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High precision integration for dynamic structural systems with holonomic constraints

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Abstract. This paper presents a high precision integration method for the dynamic response analysis of structures with holonomic constraints. A detail recursive scheme suitable for algebraic and differential equations (ADEs) which incorporates generalized forces is established. The matrix exponential involved in the scheme is calculated precisely using 2^N algorithm. The Taylor expansions of the nonlinear term concerned with state variables of the structure and the generalized constraint forces of the ADEs are derived and consequently, their particular integrals are obtained. The accuracy and effectiveness of the present method is demonstrated by two numerical examples, a plane truss with circular slot at its tip point and a slewing flexible cantilever beam which is currently interesting in optimal control of robot manipulators.

Key words: dynamic structures; holonomic constraints; matrix exponential; algebraic and differential equations; numerical integration.

1. Introduction

Aerospace, civil and mechanical engineering structures are quite often subject to prescribed constraints. These constraints can be non-linear, time dependent and holonomic due to restrictions on configurations, phase plane trajectories, and workspace of the structures. These constraints can not simply be eliminated and the governing equation of the dynamic structural system is a mixture of both algebraic and differential equations (ADEs).

The constrained dynamic problems have received increasing attentions in recent years with a large proportion of the literature focusing on flexible multibody systems (Bae and Haug 1987, Li and Sankar 1992, Amirouche 1992, Yang 1992, Barauskas 1994). Yang (1992) presented a formulation for predicting the natural frequencies of constrained structural systems. Dynamic structures with unilateral constraints under impact have been considered by Barauskas (1994). Recent studies on optimal control of space structures, such as antenna, solar panels also involve

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coupled equations of rigid motion and flexible vibration with prescribed constraints. To deal with local nonlinearities in a structure, Liu, *et al.* (1994) developed an alternative substructuring approach which results in a set of ADEs. A Newmark scheme-based predictor-corrector algorithm (NPC) was presented for the integration of the resulting ADEs.

For some time, implicit direct integration methods have been used for the solution of the ADEs (Amirouch 1992, Nikravesh 1984). These methods, however, can not maintain the property of conservation of a conservative Hamilton system and induce errors due to the use of difference approximation techniques. The accuracy of these methods denoted as $O(\tau)^p$, depends strongly on the time step size τ. In addition, the key parts of these methods, e.g. the effective stiffness matrices also rely heavily on τ. This implies that there is a little flexibility to improve the computational accuracy when using these schemes. Further, since the structures are characterised by the holonomic constraints, highly nonlinearities can arise, and direct use of the constraints can result in instability problems (Liu 1996, Baumgarte 1972). It is therefore essential that efficient integration methods with high accuracy are needed in the solution of these ADEs. Recently, Zhong (1993) presented a high precision integration method which can maintain the conservation of a dynamic Hamilton system. The method has been extended to the numerical integration of differential Riccati equation and two-point boundary value problems (Zhong and Williams 1994, Zhong 1994).Lin and his co-workers (1995) have applied the method to non-stationary random seismic response of structures with very large time steps. It is therefore foreseeable that the method could result in desirable results for ADEs.

The present paper extends Zhong's work to the response analysis of constrained structures and establishes a recursive scheme for the solution of ADEs. The matrix exponential is calculated by using the 2^N algorithm. The generalised forces in the ADEs are separated and expressed as Taylor series and their particular integrals are obtained. The accuracy and the effectiveness of the present method is demonstrated via two examples.

2. Equations of motion of constrained structure

A general set of r *independent* prescribed holonomic constraints of dynamic structure can be written as

$$\boldsymbol{\Phi}(t,\ \boldsymbol{D})=\mathbf{0} \tag{1}$$

This equation imposes r restrictions on $\boldsymbol{D}$, and $\boldsymbol{\Phi}$ is presumed to have two continuous derivatives with respect to each of its two arguments. Differentiating the constraints of Eq. (1) with respect to time gives velocity and acceleration equation as, respectively

$$\dot{\boldsymbol{\Phi}}=\mathbf{0}\Longrightarrow\boldsymbol{\Phi}_D\dot{\boldsymbol{D}}+\boldsymbol{\Phi}_t=\mathbf{0} \tag{2}$$

and

$$\begin{aligned}\ddot{\boldsymbol{\Phi}}=\mathbf{0}&\Longrightarrow\boldsymbol{\Phi}_D\ddot{\boldsymbol{D}}+(\boldsymbol{\Phi}_D\dot{\boldsymbol{D}})_D+2\boldsymbol{\Phi}_{Dt}\dot{\boldsymbol{D}}+\boldsymbol{\Phi}_{tt}=\mathbf{0}\\&\Longrightarrow\boldsymbol{\Phi}_D\ddot{\boldsymbol{D}}+\boldsymbol{R}=\mathbf{0}\end{aligned} \tag{3}$$

where

$$\boldsymbol{R}=[\partial(\boldsymbol{\Phi}_D\dot{\boldsymbol{D}})/\partial\boldsymbol{D}+2\partial\boldsymbol{\Phi}_D/\partial t]\dot{\boldsymbol{D}}+\partial^2\boldsymbol{\Phi}/\partial t^2$$

Let $E=\partial\boldsymbol{\Phi}/\partial\boldsymbol{D}$, then, after incorporation of an arbitrary Lagrangian Multiplier vector $\lambda=(\lambda_1,$

$\lambda_2, \cdots, \lambda_r$), the equation of motion of the constrained structure becomes

$$\boldsymbol{M\ddot{D}}+\boldsymbol{C\dot{D}}+\boldsymbol{KD}+\boldsymbol{E}^T\boldsymbol{\lambda}=\boldsymbol{P} \tag{4}$$

where $\boldsymbol{K}$, $\boldsymbol{M}$ and $\boldsymbol{C}$ are *time-invariant* stiffness, mass and damping matrices of the structure, while $\boldsymbol{P}$ is the vector of applied loads. The term $\boldsymbol{E}^T\boldsymbol{\lambda}$ is the *generalized constraint force* which enforces the constraint Eq. (1) being satisfied.

For some nonlinear structures, Eq. (4) can be modified by introducing the nonlinear term $\boldsymbol{Q}$ without changing the properties of $\boldsymbol{K}$, $\boldsymbol{M}$ and $\boldsymbol{C}$, which will keep the advantages of the time-invariant parameter matrices, i.e.

$$\boldsymbol{M\ddot{D}}+\boldsymbol{C\dot{D}}+\boldsymbol{KD}=\boldsymbol{P}-\boldsymbol{E}^T\boldsymbol{\lambda}+\boldsymbol{Q} \tag{5}$$

Since $\boldsymbol{Q}$ is separated and it appears on the right-hand side of the equation, it is referred to as the *nonlinear generalized force.* Usually $\boldsymbol{Q}$ is a function of the state variables of the structure.

Eqs. (5) and (1) together with the following initial conditions

$$\left.\begin{array}{r}\boldsymbol{S}_1(t_0, \boldsymbol{D})=\boldsymbol{0}\\ \boldsymbol{S}_2(t_0, \boldsymbol{D}, \dot{\boldsymbol{D}})=\boldsymbol{0}\end{array}\right\} \tag{6}$$

define an initial value problem that governs the dynamic behaviour of the structure.

It is essential that the first equation of Eq. (6) and the constraint Eq. (1), evaluated at t_0, uniquely determine the initial displacement $\boldsymbol{D}(t_0)$. Similarly, the initial velocity $\dot{\boldsymbol{D}}(t_0)$ must be uniquely determined by Eq. (2) and the second equation of Eq. (6).

From Eq. (5), one has

$$\ddot{\boldsymbol{D}}=\boldsymbol{M}^{-1}(\boldsymbol{P}-\boldsymbol{E}^T\boldsymbol{\lambda}+\boldsymbol{Q}-\boldsymbol{KD}-\boldsymbol{C\dot{D}}) \tag{7}$$

Substitution of Eq. (7) into Eq. (3) gives

$$\boldsymbol{EM}^{-1}\boldsymbol{E}^T\boldsymbol{\lambda}=\boldsymbol{EM}^{-1}(\boldsymbol{P}+\boldsymbol{Q}-\boldsymbol{KD}-\boldsymbol{C\dot{D}})+\boldsymbol{R} \tag{8}$$

If $\boldsymbol{M}$ is positive definite, which is normally the case, and if the constraints are independent, so that $\boldsymbol{E}$ is of full rank, then $\boldsymbol{EM}^{-1}\boldsymbol{E}^T$ is positive definite and $\boldsymbol{\lambda}$ can be determined uniquely from Eq. (8).

It should be noted that Eqs. (5) and (8) are coupled to each other and cannot be solved separately by the existing integration methods, such as Newmark, Houbolt, and Runge-Kutta schemes. Using the NPC method (Liu, Williams and Kennedy 1994), an estimated value of λ can be obtained from Eq. (8) due to predictions of $\boldsymbol{D}$ and $\dot{\boldsymbol{D}}$. The corrections to $\boldsymbol{D}$ can then be found by substituting this approximate $\boldsymbol{\lambda}$ into Eq. (5). In spite of its availability, the method, however cannot keep the conservation of a Hamilton system and always causes errors due to the difference approximations. Zhong's (1993) high precision integration method has been found applicable to both initial and boundary value problems and more important, this method can give a nearly exact solution for many of the linear problems and maintain the conversation of a conservative Hamilton system. The method is therefore used here to the solution of the ADEs.

3. Formulation of the high precision integration scheme

To derive the scheme, it is necessary to transform the Eq. (5) into a first-order form. By introducing the following variables $\boldsymbol{u}=\boldsymbol{M}\dot{\boldsymbol{D}}+\boldsymbol{C}\boldsymbol{D}/2$ and $v=\{\boldsymbol{D}^T, \boldsymbol{u}^T\}^T$, the Eq. (5) can be written as

$$\dot{v}=\boldsymbol{H}v+f \tag{9}$$

where $f=\begin{Bmatrix} \boldsymbol{0} \\ \boldsymbol{P}-\boldsymbol{E}^T\lambda+\boldsymbol{Q} \end{Bmatrix}$, $\boldsymbol{H}=\begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{A}_2 \\ \boldsymbol{A}_3 & \boldsymbol{A}_4 \end{bmatrix}$,

$$\boldsymbol{A}_1=-\boldsymbol{M}^{-1}\boldsymbol{C}/2,\ \boldsymbol{A}_2=\boldsymbol{M}^{-1},\ \boldsymbol{A}_3=\boldsymbol{C}\boldsymbol{M}^{-1}\boldsymbol{C}/4-\boldsymbol{K},\ \text{and}\ \boldsymbol{A}_4=-\boldsymbol{C}\boldsymbol{M}^{-1}/2.$$

Since $\boldsymbol{H}$ is a constant matrix, from the theory of ordinary differential equation, the homogeneous solution of Eq. (9) can be expressed as

$$v^h(t)=\exp\ [\boldsymbol{H}\cdot(t-t_0)]\boldsymbol{c}_0 \tag{10}$$

in which $\boldsymbol{c}_0$ is constant vector to be determined by initial conditions. Let v^p be the particular integral of the inhomogeneous term, then the general solution of Eq. (9) has the form

$$v=v^h+v^p=\exp\ [\boldsymbol{H}\cdot(t-t_0)]\boldsymbol{c}_0+v^p \tag{11}$$

The vector $\boldsymbol{c}_0$ can then be readily derived from Eq. (11) by taking $t=t_0$ as $\boldsymbol{c}_0=v(t_0)-v^p(t_0)$, so that

$$v=\exp\ (\boldsymbol{H}t)\ [(v(t_0)-v^p(t_0)]+v^p \tag{12}$$

Let τ be the constant time step size. Then the explicit recursive formula for the general solution at $(k+1)$th time step, i.e. $t=t_0+(k+1)\tau$, is

$$v_{k+1}=\exp\ (\boldsymbol{H}\cdot\tau)[v_k-v_k^p]+v_{k+1}^p.\ (k=0,1,2,\ \cdots) \tag{13}$$

The key part of the precise integration is the calculation of exp $(\boldsymbol{H}\cdot\tau)$ because the accuracy of Eq. (13) depends on how accurate the matrix exponential exp $(\boldsymbol{H}\cdot\tau)$ can be evaluated. Because of their wide applications, the matrix exponential is described in many literature (see Golub and Loan 1983). The efficient and precise calculation method for exp $(\boldsymbol{H}\cdot\tau)$, i.e. the 2^N algorithm was presented in the paper by Zhong (1993) and it has the form

$$\exp\ (\boldsymbol{H}\cdot\tau)=[\exp\ (\boldsymbol{H}\cdot\tau/n)]^n=[\exp\ (\boldsymbol{H}\cdot\Delta t)]^n \tag{14}$$

where $\Delta t=\tau/n$, $n=2^N$ and N is a positive integer.

The Δt will be a very small interval even for a small value of N, for example take $N=20$, then $\Delta t=\tau/1048576$. From the point of view of structural dynamics, this value of Δt is much less than the highest modal period of any conventional idealised structure.

Further calculation of exp $(\boldsymbol{H}\cdot\tau)$ concerns the Taylor expansion of exp $(\boldsymbol{H}\cdot\Delta t)$ that may be written as

$$\exp\ (\boldsymbol{H}\cdot\Delta t)\approx\boldsymbol{I}+\boldsymbol{H}\Delta t+\frac{(\boldsymbol{H}\Delta t)^2}{2!}+\frac{(\boldsymbol{H}\Delta t)^3}{3!}+\frac{(\boldsymbol{H}\Delta t)^4}{4!}\approx\boldsymbol{I}+\boldsymbol{R}_0 \tag{15}$$

where $\boldsymbol{R}_0=\boldsymbol{H}\Delta t+\frac{(\boldsymbol{H}\Delta t)^2}{2!}+\frac{(\boldsymbol{H}\Delta t)^3}{3!}+\frac{(\boldsymbol{H}\Delta t)^4}{4!}$.

Let $\boldsymbol{R}_i=2\boldsymbol{R}_{i-1}+\boldsymbol{R}_{i-1}\boldsymbol{R}_{i-1}$ $(i=1,\ 2,\ \cdots,\ N)$, then

$$\exp\ (\boldsymbol{H}\cdot\tau)=[\exp\ (\boldsymbol{H}\Delta t)]^{2N}\approx[(\boldsymbol{I}+\boldsymbol{R}_0)]^{2N}=\boldsymbol{I}+\boldsymbol{R}_N \tag{16}$$

The truncated error of the expansion Eq. (15) can be estimated as $(\boldsymbol{H}\cdot\Delta t)^5/5!$. Suppose $\boldsymbol{H}=\boldsymbol{Y}_r[\mu]\boldsymbol{Y}_l^T$, where $\boldsymbol{Y}_l^T\boldsymbol{Y}_r=\boldsymbol{I}$, $\boldsymbol{Y}_l$ and $\boldsymbol{Y}_r$, are respectively, the left and right eigenvector matrices of $\boldsymbol{H}$, and $[\mu]=diag\ \{\mu_1^2\ \mu_2^2\ \cdots\}$ is the eigenvalue matrix, and it follows , by using expansion Eq. (15), that

$$\exp\ (\boldsymbol{H}\cdot\Delta t)\approx \boldsymbol{Y}_r(\boldsymbol{I}+[\mu]\Delta t+\frac{1}{2!}[\mu]^2\Delta t^2+\frac{1}{3!}[\mu]^3\Delta t^3+\frac{1}{4!}[\mu]^4\Delta t^4)\boldsymbol{Y}_l^T \tag{17}$$

Then the truncated error from the ith eigensolution corresponding to μ_i is $(\mu_i\Delta t)^5/5!$. Let ε be the allowed truncated error, then $|\mu_i|\Delta t<(120\varepsilon)^{1/5}$ or $|\mu_i|\tau<2^N(120\varepsilon)^{1/5}$. If inherent damping is not considered, $|\mu_i|$ is, in fact the ith angular frequency of the structure, i.e. $|\mu_i|=\omega_i$. This gives, by substituting $\omega_i=\frac{2\pi}{T_i}$, where T_i is the ith natural period of the structure, that

$$\frac{\tau}{T_i}<2^{N-1}\frac{1}{\pi}(120\,\varepsilon)^{1/5} \tag{18}$$

Since 2^{N-1} is a very large number, the step size τ can take a very large value even for a small number ε. When N is large enough, the accuracy will not be dominated by τ in the sense of numerical computation. For example, if take $\varepsilon=10^{-17}$, which exceeds the precision the computer can represent, and $N=20$, then $\frac{\tau}{T_i}<160$. This means that there is no significant truncated error can be induced by Eq. (15) even time step size is 160 times of the ith natural period of the structure. The higher modes should be considered, but in practice, the contributions of the high modes to the solution will be damped out because of inherent damping of the structure.

So it can be concluded that when $N=20$ the computed v^h is nearly the exact homogeneous solution, thus the accuracy of the general solution of Eq. (9) mainly depends on the accuracy of the particular solution v^p. For unconstrained linear structure under simple dynamic loads, the particular solution can be found analytically and in this case the general solution can be considered as the exact solution of the structure. But for constrained structure the situation will be much complicated and the particular integrals for the non-linear terms are only the approximation to the exact ones.

4. Particular solution of the ADEs

The particular solution v^p is the algebraic sum of the particular integrals of the applied force $\boldsymbol{P}(t)$ denoted as $\boldsymbol{x}(t)$, generalized constraint force $\boldsymbol{E}^T\lambda$, as $\boldsymbol{y}(t)$, and the generalized force $\boldsymbol{Q}$, as $\boldsymbol{z}(t)$. Here the undetermined coefficients method is used to find these particular solutions. It should be pointed that the calculation of $\boldsymbol{x}(t)$ is independent of the integration scheme, this means that the accuracy of the general solution of the structure can be conveniently improved *via* the improvement of the accuracy of $\boldsymbol{P}(t)$, making it possible to incorporate an adaptive mechanism in the algorithm.

4.1. Particular integrals of the applied force P(t)

(i) If $\boldsymbol{P}(t)=\boldsymbol{r}_0+\boldsymbol{r}_1t+\boldsymbol{r}_2t^2+\cdots\cdots+\boldsymbol{r}_mt^m$, where r_i $(i=0,1,\cdots,\ m)$ are constant vectors, then we take

$$\boldsymbol{x}(t)=\boldsymbol{a}_0+\boldsymbol{a}_1 t+\boldsymbol{a}_2 t^2+\cdots\cdots+\boldsymbol{a}_m t^m \tag{19}$$

in which

$$\begin{aligned}
\boldsymbol{a}_m&=-\boldsymbol{H}^{-1}\boldsymbol{r}_m\\
\boldsymbol{a}_{m-1}&=\boldsymbol{H}^{-1}(m\boldsymbol{a}_m-\boldsymbol{r}_{m-1})\\
\boldsymbol{a}_{m-2}&=\boldsymbol{H}^{-1}[(m-1)\boldsymbol{a}_{m-1}-\boldsymbol{r}_{m-2}]\\
&\vdots\\
&\vdots\\
\boldsymbol{a}_2&=\boldsymbol{H}^{-1}(3\boldsymbol{a}_3-\boldsymbol{r}_2)\\
\boldsymbol{a}_1&=\boldsymbol{H}^{-1}(2\boldsymbol{a}_2-\boldsymbol{r}_1)\\
\boldsymbol{a}_0&=\boldsymbol{H}^{-1}(\boldsymbol{a}_1-\boldsymbol{r}_0)
\end{aligned} \tag{20}$$

(ii) If $\boldsymbol{P}(t)=\boldsymbol{r}_1 \sin\theta t+\boldsymbol{r}_2 \cos\theta t$, where θ is constant, and $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are constant vectors, then we take

$$\boldsymbol{x}(t)=\boldsymbol{a}_1 \sin\theta t+\boldsymbol{a}_2 \cos\theta t \tag{21}$$

where

$$\left.\begin{aligned}
\boldsymbol{a}_1&=(\theta\boldsymbol{I}+\boldsymbol{H}^2)^{-1}(\theta\boldsymbol{r}_2-\boldsymbol{H}\boldsymbol{r}_1)\\
\boldsymbol{a}_2&=-(\theta\boldsymbol{I}+\boldsymbol{H}^2)^{-1}(\theta\boldsymbol{r}_1-\boldsymbol{H}\boldsymbol{r}_2)
\end{aligned}\right\} \tag{22}$$

(iii) If $\boldsymbol{P}(t)=\boldsymbol{r}e^{bt}$, where b is a constant and $\boldsymbol{r}$ a constant vector, then we take

$$\boldsymbol{x}(t)=\boldsymbol{a}e^{bt} \tag{23}$$

where $\boldsymbol{a}=(b\boldsymbol{I}-\boldsymbol{H})^{-1}\boldsymbol{r}$.

If $\boldsymbol{P}(t)$ is the sum of any two or all of above special forms, the particular integral is then the appropriate sum of the individual particular integrals.

The $\boldsymbol{x}(t)$ is exact integral if the above $\boldsymbol{P}(t)$ is the exact expression of the applied force, otherwise, e.g. if the real applied force is fitted by the above special forms, then $\boldsymbol{x}(t)$ is only an approximation of the particular integral of the applied force.

4.2. Particular integrals of the generalized forces

Expending the constraint force $\boldsymbol{E}^T\boldsymbol{\lambda}$ at t_k one has

$$(\boldsymbol{E}^T\boldsymbol{\lambda})|_t \approx (\boldsymbol{E}^T\boldsymbol{\lambda})\Big|_{t_k}+\frac{d(\boldsymbol{E}^T\boldsymbol{\lambda})}{dt}\Big|_{t_k}(t-t_k)+\frac{1}{2}\frac{d^2(\boldsymbol{E}^T\boldsymbol{\lambda})}{dt^2}\Big|_{t_k}(t-t_k)^2 \tag{24}$$

where

$$\frac{d(\boldsymbol{E}^T\boldsymbol{\lambda})}{dt}=\frac{d\boldsymbol{E}^T}{dt}\boldsymbol{\lambda}+E^T\frac{d\boldsymbol{\lambda}}{dt}$$

and

$$\frac{d^2(\boldsymbol{E}^T\boldsymbol{\lambda})}{dt^2}=\frac{d^2\boldsymbol{E}^T}{dt^2}\boldsymbol{\lambda}+2\frac{d\boldsymbol{E}^T}{dt}\frac{d\boldsymbol{\lambda}}{dt}+E^T\frac{d^2\boldsymbol{\lambda}}{dt^2}.$$

The multiplier $\boldsymbol{\lambda}$ can be found from Eq. (8), as

$$\boldsymbol{\lambda}=\boldsymbol{S}[\boldsymbol{E}\boldsymbol{M}^{-1}(\boldsymbol{P}+\boldsymbol{Q}-\boldsymbol{K}\boldsymbol{D}-\boldsymbol{C}\dot{\boldsymbol{D}})+\boldsymbol{R}] \tag{25}$$

and consequently

$$\dot{\lambda}=\dot{G}(P+Q-KD-C\dot{D})+G(\dot{P}+\dot{Q}-K\dot{D}-C\ddot{D})+\dot{S}R+S\dot{R} \tag{26}$$

$$\ddot{\lambda}=\ddot{G}(P+Q-KD-C\dot{D})+2\dot{G}(\dot{P}+\dot{Q}-K\dot{D}-C\ddot{D})$$
$$+G(\ddot{P}+\ddot{Q}-K\ddot{D}-C\dddot{D})+\ddot{S}R+2\dot{S}\dot{R}+S\ddot{R} \tag{27}$$

where $S=(EM^{-1}E^T)^{-1}$ and $G=SEM^{-1}$
Similarly, the generalized force Q can be expanded as

$$Q(t)=Q(t_k)+\dot{Q}(t_k)(t-t_k)+\frac{1}{2}\ddot{Q}(t_k)(t-t_k)^2 \tag{28}$$

The high order derivative $\dddot{D}$ involved in Eqs. (27) and (28) can be calculated approximately using numerical differentiation approaches. In this paper, the following three point formula is employed

$$\dddot{D}_k=\frac{1}{2\tau}(\ddot{D}_{k-2}-4\ddot{D}_{k-1}+3\ddot{D}_k) \tag{29}$$

The Eqs. (24) and (28) , which can be used to evaluated the particular integrals as described in Section 4.1, are quadratic approximations of $E^T\lambda$ and Q, respectively. However, it is possible to use higher order approximation of the nonlinear terms for some special problems.

In summary, the particular integral of $E^T\lambda$ is

$$y(t)=a_0+a_1t+a_2t^2 \tag{30}$$

where

$$a_2=-\frac{1}{2}H^{-1}\frac{d^2(E^T\lambda)}{dt^2}$$

$$a_1=H^{-1}\left[2a_2-\frac{d(E^T\lambda)}{dt}+\frac{d^2(E^T\lambda)}{dt^2}t_k\right]$$

$$a_0=H^{-1}\left[a_1-E^T\lambda+\frac{d(E^T\lambda)}{dt}t_k-\frac{1}{2}\frac{d^2(E^T\lambda)}{dt^2}t_k^2\right]$$

and the particular integral of Q is

$$z(t)=a_0+a_1t+a_2t^2 \tag{31}$$

where

$$a_2=-\frac{1}{2}H^{-1}\ddot{Q}$$

$$a_1=H^{-1}[2a_2-\dot{Q}+\ddot{Q}t_k]$$

$$a_0=H^{-1}\left[a_1-Q+\dot{Q}t_k-\frac{1}{2}\ddot{Q}t_k^2\right].$$

5. Algorithm and numerical examples

Fig. 1 is the flowchart of the algorithm. A demonstration of the effectiveness of the proposed method is provided by a plane 14-bar truss and a slewing flexible beam. No inherent damping is considered for both of the two examples.

Since instability may rise due to the use of acceleration constraint Eq. (3), a modified constraint is adopted. For the purpose, we introduce penalty functions $|\dot{\Phi}|^p$ and $|\Phi|^q$ to damp out errors in the integration. The modified constraint is then represented as

$$\boldsymbol{E}\ddot{\boldsymbol{D}}+\boldsymbol{R}+\kappa_1 \operatorname{sgn}(\dot{\boldsymbol{\Phi}})|\dot{\boldsymbol{\Phi}}|^p+\kappa_1 \operatorname{sgn}(\boldsymbol{\Phi})|\boldsymbol{\Phi}|^q=\mathbf{0} \tag{32}$$

where sgn is a signum function, κ_1 and κ_2 are positive real numbers, and p and q positive integers.

The accuracy of the present method is assessed by the violations of the holonomic constraints and the comparison with the NPC is made.

5.1. Plane truss structure

Fig. 2(a) shows a plane truss with 14 bars. The properties of the bars are: Young's modulus $E=2.1\times10^{11}\,N/\mathrm{m}^2$, density $\rho=7800.0\,\mathrm{kg/m}^3$, and cross-section area $A=9.0\times10^{-4}\,\mathrm{m}^2$. Two loading cases, one $p(t)=10000 \sin 20t(N)$ and the other $p(t)=10000\ (N)$, are considered. Two constraints

$$\text{(i)}\quad \Phi\equiv(u_a-e_x)^2+(v_a-e_y)^2-r^2=0$$

where u_a and v_a are the displacements of node a in X and Y directions, respectively, and $e_x=0.05$ m, $e_y=0.0$ and $r=0.05$ m, and

$$\text{(ii)}\quad \Phi\equiv v_a-0.005 \sin 2\pi t=0$$

are considered, respectively. The constraint (i) defines a circular slot at $(e_x+5.0,\ e_y+1.0)$ with radius r (see Fig. 2(b)) while the constraint (ii) enforces v_a to vary sinusoidally.

To investigate the accuracy and effectiveness of the present method, the constraint error, represented by ε, is defined, e.g. for the constraint (i)

$$\varepsilon=\left[\sqrt{(u_a-e_x)^2+(v_a-e_y)^2-r}\right]/r*100\%$$

and for the constraint (ii) simply

$$\varepsilon=v_a-0.005 \sin 2\pi t.$$

The static initial condition, i.e., $\boldsymbol{D}(0)=\dot{\boldsymbol{D}}(0)=\mathbf{0}$ is considered for the dynamic truss with the constraint (i). For the constraint (ii), let $\boldsymbol{D}(0)=\mathbf{0}$ and make $\dot{\boldsymbol{D}}(0)$ being consistent with this constraint at $t=0$.

Figs. 3 and 4 show the percentage errors of the constraint (i) for the two loading cases. It is shown that the present method gives much smaller errors than NPC does when the modified Eq. (32) is used, i.e. $p=q=1$, $\kappa_1=1000.0$ and $\kappa_2=100.0$. Both of the two methods are unstable if the stabilization term $\kappa_1 \operatorname{sgn}(\dot{\boldsymbol{\Phi}})|\dot{\boldsymbol{\Phi}}|^p+\kappa_2 \operatorname{sgn}(\boldsymbol{\Phi})|\boldsymbol{\Phi}|^q$ is ignored.

Tables 1 shows a comparison of the results obtained by the present method and that by NPC method for the constraint (ii). It can be seen that the results by present method are much

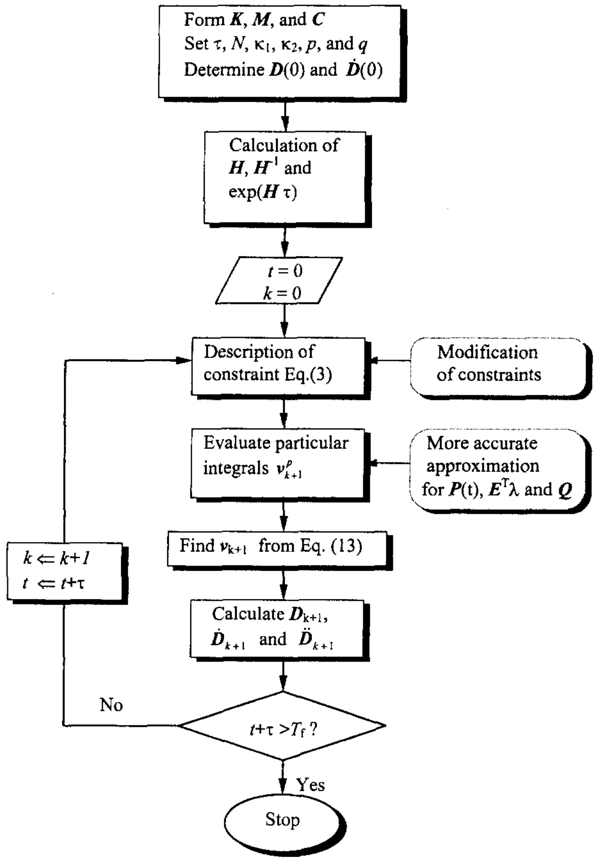

Fig. 1 Scheme for high precision integration of constrained dynamic structures.

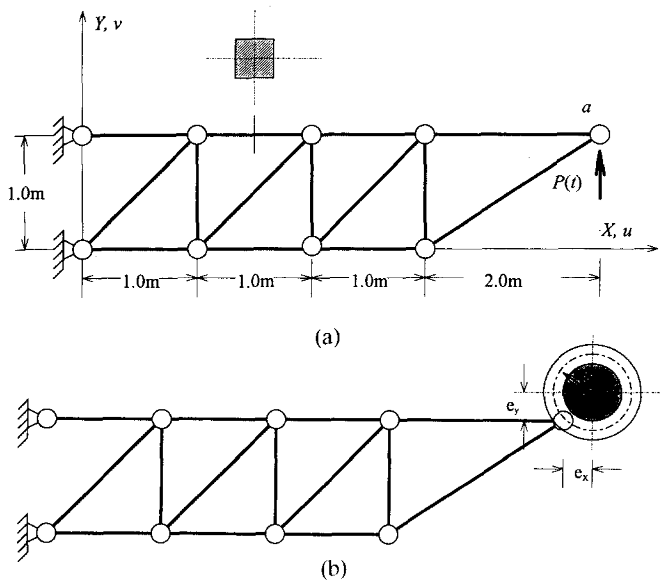

Fig. 2 Constrained truss.
(a) Plane truss; (b) circular slot constraint for node a as the constraint (I).

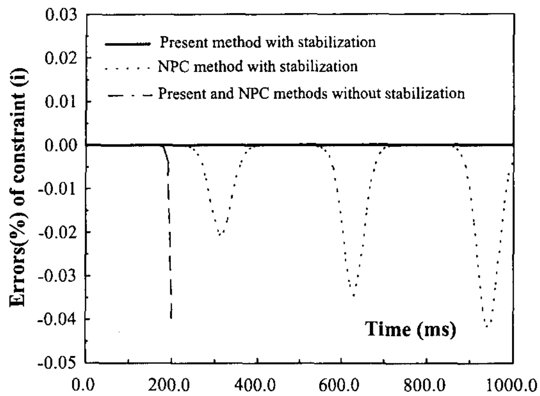

Fig. 3 Percentage errors of the constraint (i) subject to $p(t)=10000 \sin 20t$ (N) with $N=20$ and $\tau=0.0001$ sec.

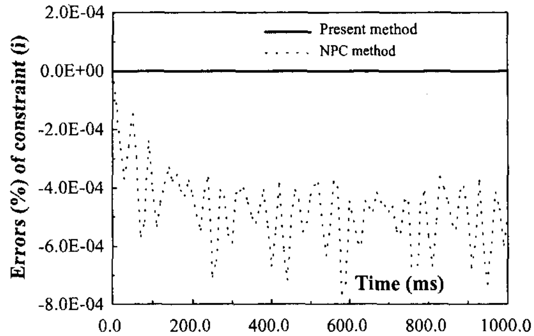

Fig. 4 Percentage errors of the constraint (i) subject to $p(t)=10000$ (N) with $N=20$ and $\tau=0.0001$ sec.

Table 1 Comparison of errors of constraint (ii)
($\tau=0.0001$ sec, $N=20$, $p=q=2$, $\kappa_1=10^5$, $\kappa_2=10^8$)

Time (ms)	Violation of Φ by present method	Violation of Φ by NPC method	
	Same results for the two loading cases	$p=10^4 \sin 20t(N)$	$p=10^4$ (N)
200	1.2×10^{-11}	9.5×10^{-9}	2.5×10^{-6}
400	6.2×10^{-11}	1.2×10^{-8}	-6.3×10^{-6}
600	1.3×10^{-11}	8.0×10^{-9}	-6.3×10^{-6}
800	3.0×10^{-12}	1.4×10^{-8}	4.3×10^{-6}
1000	7.8×10^{-11}	9.9×10^{-9}	5.1×10^{-6}

Table 2 Comparison of errors of the constraint (ii) subject to $p(t)=10000$ (N). ($\tau=0.0001$ sec, $N=20$, and $\kappa_1=\kappa_2=0$)

Time (ms)	Violation of Φ by present method	Violation of Φ by NPC method
200	7.8×10^{-7}	6.7×10^{-3}
400	1.6×10^{-6}	1.3×10^{-2}
600	2.3×10^{-7}	2.0×10^{-2}
800	3.1×10^{-7}	2.7×10^{-2}
1000	3.8×10^{-7}	3.3×10^{-2}

more accurate than those obtained by NPC. In the computation the higher order derivative of the Lagrangian multiplier $\ddot{\boldsymbol{\lambda}}$ is evaluated by numerical differentiation and used to approximate the generalised constraint force in Eq. (24) up to the third term.

The results without stabilization are depicted in Table 2. As expected, the result from the present method is accurate and makes the constraint satisfied well whilst the result by NPC is unacceptable for its poor accurate.

5.2. Slewing flexible uniform beam

The slewing flexible beam models systems such as spacecraft antenna, solar panel, robot manipulators and crane arm for the optimal control purposes (Liu and Yang 1993, Liu and Wu 1996). The present flexible beam, as shown in Fig. 5, is attached to a rigid hub and driven by a motor. During rotation, the beam undergoes both nonlinear rigid-body rotational maneuver and flexible body vibration. The beam is idealized with 5 beam elements and its governing equation is derived based on the second kind Lagrangian equation.

The material properties of the uniform beam with cross section 0.03 m$\times$0.03 m and length $l=1.0$ m are given as: Young's modulus $E=2.1\times10^{11}$ N/m^2 density $\rho=7800.0$ kg/m^3. The mass of the hub is omitted. There is no external applied force. Suppose that the controlled movement of the slewing angle θ is described by

$$\theta=\begin{cases}50.0\ t^2\ \text{rad} & \text{for } t\leq 0.1 \text{ sec}\\ -50.0(0.02-0.4\ t+t^2)\ \text{rad} & \text{for } 0.1\leq t\leq 0.2 \text{ sec}\end{cases}$$

This represents a rotation with angular acceleration 100.0 rad/sec^2 from 0.0 to 0.1 sec, and then with angular deceleration 100.0 rad/sec^2 from 0.1 to 0.2 sec. Differentiating with respect time once and twice gives

$$\dot{\theta}=\begin{cases}100.0\ t\ \text{rad/sec} & \text{for } t\leq 0.1 \text{ sec}\\ -100.0(-0.2+t)\ \text{rad/sec} & \text{for } 0.1\leq t\leq 0.2 \text{ sec}\end{cases}$$

and

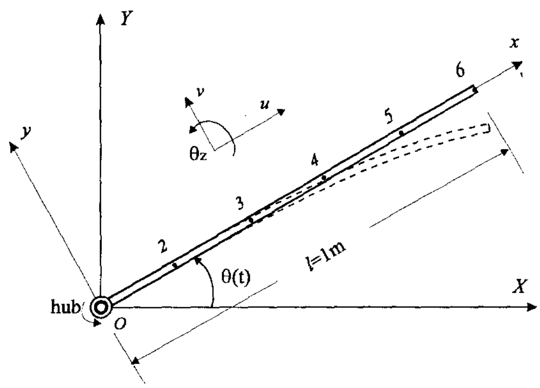

Fig. 5 Model of slewing flexible uniform beam.

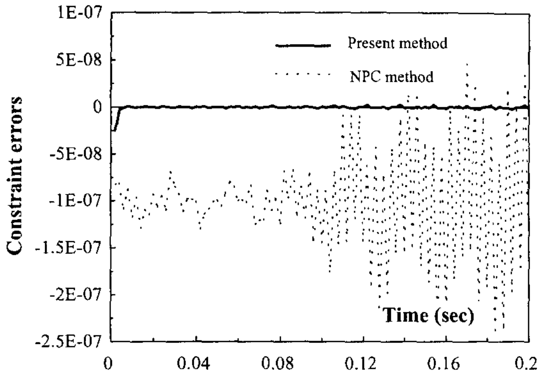

Fig. 6 Comparison of the constraint errors for the slewing beam with $N=20$ and $\tau=0.001$ sec.

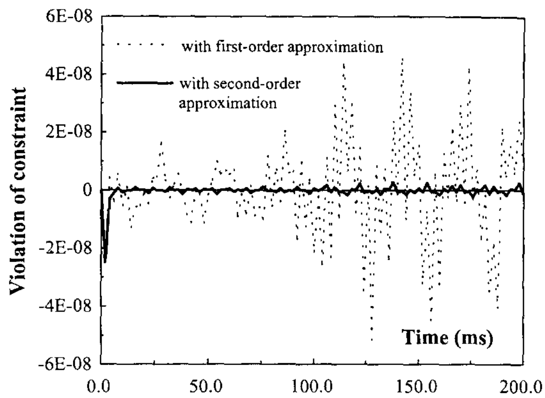

Fig. 7 Constraint errors due to different approximations of generalized forces for the slewing beam ($N=20$ and $\tau=0.001$ sec).

$$\ddot{\theta}=\begin{cases}100.0 \text{ rad/sec}^2 & \text{for } t\leq 0.1 \text{ sec}\\ -100.0 \text{ rad/sec}^2 & \text{for } 0.1\leq t\leq 0.2 \text{ sec}\end{cases}$$

A constraint enforcing the y-direction displacements at points 5 and 6 to satisfy $\Phi\equiv v_5-v_6=0$ is considered. And for this constraint, error $\varepsilon=v_5-v_6$ is employed to assess the integration accuracy.

Static initial condition is considered.

Fig. 6 is the variations of constraint errors by the present and NPC methods. The stabilization parameters $p=q=2$, $\kappa_1=10^4$ and $\kappa_2=10^7$ are used. With the estimated $\ddot{\boldsymbol{D}}$ from Eq. (29), the second-order expansion of Eqs. (24) and (28) are evaluated and used in the computations. As expected, the results by present method is much better than that by NPC.

The influence of the expansion of generalized forces on the accuracy of integration is demonstrated by Fig. 7, where the results are evaluated by using first- and second-order approximations respectively. The second-order information is obtained simply by the numerical differentiation formula similar to Eq. (29). It can be readily found that the result with second-order expansion is much better than that with first-order expansion. This gives an idea that the accuracy of the present method can be improved from time to time without reduction of the time step size and reconstruction of the matrix exponential which is considered as the core of the method.

6. Conclusions

The high precision method for the solution of constrained structure has been presented. In the method, the nonlinear terms and the constraint force of the ADEs are separated and referred to as the right hand term, which weakens the linkage between the algebraic and the differential equations and makes it easy to solve the ADEs. The 2^N algorithm is used to calculate the matrix exponential. This led to an exact homogeneous solution in the sense of numerical computation, and therefore a high precise general solution has been achieved. The examples have shown the high accuracy of the proposed method.

Further research is the evaluation of better approximation of the generalized force as well as the constraint force. Since the calculation of the particular integrals of inhomogeneous terms is independent of the key part of the integration scheme, an adaptive algorithm could be expected through the improvement in the approximation. We leave this for future investigation.

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