Structural Engineering and Mechanics, Vol. 5, No. 5 (1997) 523-528
DOI: http://dx.doi.org/10.12989/sem.1997.5.5.523

Bracing of structures to prescribed buckling loads

James Barbato† and Ray Lawther‡

School of Civil Engineering, The University of New South Wales, Sydney 2052, Australia

Abstract. Stiffness and flexibility equations are combined in the buckling analysis of a braced structure - stiffness for the original structure and flexibility for the bracing. Choosing a flexibility formulation for the bracing gives a very compact computational problem. It also gives theoretical insights into the behaviour of the braced structure.

Key words: Buckling, eigenvalues & eigenvecters, bracing, shiffess, flexibility.

1. Introduction

Stiffness analysis is the most commonly used technique in structural analysis, and is near to universal in buckling analyses, where the geometric basis of the method connects naturally with the geometric nonlinearity of the problem. Bracing is adding support to a structure to increase the buckling load, and this is most readily done through further stiffness relations. An alternative is to use flexibility equations for the additions, producing a computational problem that is very compact - the size of the problem is the size of the bracing, which may be as small as a single freedom eigenvalue equation. Further, theoretical benefits include new insights into existing theorems as well as results that are thought to be new. This paper considers a single unloaded brace, and the conditions under which it will achieve its theoretically maximum effectiveness.

2. Formulation

Buckling equations of a structure are assumed to be written in the algebraic eigenvalue form

$$\boldsymbol{K}(P)\boldsymbol{u}=(\boldsymbol{K}-P\boldsymbol{S})\boldsymbol{u}=0 \tag{1}$$

where the stiffness matrix $\boldsymbol{K}(P)$ depends on the loading, represented by a single parameter P. Here $\boldsymbol{K}(P)$ is a linear combination of the conventional stiffness $\boldsymbol{K}$ and the stability matrix, or geometric stiffness matrix $\boldsymbol{S}$, which is the form commonly produced in finite element formulations, amongst others. These eigenvalue equations are solved for associated pairs $(P, \boldsymbol{u})$ which give a buckling load and a buckling shape or mode.

† Civil Engineer
‡ Senior Lecturer

A single, or *rank 1*, brace will add stiffness to a single displacement u_b, related to the structure displacements by

$$u_b = \boldsymbol{g}\boldsymbol{u} \tag{2}$$

and related to the force in the bracing, N_b, by

$$u_b = fN_b \tag{3}$$

Contragredience gives the bracing forces in the original coordinates as

$$\bar{\boldsymbol{P}} = \boldsymbol{g}^T N_b \tag{4}$$

and with the total of the internal forces applied to the nodes now $\boldsymbol{K}(P)\boldsymbol{u} + \bar{\boldsymbol{P}}$, the equilibrium equations of the braced structure are

$$\boldsymbol{K}(P)\boldsymbol{u} + \boldsymbol{g}^T N_b = \boldsymbol{0} \tag{5}$$

Combining 2, 3 and 5

$$\begin{bmatrix} \boldsymbol{K}(P) & \boldsymbol{g}^T \\ \boldsymbol{g} & -f \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ N_b \end{bmatrix} = \boldsymbol{0} \tag{6}$$

The bracing is assumed to be unloaded prior to buckling. Being unloaded, it does not change the load path of the structure, leaving both f and $\boldsymbol{S}$ constant. The geometry of the bracing connections, given by $\boldsymbol{g}$, is similarly assumed to be independent of load. More general forms of bracing (and $\boldsymbol{K}(P)$) are considered by Lawther (1995).

Typical examples of rank 1 bracing are provided by translational or moment springs bracing a single freedom, and single truss members. The rank of a brace is the number of different ways that it can be stressed, and rank 1 bracing is the basic building block for bracing of any complexity.

Eq. (6) describes the new, braced structure in a mixed formulation. The original structure is included through the original stiffness matrix which is unaltered, and occupies the leading diagonal submatrix partition, and the bracing is included through a flexibility relation that augments the original.

3. Assessing stability

Peters and Wilkinson (1969) presented an algorithm for the mode count of a structure with stiffness $K(P) = \boldsymbol{K} - P\boldsymbol{S}$, giving the number of buckling modes between 0 and a trial value of P as

$$m(P) = s(\boldsymbol{K}(P)) \tag{7}$$

Here $s(\hat{\ })$ is the negative Sylvester inertia of a symmetric matrix (Parlett 1980), usually computed by a count of the negative terms on the diagonal after a Gauss-reduction to triangular form, leading to the common name of *sign count*. The mode count of Eq. (7) is only valid if $m(0) = 0$, that is, the unloaded assemblage is a structure.

Williams and Anderson (1983) show how mode counting can be applied when Lagrange Multipliers are used, and this work directly gives the mode count of the formulation in Eq. (6)

as

$$m = s(\boldsymbol{K}) + s(R) - 1 \tag{8}$$

where R is the lower right partition of the matrix in Eq. (6) after a partial Gauss reduction has converted $\boldsymbol{K}$ to triangular form, and is given symbolically as

$$R = -f - \boldsymbol{g}\boldsymbol{K}^{-1}\boldsymbol{g}^T \tag{9}$$

$s(\boldsymbol{K}) + s(R)$ is the sign count of the fully triangulated matrix of Eq. (6), and R is a scalar, so $s(R)$ is either 0 or 1, giving

$$m = m(\boldsymbol{K}) - 1 \text{ or } m = m(\boldsymbol{K}) \tag{10}$$

leading to the interleaving (strictly, non-overlapping) property: For any given load level P, the effect of an unloaded rank 1 brace is to decrease the mode count by 1, or leave it unaltered. Alternatively, the nth load of the braced structure must fall on or between the nth and $(n+1)$th loads of the unbraced structure.

This interleaving property is well known, but Eq. (10) gives a marvellously simple proof.

4. Sufficient bracing stiffness

A brace will affect the mode count at the point where $s(R)$ increments, from Eq. (8), and for a rank 1 brace this is when $R=0$. Eq. (9) gives the required bracing stiffness k as

$$1/k = f = -\boldsymbol{g}\boldsymbol{K}^{-1}\boldsymbol{g}^T \tag{11}$$

The bracing stiffness required to brace a mode (reduce the mode count) is

$$k = \frac{1}{-\boldsymbol{g}\boldsymbol{K}^{-1}\boldsymbol{g}^T} \tag{11a}$$

and at this stiffness the braced structure buckles in mode $m(\boldsymbol{K})$. A negative result from Eq. (11a) contradicts the condition that f is positive and shows that the bracing is not possible.

Eq. (11) is an eigenvalue problem in a single freedom (rank r bracing gives an $r \times r$ eigenvalue problem), and applications of this equation can be found in Lawther (1995). The interleaving property shows that the full potential of a single brace is to increase a given buckling load to the next, and Eq. (11) will determine if a given connection $\boldsymbol{g}$ is able to achieve this but it gives no help in deciding how the connections $\boldsymbol{g}$ should be chosen. This is the subject of the next section.

5. Necessary conditions for bracing to full potential

$\boldsymbol{K} - P\boldsymbol{S}$ is assumed to be positive definite for some value of P, here taken as 0 without loss of generality (this is the same condition as the assumption $m(0)=0$ used in Eq. (7)), guaranteeing that both $\boldsymbol{K}$ and $\boldsymbol{S}$ can be diagonalised by the same real matrix transformation (Parlett 1980). Let such a transform be $\boldsymbol{T}$. Then

$$\boldsymbol{T}^T(\boldsymbol{K} - P\boldsymbol{S})\boldsymbol{T}\boldsymbol{T}^{-1}\boldsymbol{u} = 0 \text{ or } (\boldsymbol{D}^K - P\boldsymbol{D}^S)\boldsymbol{z} = 0 \tag{12}$$

where $\boldsymbol{D}^K=\boldsymbol{T}^T\boldsymbol{KT}$ and $\boldsymbol{D}^S=\boldsymbol{T}^T\boldsymbol{ST}$ are diagonal matrices, with all D_{ii}^K positive, and $\boldsymbol{z}=\boldsymbol{T}^{-1}\boldsymbol{u}$. We have a complete set of eigensolutions $\{D_{ii}^K, D_{ii}^S, \boldsymbol{z}_i=\boldsymbol{e}_i=i\text{th base vector}\}$ in the sense described by Parlett, and a complete set $\{1/P=D_{ii}^S/D_{ii}^K, \boldsymbol{e}_i\}$ in the conventional sense, but $\{P=D_{ii}^K/D_{ii}^S, \boldsymbol{e}_i\}$ will be incomplete if D_{ii}^S is zero. Modes $\boldsymbol{e}_i$ with $D_{ii}^S\neq 0$ will be called finite modes, and the others infinite modes. The finite modes will be further classified positive or negative according to the sign of D_{ii}^S.

The vectors $\boldsymbol{z}_i=\boldsymbol{e}_i$ combine to give $\boldsymbol{Z}=\boldsymbol{I}$, and with $\boldsymbol{Z}=\boldsymbol{T}^{-1}\boldsymbol{U}$, $\boldsymbol{U}=\boldsymbol{T}$, i.e., the columns of $\boldsymbol{T}$ are the eigenvectors $\boldsymbol{u}_i$ of Eq. (2), to any scale, and in any order. Further stipulating that $\boldsymbol{T}^T\boldsymbol{KT}=\boldsymbol{I}$, and the terms of D_{ii}^S are ordered positive followed by negative followed by zeros, with the positive and negative terms ordered in decreasing magnitude, produces $\boldsymbol{T}$ which is unique to within repeated solutions and the directions chosen for the eigenvectors. With $\boldsymbol{T}$ $(=\boldsymbol{U})$ so defined, P_1 is the lowest positive buckling load, P_2 is the next lowest positive load, and so on, followed by the negative loads starting with the smallest, and finally followed by the infinite modes.

With bracing described by $\boldsymbol{g}$ and k, the added stiffness is $\boldsymbol{K}_b=\boldsymbol{g}^T k\boldsymbol{g}$, which transforms to $\boldsymbol{U}^T\boldsymbol{g}^T k\boldsymbol{gU}$. The way that the brace connects to the freedoms is described by $\boldsymbol{g}$, and

$$\alpha=\boldsymbol{gU} \tag{13}$$

similarly describes the way that it connects to the modes: a mode $\boldsymbol{u}_i$ and a constraint $\boldsymbol{g}$ are said to connect if $\alpha_i=\boldsymbol{gu}_i\neq 0$. This connection is purely geometric, depending on the modes and the geometry of the bracing. A second interaction, which depends on the loads as well, will be introduced later.

$\boldsymbol{K}(P)$ is in the algebraic form of Eq. (1), and $\boldsymbol{g}=\alpha\boldsymbol{U}^{-1}$, giving

$$1/k=-\alpha\boldsymbol{U}^{-1}(\boldsymbol{K}-P\boldsymbol{S})^{-1}\boldsymbol{U}^{-T}\alpha^T.$$

With $\boldsymbol{U}^T(\boldsymbol{K}-P\boldsymbol{S})\boldsymbol{U}=\boldsymbol{I}-P\boldsymbol{D}^S$,

$$(\boldsymbol{K}-P\boldsymbol{S})^{-1}=\boldsymbol{U}(\boldsymbol{I}-P\boldsymbol{D}^S)^{-1}\boldsymbol{U}^T$$

giving

$$1/k=-\alpha(\boldsymbol{I}-P\boldsymbol{D}^S)^{-1}\alpha^T=-\sum\frac{\alpha_i^2}{1-PD_{ii}^S}\,. \tag{14}$$

Also, $P_i=1/D_{ii}^S$ for the finite modes, and $D_{ii}^S=0$ for the infinite modes, giving

$$1/k=-\sum\frac{P_i\alpha_i^2}{P_i-P} \tag{15}$$

where it is understood that $P_i/(P_i-P)=1$ for the infinite modes.

To brace the fundamental mode, $\alpha_1\neq 0$, and to brace it to a load P, Eq. (15) must return a non-negative value of k, i.e.,

$$\sum\frac{P_i\alpha_i^2}{P_i-P}\leq 0.$$

For $P<P_2$ this gives

$$-\frac{P_i\alpha_1^2}{P_1-P}\geq\sum_{i=2..}\frac{P_i\alpha_i^2}{P_i-P} \tag{16}$$

where all terms in the summation are positive (even if P_i is not). $|P_i\alpha_i^2/(P_i-P)|$ measures an interaction between a load P and a buckling load P_i with modal bracing connection α_i.

To brace the fundamental load from P_1 to P, the interaction with mode 1 must be greater than the interaction with all the other modes combined.

Note that as $P \rightarrow P_2$ the interaction with mode 2 becomes infinite, unless $\alpha_2=0$.

To brace the fundamental mode from P_1 to P_2

(i) The brace must connect to mode 1

(ii) The brace must not connect to mode 2

(iii) The interaction with mode 1 must be at least the combined interactions with all other modes, i.e.,

$$\frac{-P_i\alpha_1^2}{P_1-P_2} \geq \sum_{i=3..} \frac{P_i\alpha_i^2}{P_i-P_2}$$

If the brace does interact with mode 2, Eq. (16) places a limit on the strength that can be reached:

$$P \leq \frac{P_1P_2(\alpha_1^2+\alpha_2^2)}{P_1\alpha_1^2+P_2\alpha_2^2} \tag{17}$$

Condition (iii) is interesting. The effectiveness in bracing the first mode depends on the interactions with other modes, and a brace which connects to the first mode may not fully brace it because it is too good at bracing the third or higher, and this is irrespective of the brace stiffness. It seems odd, but its true.

6. Examples

The Euler strut shown in Fig. 1 has a fundamental mode of a half sine wave, with the second mode a full wave, as shown, at loads $\tilde{P}=PL^2/EI=9.870$ and 39.48 (π^2 and $4\pi^2$) respectively. The second mode has a point of zero translational displacement (a node) at midspan, and placing a conjugate support here, as shown in Fig. 1b, necessarily produces a structure with the full wave as one of its buckling modes. With fundamental and second modes at $\tilde{P}=39.48$ and 80.76 this full wave mode is now the fundamental, and full bracing has been achieved.

Supporting a point of zero displacement satisfies condition (ii) in a simple and direct way, but it can be satisfied in others. In Fig. 1c a brace enforces equal rotations at the ends of the strut, producing a double fundamental mode at $\tilde{P}=39.48$ - full bracing has been produced, but 'only just'. And bracing the midspan rotation to be equal but opposite to an end rotation (Fig. 1d) produces $\tilde{P}=23.27$ and 39.48, failing to achieve the full potential of a single brace.

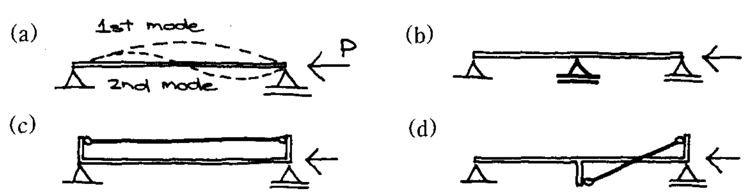

Fig. 1 Euler strut with various braces.

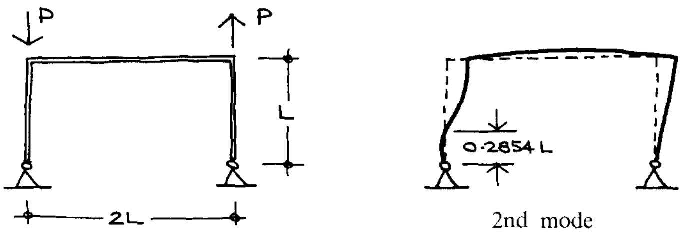

Fig. 2 Portal frame. Supporting a node does not give full bracing.

Respectively, the brace in each of Fig. 1b, c and d satisfies (iii) as an inequality, as an equality, and fails to satisfy the condition.

Supporting a node of the second mode generally achieves full bracing of the first mode, as in the example above. Despite this generality, condition (iii) warns that it is not necessarily universal. A rare counter example is shown in Fig. 2. Ubraced, this portal has buckling loads $\tilde{P} = 10.71$ and 40.40. If a support is introduced at the node of the second mode, at $0.2854L$ above the left support, the buckling loads are $\tilde{P} = 26.90$ and 40.40.

7. Concluding remarks

It is well known that the maximum effect of a single brace on a structure is to increase its fundamental buckling load up to its second load. A mixed stiffness and flexibility analysis has allowed the development of necessary and sufficient conditions to achieve this.

References

Lawther, R. (1995), "Flexible bracing," *ICSSD95 Conference*, Sydney, 91-94.

Parlett, Beresford N. (1980), *The Symmetric Eigenvalue Problem*, Prentice Hall, Englewood Cliffs, N.J.

Peters, G. and Wilkinson, J.H. (1969), "Eigenvalues of $Ax = \lambda Bx$ with band symmetric A and B", *Comp. J.*, **12**, 398-404.

Williams, F.W. and Anderson, M.S. (1983), "Incorporation of Lagrangian multipliers into an algorithm for finding exact natural frequencies or critical buckling loads," *Int. J. Mech. Sci.*, **25**(8), 579-584.