect of the load disturbance $q\ f$ disappeared.

The last attempt was to aim at BP2 from a non-equilibrium point ⑥ at which the frame



Bypass (⓪～②) *Bypass* (③～⑤) *Homotopy Path* (⑥～⑧)

Fig. 5 Configuration on the global path.

deformation was such that all the nodal displacements, but not those near the frame center, were zero (Fig. 5). The second eigenpair was used in the homotopy path Eqs. (21), (22). The defined homotopy path way was first directed toward BP2 and it turned round later unexpectedly to ⑧ (=BP5). The computed homotopy path (⑥→⑦→⑧) in the (p-Δ)-plot (Fig. 4) is qualitatively similar to the primary path for the reference load vector $\boldsymbol{e}$. This is due to the fact that the load mode represented by $\boldsymbol{E}_A$ at the starting point ⑥ is alike to $\boldsymbol{e}$.

7. Concluding remarks

For nonlinear stability analysis, global and local procedures to precisely compute stability points have been described. Except the usual equilibrium path, bypass and homotopy path may be now chosen in stability analysis. The starting point may be far away from the target and it may be an equilibrium point or a non-equilibrium point. Differently from the usual bisection or bracketing technique, the local pinpointing procedure can shoot the target directly by Newton-Rapshon iteration, in which the eigenvalue sensitivity formula is used. Path-switching worked well too. All bifurcation examples computed by the authors (Fujii and Choong 1992, Fujii and Okazawa 1994, Fujii and Ramm 1995, Okazawa and Fujii 1996, Fujii and Okazawa 1997) now have been nonlinear elastic. The future topic in computational stability theory seems to be bifurcation instability in elasto-plastic materials.

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