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Exact solutions of variable-arc-length elasticas under moment gradient

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Abstract. This paper deals with the bending problem of a variable-arc-length elastica under moment gradient. The variable arc-length arises from the fact that one end of the elastica is hinged while the other end portion is allowed to slide on a frictionless support that is fixed at a given horizontal distance from the hinged end. Based on the elastica theory, exact closed-form solution in the form of elliptic integrals are derived. The bending results show that there exists a maximum or a critical moment for given moment gradient parameters; whereby if the applied moment is less than this critical value, two equilibrium configurations are possible. One of them is stable while the other is unstable because a small disturbance will lead to beam motion.

Key words: elliptic-integrals; large deflections; variable-arc-length bars; beams; elasticas.

1. Introduction

In deep offshore engineering operations, a long vertical pipe or a marine riser stretching from the sea floor to sea surface may be considered as an elastica. In the analysis of such an elastica, the water depth may be considered as the span length and is assumed to be known while the total arc-length of the elastica in the displaced position may be unknown. Owing to the variableness of the arc-length, the determination of an equilibrium position of the elastica under a given loading condition is most challenging.

Elasticas with constant arc-lengths have been extensively researched and classical solutions under a variety of loading and boundary condition can be found in classical text books, (e.g., Love 1944, Frisch-Fay 1962, and Britvec 1973) and in research papers, (e.g., Barten 1944, Bisshop and Drucker 1945, Conway 1947, 1956, Schile and Sierakowski 1967, Lau 1982, Seide 1984, Bottega

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1991, and Navaee and Elling 1992, 1993). In contrast, a smaller number of papers have been published on elasticas with variable arc-lengths, (e.g., Chucheepsakul, *et al.* 1994, 1995, 1996 and Wang, *et al.* 1997). In almost all of these elastica problems, the formulation is based on the elastica theory for which exact closed-form solutions may be obtained by using the elliptic integral method. Recently, Chucheepsakul, *et al.* (1994) solved the large deflection of beams under moment gradient using both the elliptic integral method and the shooting-optimization method (Wang and Kittpornchai 1992). However, later studies (Chucheepsakul, *et al.* 1995, and Thepphitak 1995) on variable arc length elastica problems showed that there are two possible equilibrium configurations for a given loading condition. One of this configuration corresponds to a stable equilibrium while the other an unstable equilibrium. The latter implies that a slight perturbation will cause the elastica departs from its equilibrium state. In Chucheepsakul, *et al.* (1994), only the stable equilibrium state solutions were presented. For completeness, this paper presents the exact solutions for the unstable equilibrium state. Further, the former solutions given in Chucheepsakul, *et al.* (1994) contain some non-elliptic integral functions which will now converted into fully elliptic integral form for greater computational accuracy. Also, this study gives the maximum or critical moment values in which the elastica can withstand.

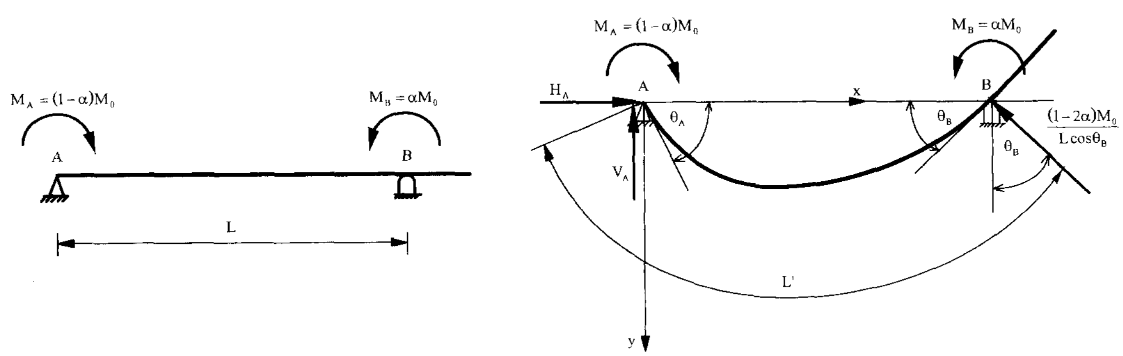
2. Elastica formulation

The equilibrium configuration of an elastica of fixed span length L with variable arc-length is shown in Fig. 1. It is hinged at support A and cantilevered over a frictionless support B. The beam is subjected to end moments $M_A=(1-\alpha)\ M_0$ and $M_B=\alpha M_0$, where α is the scaling parameter. When $\alpha=1$, the beam is subjected to only moment M_B. For $\alpha=0$, the beam is under moment M_A and when $\alpha=1/2$, the beam is under uniform moment along in fixed span L.

Consider a free body diagram of a segment of the deflected beam as shown in Fig. 2. The equilibrium of moments at point J gives

$$M=-(1-2\alpha)\left(\frac{M_0}{L}\right)\ x+(1-2\alpha)\left(\frac{M_0}{L}\right)\ \tan\theta_B y+(1-\alpha)\ M_0 \tag{1}$$

The moment-curvature relation and the geometric relations, are given by



(a) Undeflected shape (b) Deflected shape

Fig. 1 Variable-arc-length elasticas under moment gradient.

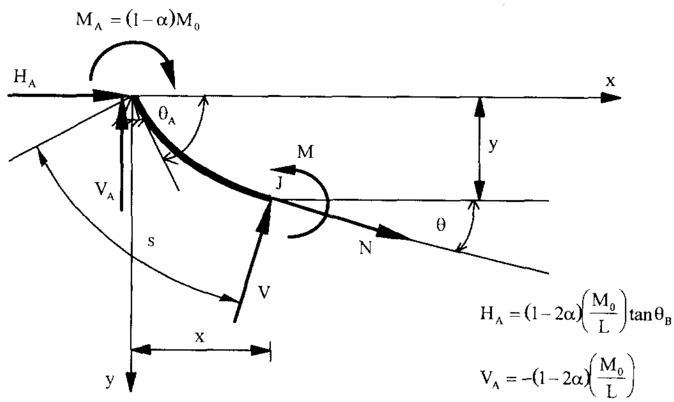


Fig. 3 Variations of the moment parameter, $\bar{M}$ with respect to the slopes θ_A and θ_B.

$$M = -EI\frac{d\theta}{ds};\ \frac{dx}{ds} = \cos\theta;\ \text{and}\ \frac{dy}{ds} = \sin\theta \tag{2a, b, c}$$

In view of Eq. (2) and some algebraic manipulations, Eq. (1) may be written as

$$\frac{EI}{2}\left(\frac{d\theta}{ds}\right)^2 = (1-2\alpha)\left(\frac{M_0}{L}\right)\sin\theta + (1-2\alpha)\left(\frac{M_0}{L}\right)\tan\theta_B\cos\theta + C \tag{3}$$

The integrating constant C in Eq.(3) can be evaluated by using the boundary condition at either A or at B. Using the boundary condition at A, where $\theta = \theta_A$ and $\frac{d\theta}{ds} = \frac{-(1-\alpha)M_0}{EI}$, one obtains

$$C = \frac{(1-\alpha)^2(M_0)^2}{2EI} - (1-2\alpha)\left(\frac{M_0}{L}\right)\sin\theta_A - (1-2\alpha)\left(\frac{M_0}{L}\right)\tan\theta_B\cos\theta_A \tag{4}$$

while the boundary condition at B, in which $\theta = \theta_B$ and $d\theta/ds = -\alpha M_0/EI$, gives

$$C = \frac{(\alpha M_0)^2}{2EI} \tag{5}$$

Eqs. (4) and (5) yields the following relation between θ_A and θ_B

$$\bar{M} = 2(\sin\theta_A + \tan\theta_B\cos\theta_A) \tag{6}$$

where $\bar{M} = M_0L/EI$. By substituting Eq. (5) into Eq. (3) and taking the square root of Eq. (3), one obtains the following curvature expression

$$\frac{d\theta}{ds} = -\left(\frac{1}{L}\right)\sqrt{2\bar{M}}\sqrt{\mu_1 + \mu_2\sin\theta + \mu_3\cos\theta} \tag{7}$$

where

$$\mu_1 = \alpha^2(\sin\theta_A + \tan\theta_B\cos\theta_A);\ \mu_2 = (1-2\alpha);\ \mu_3 = (1-2\alpha)\tan\theta_B. \tag{8a-c}$$

Combining Eqs. (2a), (2b), and Eq. (7), and after integration, yields the following expressions

$$\frac{s}{L}=\bar{s}=\int_{\theta_A}^{\theta}\frac{-\lambda d\theta}{\sqrt{\mu_1+\mu_2\sin\theta+\mu_3\cos\theta}} \tag{9}$$

$$\frac{x}{L}=\bar{x}=\int_{\theta_A}^{\theta}\frac{-\lambda\cos\theta d\theta}{\sqrt{\mu_1+\mu_2\sin\theta+\mu_3\cos\theta}} \tag{10}$$

$$\frac{y}{L}=\bar{y}=\int_{\theta_A}^{\theta}\frac{-\lambda\sin\theta d\theta}{\sqrt{\mu_1+\mu_2\sin\theta+\mu_3\cos\theta}} \tag{11}$$

where $\lambda=\frac{1}{\sqrt{2M}}$.

3. Exact elastica solutions

The elastica solutions of Eqs. (9)-(11) may be categorized into four cases, depending on the parameters μ_1, μ_2 and μ_3, as follows.

Case I: $\frac{\mu_1}{\sqrt{\mu_2^2+\mu_3^2}}<1$ and $\mu_2\leq 0$

$$\bar{s}=\begin{cases}\lambda\eta_1\{F(\Phi_1,\ k)-F(\Phi_2,\ k)\}, & \text{if } \theta\geq\gamma_1\\ \lambda\eta_1\{F(\Phi_1,\ k)+F(\Phi_2,\ k)\}, & \text{if } \theta<\gamma_1\end{cases} \tag{12}$$

$$\bar{x}=\begin{cases}\lambda\eta_2[2\{E(\Phi_1,\ k)-E(\Phi_2,\ k)\}-\{F(\Phi_1,\ k)-F(\Phi_2,\ k)\}+\eta_3\{\cos\Phi_1-\cos\Phi_2\}], & \text{if } \theta\geq\gamma_1\\ \lambda\eta_2[2\{E(\Phi_1,\ k)+E(\Phi_2,\ k)\}-\{F(\Phi_1,\ k)+F(\Phi_2,\ k)\}+\eta_3\{\cos\Phi_1-\cos\Phi_2\}], & \text{if } \theta<\gamma_1\end{cases} \tag{13}$$

$$\bar{y}=\begin{cases}\lambda\eta_4[2\{E(\Phi_1,\ k)-E(\Phi_2,\ k)\}-\{F(\Phi_1,\ k)-F(\Phi_2,\ k)\}-\eta_5\{\cos\Phi_1-\cos\Phi_2\}], & \text{if } \theta\geq\gamma_1\\ \lambda\eta_4[2\{E(\Phi_1,\ k)+E(\Phi_2,\ k)\}-\{F(\Phi_1,\ k)+F(\Phi_2,\ k)\}-\eta_5\{\cos\Phi_1-\cos\Phi_2\}], & \text{if } \theta<\gamma_1\end{cases} \tag{14}$$

where E and F are the elliptic integrals of the first and second kind (Byrd and Friedman 1971) respectively, and

$$\Phi_1=\sin^{-1}\sqrt{\frac{\mu_2^2+\mu_3^2-\mu_2\sin\theta_A-\mu_3\cos\theta_A}{\mu_1+\sqrt{\mu_2^2+\mu_3^2}}};\quad \Phi_2=\sin^{-1}\sqrt{\frac{\mu_2^2+\mu_3^2-\mu_2\sin\theta-\mu_3\cos\theta}{\mu_1+\sqrt{\mu_2^2+\mu_3^2}}}$$

$$k=\sqrt{\frac{\mu_1+\sqrt{\mu_2^2+\mu_3^2}}{2\sqrt{\mu_2^2+\mu_3^2}}};\quad \gamma_1=\sin^{-1}\left(\frac{\mu_2}{\sqrt{\mu_2^2+\mu_3^2}}\right);\quad \eta_1=\frac{\sqrt{2}}{(\mu_2^2+\mu_3^2)^{1/4}};\quad \eta_2=\frac{\sqrt{2}\,\mu_3}{(\mu_2^2+\mu_3^2)^{3/4}};$$

$$\eta_3=\frac{2k\mu_2}{\mu_3};\quad \eta_4=\frac{\sqrt{2}\,\mu_2}{(\mu_2^2+\mu_3^2)^{3/4}};\quad \eta_5=\frac{2k\mu_3}{\mu_2}. \tag{15a-i}$$

Case II: $\frac{\mu_1}{\sqrt{\mu_2^2+\mu_3^2}}<1$ and $\mu_2>0$

$$\bar{s}=\begin{cases}-\lambda\eta_1\{F(\Phi_1,\ k)-F(\Phi_2,\ k)\}, & \theta\leq\theta_A\leq\gamma_1\\ \lambda\eta_1\{F(\Phi_1,\ k)+F(\Phi_2,\ k)\}, & \theta\leq\gamma_1\leq\theta_A\\ \lambda\eta_1\{F(\Phi_1,\ k)-F(\Phi_2,\ k)\}, & \gamma_1\leq\theta\leq\theta_A\end{cases} \tag{16}$$

$$\bar{x}=\begin{cases}\lambda\eta_2[\{F(\Phi_1,\ k)-F(\Phi_2,\ k)\}-2\{E(\Phi_1,\ k)-E(\Phi_2,\ k)\}+\eta_3\{\cos\Phi_1-\cos\Phi_2\}], & \text{if } \theta\le\theta_A\le\gamma_1\\ \lambda\eta_2[2\{E(\Phi_1,\ k)+E(\Phi_2,\ k)\}-\{F(\Phi_1,\ k)+F(\Phi_2,\ k)\}+\eta_3\{\cos\Phi_1-\cos\Phi_2\}], & \text{if } \theta\le\gamma_1\le\theta_A\\ \lambda\eta_2[2\{E(\Phi_1,\ k)-E(\Phi_2,\ k)\}-\{F(\Phi_1,\ k)-F(\Phi_2,\ k)\}+\eta_3\{\cos\Phi_1-\cos\Phi_2\}], & \text{if } \gamma_1\le\theta\le\theta_A\end{cases}\tag{17}$$

$$\bar{y}=\begin{cases}\lambda\eta_4[\{F(\Phi_1,\ k)-F(\Phi_2,\ k)\}-2\{E(\Phi_1,\ k)-E(\Phi_2,\ k)\}-\eta_5\{\cos\Phi_1-\cos\Phi_2\}], & \text{if } \theta\le\theta_A\le\gamma_1\\ \lambda\eta_4[2\{E(\Phi_1,\ k)+E(\Phi_2,\ k)\}-\{F(\Phi_1,\ k)+F(\Phi_2,\ k)\}-\eta_5\{\cos\Phi_1-\cos\Phi_2\}], & \text{if } \theta\le\gamma_1\le\theta_A\\ \lambda\eta_4[2\{E(\Phi_1,\ k)-E(\Phi_2,\ k)\}-\{F(\Phi_1,\ k)-F(\Phi_2,\ k)\}-\eta_5\{\cos\Phi_1-\cos\Phi_2\}], & \text{if } \gamma_1\le\theta\le\theta_A\end{cases}\tag{18}$$

where Φ_1, Φ_2, k, γ_1, η_1, η_2, η_3, η_4 and η_5 are defined in Eqs. (15).

Case III: $\dfrac{\mu_1}{\sqrt{\mu_2^2+\mu_3^2}} < 1$ and $\mu_2 > 0$

$$\bar{s}=\begin{cases}\lambda\eta_1\{F(\Phi_1,\ k)-F(\Phi_2,\ k)\}, & \text{if } \theta\ge\gamma_1\\ \lambda\eta_1\{F(\Phi_1,\ k)+F(\Phi_2,\ k)\}, & \text{if } \theta<\gamma_1\end{cases}\tag{19}$$

$$\bar{x}=\begin{cases}\lambda\eta_2[\{E(\Phi_1,\ k)-E(\Phi_2,\ k)\}-\delta_2\{F(\Phi_1,\ k)-F(\Phi_2,\ k)\}\\ \quad+\eta_3\{\sqrt{1-k^2\sin^2\Phi_1}-\sqrt{1-k^2\sin^2\Phi_2}\}], & \text{if } \theta\ge\gamma_1\\ \lambda\eta_2[\{E(\Phi_1,\ k)-E(\Phi_2,\ k)\}-\delta_1\{F(\Phi_1,\ k)+F(\Phi_2,\ k)\}\\ \quad+\eta_3\{\sqrt{1-k^2\sin^2\Phi_1}-\sqrt{1-k^2\sin^2\Phi_2}\}], & \text{if } \theta<\gamma_1\end{cases}\tag{20}$$

$$\bar{y}=\begin{cases}\lambda\eta_4[\{E(\Phi_1,\ k)-E(\Phi_2,\ k)\}-\delta_2\{F(\Phi_1,\ k)-F(\Phi_2,\ k)\}\\ \quad-\eta_5\{\sqrt{1-k^2\sin^2\Phi_1}-\sqrt{1-k^2\sin^2\Phi_2}\}], & \text{if } \theta\ge\gamma_1\\ \lambda\eta_4[\{E(\Phi_1,\ k)-E(\Phi_2,\ k)\}-\delta_1\{F(\Phi_1,\ k)+F(\Phi_2,\ k)\}\\ \quad-\eta_5\{\sqrt{1-k^2\sin^2\Phi_1}-\sqrt{1-k^2\sin^2\Phi_2}\}], & \text{if } \theta<\gamma_1\end{cases}\tag{21}$$

where

$$\begin{aligned}&\Phi_1=\sin^{-1}\sqrt{\frac{\sqrt{\mu_2^2+\mu_3^2}-\mu_2\sin\theta_A-\mu_3\cos\theta_A}{2\sqrt{\mu_2^2+\mu_3^2}}};\\ &\Phi_2=\sin^{-1}\sqrt{\frac{\sqrt{\mu_2^2+\mu_3^2}-\mu_2\sin\theta-\mu_3\cos\theta}{2\sqrt{\mu_2^2+\mu_3^2}}}\\ &k=\sqrt{\frac{2\sqrt{\mu_2^2+\mu_3^2}}{\mu_1+\sqrt{\mu_2^2+\mu_3^2}}};\ \gamma_1=\sin^{-1}\left(\frac{\mu_2}{\sqrt{\mu_2^2+\mu_3^2}}\right);\ \delta_1=\frac{\mu_1}{\mu_1+\sqrt{\mu_2^2+\mu_3^2}};\\ &\eta_1=\frac{2}{\sqrt{\mu_1+\sqrt{\mu_2^2+\mu_3^2}}};\ \eta_2=\frac{2\mu_3\sqrt{\mu_1+\sqrt{\mu_2^2+\mu_3^2}}}{(\mu_2^2+\mu_3^2)}\quad \eta_3=\frac{\mu_2}{\mu_3};\\ &\eta_4=\frac{2\mu_2\sqrt{\mu_1+\sqrt{\mu_2^2+\mu_3^2}}}{(\mu_2^2+\mu_3^2)};\ \eta_5=\frac{\mu_3}{\mu_2}.\end{aligned}\tag{22a-i}$$

Case IV: $\dfrac{\mu_1}{\sqrt{\mu_2^2+\mu_3^2}} > 1$ and $\mu_2 > 0$

$$s=\begin{cases}-\lambda\eta_1\{F(\Phi_1,\ k)-F(\Phi_2,\ k)\},\ \theta\leq\theta_A\leq\gamma_1\\ \lambda\eta_1\{F(\Phi_1,\ k)+F(\Phi_2,\ k)\},\ \theta\leq\gamma_1\leq\theta_A\\ \lambda\eta_1\{F(\Phi_1,\ k)-F(\Phi_2,\ k)\},\ \gamma_1\leq\theta\leq\theta_A\end{cases}\tag{23}$$

$$\bar{x}=\begin{cases}\lambda\eta_2[\delta_1\{F(\Phi_1,\ k)-F(\Phi_2,\ k)\}-\{E(\Phi_1,\ k)-E(\Phi_2,\ k)\}\\ \quad+\eta_3\{\sqrt{1-k^2\sin^2\Phi_1}-\sqrt{1-k^2\sin^2\Phi_2}\}],\ \text{if}\ \theta\leq\theta_A\leq\gamma_1\\ \lambda\eta_2[\{E(\Phi_1,\ k)+E(\Phi_2,\ k)\}-\delta_1\{F(\Phi_1,\ k)+F(\Phi_2,\ k)\}\\ \quad+\eta_3\{\sqrt{1-k^2\sin^2\Phi_1}-\sqrt{1-k^2\sin^2\Phi_2}\}],\ \text{if}\ \theta\leq\gamma_1\leq\theta_A\\ \lambda\eta_2[\{E(\Phi_1,\ k)-E(\Phi_2,\ k)\}-\delta_1\{F(\Phi_1,\ k)-F(\Phi_2,\ k)\}\\ \quad+\eta_3\{\sqrt{1-k^2\sin^2\Phi_1}-\sqrt{1-k^2\sin^2\Phi_2}\}],\ \text{if}\ \gamma_1\leq\theta\leq\theta_A\end{cases}\tag{24}$$

$$\bar{y}=\begin{cases}\lambda\eta_4[\delta_1\{F(\Phi_1,\ k)-F(\Phi_2,\ k)\}-\{E(\Phi_1,\ k)-E(\Phi_2,\ k)\}\\ \quad-\eta_5\{\sqrt{1-k^2\sin^2\Phi_1}-\sqrt{1-k^2\sin^2\Phi_2}\}],\ \text{if}\ \theta\leq\theta_A\leq\gamma_1\\ \lambda\eta_4[\{E(\Phi_1,\ k)+E(\Phi_2,\ k)\}-\delta_1\{F(\Phi_1,\ k)+F(\Phi_2,\ k)\}\\ \quad+\eta_5\{\sqrt{1-k^2\sin^2\Phi_1}-\sqrt{1-k^2\sin^2\Phi_2}\}],\ \text{if}\ \theta\leq\gamma_1\leq\theta_A\\ \lambda\eta_4[\{E(\Phi_1,\ k)-E(\Phi_2,\ k)\}-\delta_1\{F(\Phi_1,\ k)-F(\Phi_2,\ k)\}\\ \quad-\eta_5\{\sqrt{1-k^2\sin^2\Phi_1}-\sqrt{1-k^2\sin^2\Phi_2}\}],\ \text{if}\ \gamma_1\leq\theta\leq\theta_A\end{cases}\tag{25}$$

where Φ_1, Φ_2, k, γ_1, δ_1, η_1, η_2, η_3, η_4 and η_5 are defined in Eqs. (22).

4. Special case; $\alpha=0.5$

In the special case of $\alpha=0.5$ (i.e., when the beam is under uniform moment over its fixed span L), the parameters $\mu_1=\bar{M}/8$, $\mu_2=\mu_3=0$. The simplification of the μ perameter for Eqs. (9)-(10) yields.

$$s=-\frac{2}{\bar{M}}(\theta-\theta_A)\tag{26}$$

$$\bar{x}=\frac{2}{\bar{M}}(\sin\theta_A-\sin\theta)\tag{27}$$

$$\bar{y}=\frac{2}{\bar{M}}(\cos\theta-\cos\theta_A)\tag{28}$$

Eqs. (26)-(28) implies that the elastica has a circular arc configuration.

By applying the boundary condition at B and setting $\theta=-\theta_B$, Eqs. (27) and (28) give

$$\bar{M}=4\sin\theta_A\tag{29}$$

and

$$\bar{L}=\frac{4\,\theta_A}{\bar{M}}=\frac{\theta_A}{\sin\theta_A}\tag{30}$$

5. Solution procedure

For a given value of $\bar{M}$, two unknowns (θ_A and θ_B) are to be solved in order to obtain the equilibrium configuration. The two equations needed come from Eqs. (6) and (10) with $x=1$ and $\theta=-\theta_B$, i.e.,

$$\int_{\theta_A}^{-\theta_B} \frac{-\lambda\cos\theta d\,\theta}{\sqrt{\mu_1+\mu_2\sin\theta+\mu_3\cos\theta}}=1 \tag{31}$$

In solving these two nonlinear equations, an iterative procedure is required and it is terminated

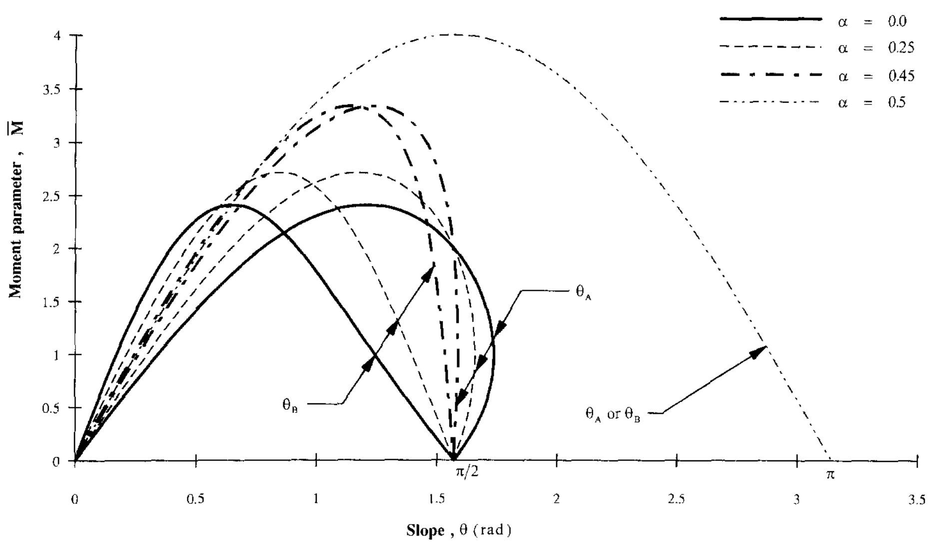
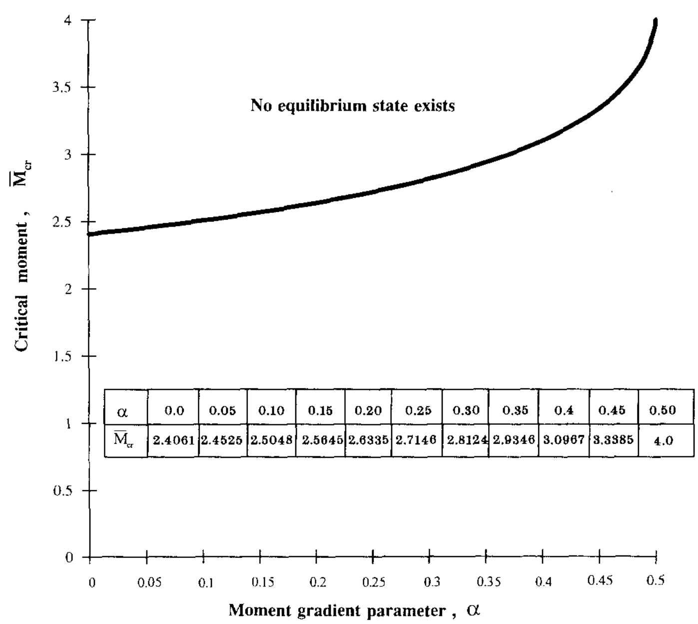


Fig. 2 Free body of a segment of elastica.



α	0.0	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.4	0.45	0.50
$\overline{M}_{cr}$	2.4061	2.4525	2.5048	2.5645	2.6335	2.7146	2.8124	2.9346	3.0967	3.3385	4.0

Fig. 4 Variations of the critical moment parameter, $\overline{M}_{cr}$ with respect to the moment gradient parameter, α.

when the obtained results satisfy the specified tolerance. In case where $\alpha \leq 0.5$ and if the assigned value of $\bar{M}$ is greater than $\bar{M}_{cr}$ (critical moment or the maximum moment value), the equilibrium solution does not exist. Thus, using aforementioned procedure may be not convenient for solving the problem. A more direct method of solution is recommended, in which only one equation is required to solve the problem. By setting $y=0$ and $\theta=-\theta_B$ in Eq. (11), one obtains

$$\int_{\theta_A}^{-\theta_B} \frac{-\sin\theta d\theta}{\sqrt{\mu_1+\mu_2\sin\theta+\mu_3\cos\theta}}=0 \tag{32}$$

This equation can be transformed into elliptic integral forms as given in Eqs. (14), (18), (21), and (25). If θ_B is given, one can solve θ_A from these equations and $\bar{M}$ can be determined from Eq. (6).

The solution steps are summarized as follows:

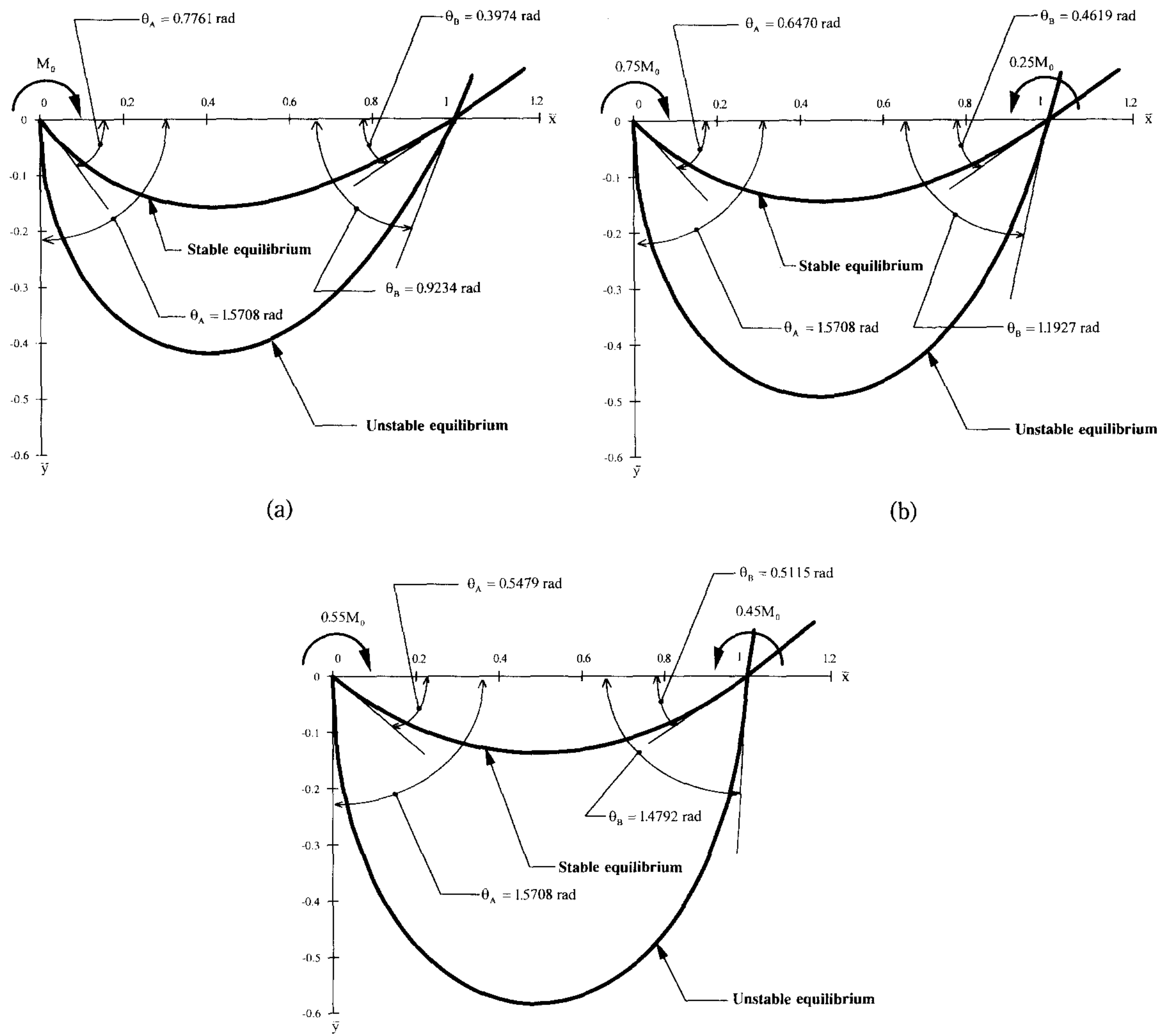
1. Assign a value for θ_B, $(0 \leq \theta_B \leq \pi/2)$, and set the initial value of θ_A to be zero.
2. Solve θ_B in Eq. (31) by the Newton-Raphson iterative process based on the value of θ_A given in Step 1.
3. Evaluate $\bar{M}$ from Eq. (6) for the values of θ_A and θ_B obtained from step 2.
4. Add an increment $\Delta\theta_B$ to θ_B to obtain the new value of θ_B.
5. Repeat steps 2-4 and construct the curves of $\bar{M}$ versus θ_A and θ_B.

To solve the deflection $\bar{y}$ and $\bar{L}$ at a distance $\bar{x}$ it is necessary to determine θ at x first. At any distance $\bar{x}$ assigned and the angles θ_A and θ_B found, the value of θ is determined from Eqs. (13), (17), (20), and (24). Then the corresponding values of $\bar{L}$ and y are determined from Eqs. (12), (16), (19), or (23) and from Eqs. (14), (18), (21), and (25) respectively.

6. Numerical results

Fig. 3 shows the variations of $\bar{M}$ with respect to the end slopes θ_A and θ_B for $\alpha=0$, 0.25 and 0.45. There is a peak value of $\bar{M}$ for each value of α which is less than 0.5. This peak value is known as the maximum or critical moment, $\bar{M}_{cr}$, and it can be determined numerically using the Dichotomous search algorithm (Kempf 1987) during the solution procedure in step 3. Fig. 4 shows the numerical values of $\bar{M}_{cr}$ and the plots of these values for different values of α. At $\alpha=0$, $\bar{M}_{cr}$ is equal to 2.4061 which was previously obtained in Chucheepsakul, *et al.* (1995). For $\alpha=0.5$, Eq. (29) gives a maximum value of $\bar{M}_{cr}=4$ when $\theta_A=\pi/2(0 \leq \theta_A \leq \pi)$ and from Eq. (30), the corresponding $\bar{L}=\pi/2$.

Fig. 5 shows typically, two different curves representing the stable and unstable equilibrium configurations for $\bar{M}=2$ and $\alpha=0$, 0.25 and 0.45. In the case of $\alpha=0.5$, for a given value of $\bar{M}$ which is less than 4.0, one can determined θ_A from Eq. (29) which gives two values of θ_A. The smaller θ_A gives the stable equilibrium configuration which is a circular arc of smaller total arc-length. On the other hand, the larger θ_A gives the unstable equilibrium configuration which is a circular arc with the larger total arc-length. Numerical results of the stable and unstable equilibrium configurations are given in Table 1. For the stable cases the results are obtained for every values of α whereas in the unstable cases the results are obtained for every values of α which is less than or equal to 0.5. Beyond this α value, the equilibrium results of the unstable cases are non-existent. To confirm the validity of the elliptic-integral formulation and solution, the results obtained by this method are compared with those determined from the



(c)

Fig. 5 Equilibrium configurations for $\bar{M}=2$ and $\alpha=0$, 0.25, and 0.45.

shooting-optimization method (Wang and Kittipornchai 1992). The details of this method is given in Chucheepsakul, *et al.* (1994). It was found that both methods yield the same solutions up to four decimal places; confirming the correctness of the elliptic integral expressions.

7. Conclusions

Exact solutions for elasticas with variable-arc-length under moment gradient have been presented. The solutions are given for both cases of stable and unstable equilibrium states. In addition, the critical moments for various moment gradients are given. These solutions should be useful to engineers designing offshore risers and the results may serve as benchmark values to verify convergence, validity and accuracy of numerical results obtained from computational methods.

Table 1 θ_A, θ_B and $\bar{L}$ versus $\bar{M}$ for various values of α for both stable and unstable equilibrium states

α	$\bar{M}$	θ_A		θ_B		$\bar{L}$	
		Stable	Unstable	Stable	Unstable	Stable	Unstable
0.0	1.0	0.3429	1.7378	0.1722	1.2414	1.0119	1.7848
	1.5	0.5364	1.6975	0.2711	1.0894	1.0299	1.5913
	2.0	0.7761	1.5708	0.3974	0.9234	1.0655	1.4070
0.25	1.0	0.2979	1.6590	0.2127	1.3951	1.0112	1.8480
	1.5	0.4605	1.6393	0.3288	1.3013	1.0271	1.6883
	2.0	0.6470	1.5708	0.4619	1.1927	1.0550	1.5319
	2.5	0.9009	1.4154	0.6440	1.0373	1.1131	1.3596
0.5	1.0	0.2527	2.8889	0.2527	2.8889	1.0107	11.5556
	1.5	0.3844	2.7572	0.3844	2.7572	1.0251	7.3525
	2.0	0.5236	2.6180	0.5236	2.6180	1.0472	5.2360
	2.5	0.6751	2.4665	0.6751	2.4665	1.0802	3.9463
0.75	1.0	0.2073	–	0.2920	–	1.0106	–
	1.5	0.3085	–	0.4381	–	1.0237	–
	2.0	0.4053	–	0.5828	–	1.0417	–
	2.5	0.4938	–	0.7224	–	1.0631	–
1.0	1.0	0.1618	–	0.3307	–	1.0108	–
	1.5	0.2331	–	0.4901	–	1.0232	–
	2.0	0.2917	–	0.6395	–	1.0382	–
	2.5	0.3337	–	0.7734	–	1.0533	–

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