Structural Engineering and Mechanics, Vol. 5, No. 2 (1997) 137-148
DOI: http://dx.doi.org/10.12989/sem.1997.5.2.137

Natural frequency error estimation for 3D brick elements

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Abstract. In computing eigenvalues for a large finite element system it has been observed that the eigenvalue extractors produce eigenvectors that are in some sense more accurate than their corresponding eigenvalues. From this observation the paper uses a patch type technique based on the eigenvector for one mesh quality to provide an eigenvalue error indicator. Tests show this indicator to be both accurate and reliable.

This technique was first observed by Stephen and Steven for an error estimation for buckling and natural frequency of beams and two dimensional in-plane and out-of-plane structures. This paper produces and error indicator for the more difficult problem of three dimensional brick elements.

Key words: finite elements; natural frequence; error estimation.

1. Introduction

The finite element method discretises a structure into relatively simple elements to represent a more complex model. The prime source of error in the finite element analysis is from this discretisation where the displacements of the simple elements attempt to represent a complex distorted shape.

Barlow (1976) discovered that the optimal location for the retrieval of stresses in a finite element was at the Gauss points of the elements. Hinton and Campbell (1974) used this principle to improve the stresses at the nodes in the finite element model. Zienkiewicz and Zhu (1992) took this discovery even further to produce an error estimator for the linear static analysis. Wieberg (1994) modified the same principle to improve the accuracy of the stresses over the surface of the elements by considering a patch of elements around a central patch as opposed to a patch of elements around a central node as used in the previous procedures.

It has been observed, and is demonstrated using computational examples (Stephen and Steven 1994) that the eigenvector solution to a finite element analysis is more accurate than the eigenvalue. It is the eigenvalue solution that is required in a finite element analysis, so any error indicator should use the more accurate eigenvector solution to determine an error estimate for the analysis.

In previous papers Stephen and Steven (1994) derived error measures for the buckling and natural frequency finite element analysis for beams and in-plane and out-of-plane plate elements.

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This was based on the more accurate eigenvector solution to determine an error measure for the eigenvalue representing the buckling load factor or natural frequency of a structure. This paper extends the same philosophy to derive an error measure for the natural frequency of a finite element analysis.

Previous papers used a standard weighted least squares as the interpolation function over the patch of elements. With plate elements this method must be modified to include the rotational terms in the calculation of the function. The brick elements require a three dimensional function to be interpolated from the centroid of each element being examined.

2. Error measure

To determine the error for any situation, the result must be compared to a known more accurate solution. In the case of the buckling finite element analysis the more accurate solution is achieved via the use of a patch recovery technique for each element.

The finite element buckling analysis has the eigenvector being the more accurate nodal quantity, hence a more accurate solution is achieved by interpolation over the patch using the nodal displacement quantities.

A finite element has sufficient nodal quantities to produce a displaced shape equal to its polynomial shape function. A high order polynomial function over the element which is of higher order than the shape function will produce a more accurate displaced shape. This high order polynomial function can be produced by a least squares procedure incorporating the eigenvector solution of the nodes of the element and the surrounding elements in the patch (Stephen and Steven 1994).

In three dimensional structures consisting of brick elements there can be many elements and nodes in a patch. Attempting to fit very high order polynomials to the patch often leads to singularities in the matrix for the interpolation function, or the interpolation function produces additional waves inside the element. A higher order polynomial than the element shape function and a lower order polynomial than using every node in the patch can be achieved efficiently using the method of least squares.

Eight node isoparametric brick element are examined in this paper. The shape function used for the brick element has a complete polynomial of order only one. To improve the displacement field across the element is necessary to have the order of the interpolation function at a value of two or greater. There will be at least ten terms in the resulting polynomial.

It is assumed that the higher order polynomial function on each element, derived using a weighted least squares interpolation over the patch, gives a more accurate result for the displaced shape. The calculation of a more accurate eigenvalue solution is achieved by sub-dividing each element into sub-elements with nodal displacements obtained from the higher order interpolated function. The sub-element model for a two dimensional patch example is shown in Fig. 1 for one element. The procedure is the same for three dimensional elements.

A new eigenvalue λ^* can be calculated using the Rayleigh Quotient using the sub-elements and the interpolated nodal displacements. The Rayleigh Quotient is the summation of all of the sub-element contributions for all elements. The sub-element Rayleigh Quotient is written as Eq. (1):

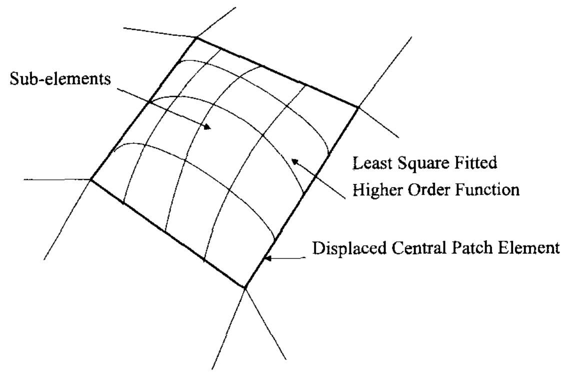

Fig. 1 Sub-elements on the original patch element.

$$\lambda^* = \frac{\sum_{p=1}^{N}\left[\sum_{i=1}^{n}\sum_{j=1}^{n} u_i (k)_{i,j} u_j\right]}{\sum_{p=1}^{N}\left[\sum_{i=1}^{n}\sum_{j=1}^{n} u_i (m)_{i,j} u_j\right]} \tag{1}$$

Where there are N elements in the finite element model and each of these elements is sub-divided into $n \times n \times n$ sub-elements. The terms $(k)_{i,j}$ and $(m)_{i,j}$ are the terms in the elastic stiffness and mass matrices respectively for the sub-divided structure. These quantities are pre and post multiplied by the interpolated eigenvector quantities u_i. Eq. (1) shows the computation procedure involved, indicating that the global elastic stiffness and mass matrices are not required to be formed for all the sub-elements. Only the local sub-element elastic stiffiness and mass matrices are required.

This result allows each element to have its patch in the model identified, the higher order function determined and the Rayleigh Quotient summations calculated. Upon the completion of this process for each element in the model the new eigenvalue can be calculated. The new eigenvalue obtained from the higher order polynomial equations on each element is more accurate than the original finite element solution due to the higher order function used to quantify the displaced shape and smoothing process involved with applying a least squares fit of the displaced shapes to a larger region than the single element of concern.

This improved solution is not exact, due to the true vibration shape being even more complex than the higher order polynomial. The improved eigenvalue may best be used as an error measure for the analysis.

The error for the finite element analysis is written in Eq. (2)

$$\varepsilon_\lambda = \frac{\lambda^{FE} - \lambda}{\lambda} \tag{2}$$

In most cases the exact natural frequency λ is unknown, so it may be replaced with the improved eigenvalue λ^* to give an error estimate based on this value to represent the actual error. This error estimate is expressed in Eq. (3).

$$\varepsilon^* = \frac{\lambda^{FE} - \lambda^*}{\lambda^*} \tag{3}$$

3. Interpolation procedure

Papers (Hinton and Campbell 1974, Zienkiewicz and Zhu 1992, Wieberg and Li 1994, Stephen and Steven 1994) use a weighted least squares technique to interpolate new piece-wise functions over the structure from the accurate quantity locations. The eigenvalue improvement techniques (Stephen and Steven 1994) require this function to be of a higher order than the finite element shape functions to obtain an improved prediction for the eigenvalue.

The method of least squares is based on choosing an initial polynomial function, as shown in Eq. (4):

$$f(x)=\alpha_0+\alpha_1 x+\alpha_2 x^2+\alpha_3 x^3+\cdots \tag{4}$$

The coefficients are determined by minimising the errors between the function and the true values. The function to be minimised is defined as Eq. (5):

$$v=\sum_{i=1}^{N} w_i(y_i-f(x_i))^2 \tag{5}$$

Where w_i is a weighting function giving greater emphasis to the accuracy of certain data points.

To minimise the error, the differential of the error function with respect to each coefficient in the chosen polynomial is set to zero. Hence there are as many simultaneous equations as there are unknown coefficients and the coefficients can be solved with simple Gaussian elimination techniques.

On a patch of elements it is only the central element of concern. To ensure that the function equals the data at the nodal points, the difference between the least squares function and the exact data is calculated. The finite element shape function is calculated for the differences at the nodes. The final interpolation function is the addition of the least squares function and the difference shape function. This guarantees the valves at the data points are satisfied.

4. Interpolation function

The brick elements examined in this paper have eight nodes. The eight nodes produce shape function equations for the three orthogonal displacement directions with eight coefficients in each function. The displacement equation, call it function $f_1(x, y, z)$, is shown following Eq. (6) for the x axis displacement. Orthogonal displacement functions for the y and z directions are similar.

$$f_1(x, y, z)=\alpha_{x0}+\alpha_{x1}x+\alpha_{x2}y+\alpha_{x3}z+\alpha_{x4}xy+\alpha_{x5}yz+\alpha_{x6}zx+\alpha_{x7}zx \tag{6}$$

These equations have a complete polynomial order of only one. A higher order function would have a complete polynomial order of two or greater. A polynomial with a complete order of two would contain ten coefficients and could be written as Eq. (7). Call Eq. (7) function $f_2(x, y, z)$.

$$f_2(x, y, z)=\alpha_{x0}+\alpha_{x1}x+\alpha_{x2}y+\alpha_{x3}z+\alpha_{x4}x^2+\alpha_{x5}xy+\alpha_{x6}y^2+\alpha_{x7}yz+\alpha_{x8}z^2+\alpha_{x9}zx \tag{7}$$

Function $f_1(x, y, z)$ contains one cubic term. This high order term could possibly cause function $f_1(x, y, z)$ to be more accurate than function $f_2(x, y, z)$ which contains squared terms. To justify the use of function $f_2(x, y, z)$ as the higher order function, a comparison must be made for both functions to an arbitrary more complex function.

Define function $f_0(x, y, z)$ as Eq. (8):

$$f_0(x, y, z)=\sin(x)\sin(y)\sin(z) \tag{8}$$

over a dube region from 0 to $\frac{\pi}{2}$ for each side. Divide this cube into $n\times n\times n$ elements and determine the coefficients for both functions $f_1(x, y, z)$ and $f_2(x, y, z)$ using a least squares fit for the nodal points in a patch given displacements calculated from function $f_0(x, y, z)$.

Compare both functions $f_1(x, y, z)$ and $f_2(x, y, z)$ to the exact solution represented by the function $f_0(x, y, z)$ by defining errors of Eq. (9) and Eq. (10):

$$\varepsilon_1=\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\int_{\Omega}[f_0(x, y, z)-f_1(x, y, z)]^2 d\Omega \tag{9}$$

$$\varepsilon_2=\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\int_{\Omega}[f_0(x, y, z)-f_2(x, y, z)]^2 d\Omega \tag{10}$$

The integration was performed numerically and number of subdivisions along each side of the cube was varied. The error measure is plotted for both functions in Fig. 2. It is obvious from Fig. 2 that function $f_2(x, y, z)$ has a lower error than function $f_1(x, y, z)$. This function has also a more rapid convergence rate making it quite suitable to use as the higher order function for patch improvement on the eigenvector solution.

5. Examples

The first example examined is that of a simple cantilevering square section. The length of the cantilever used is six times the dimension of the side of the square section. The structure was analysed for various mesh refinements to extrapolate an exact solution. An isometric view of one particular mesh is shown in Fig. 3.

The new eigenvalue was based on a $10\times10\times10$ sub-element refinement of every element. As any mesh is refined the finite element results converge to the true solution, however the

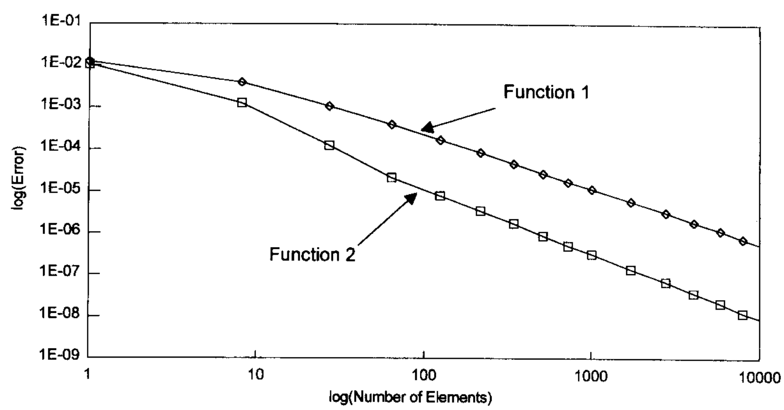

Fig. 2 Error measure plotted for both functions.

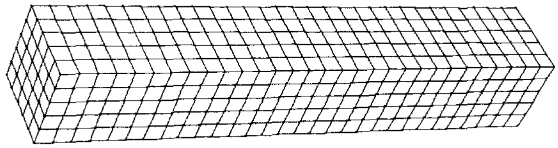

Fig. 3 Cantilevering square column example.

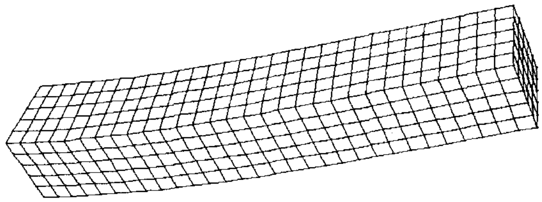

Fig. 4 Vibrating cantilevering square column-Mode 1.

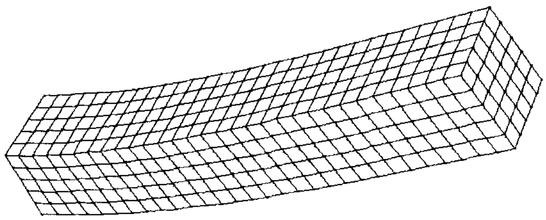

Fig. 5 Vibrating cantilevering square column-Mode 2.

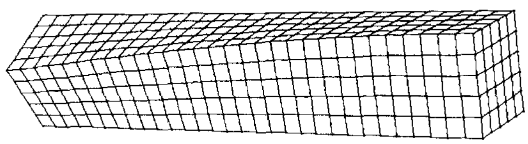

Fig. 6 Vibrating cantilevering square column-Mode 3.

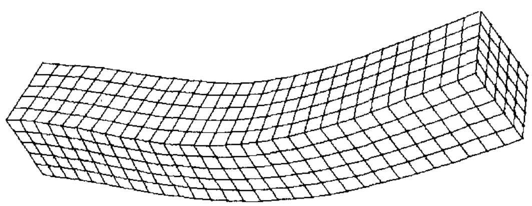

Fig. 7 Vibrating cantilevering square column-Mode 4.

more elements in the model requires longer for the analysis to solve. This refinement number was chosen such that the $10 \times 10 \times 10$ sub-element mesh was quite refined, but not too refined to cause dramatic reduction in the speed of the error calculation. The first four modes were examined. The vibrating shapes are shown for each mode in Figs. 4 to 7.

The geometry of this structure was chosen to produce modes that were quite dissimilare in

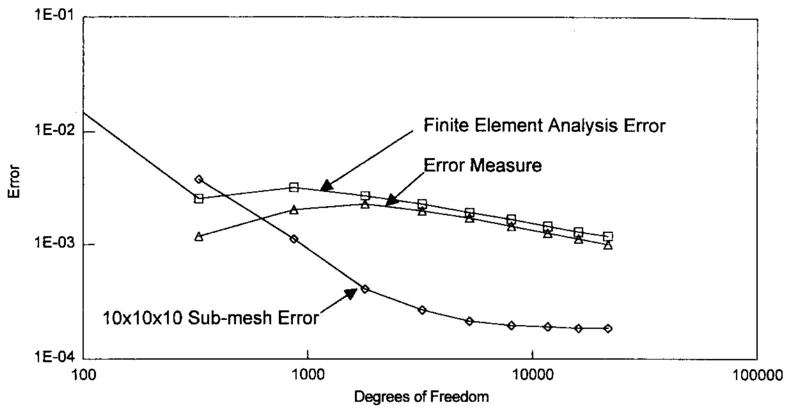

Fig. 8 Error plots for vibrating cantilever square column-Modes 1 and 2.

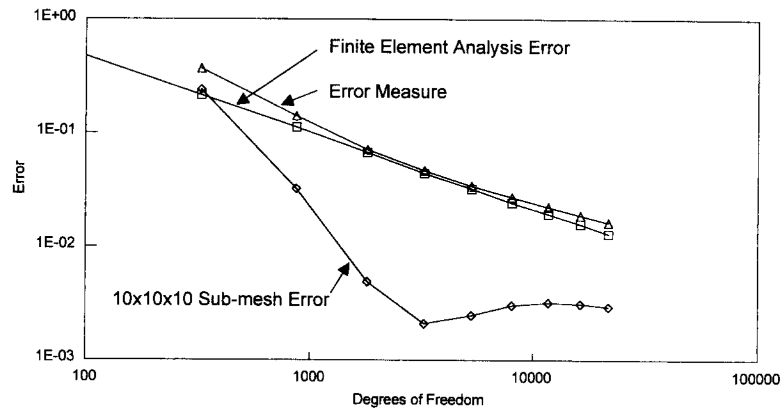

Fig. 9 Error plots for vibrating cantilever square column-Mode 3.

displacement patterns. Modes 1 and 2 are swaying displacements of the cantilever of one quarter wave length normal to the edges of the cross section, while Mode 3 is a torsional displacement about the axis of the structure and Mode 4 is a swaying displacement towards the diagonals of the cross section.

The results are plotted for the first foor modes in Figs. 8 to 10 showing the finite element analysis error, the error from the mesh subdivision and the eror measure. Modes 1 and 2 are simply a rotational transformation of each other. The eigenvalues are the same for these modes and hence the errors associated are identical.

There are three lines plotted in these figures. The Finite Element Analysis Error line represents the error in the analysis for the particular mode. The 10×10×10 Sub-mesh Error line is the error of the improved solution when the patch recovery technique is used. The Error Measure line is the approximation to the finite element analysis error using the improved eigenvalue derived from the patch recovery technique.

The first three modes produce an improved eigenvector that is far more accurate than the finite element analysis result. As such the error measure for the finite element analysis result using the improved eigenvalue is quite accurate. However the fourth mode does not produce

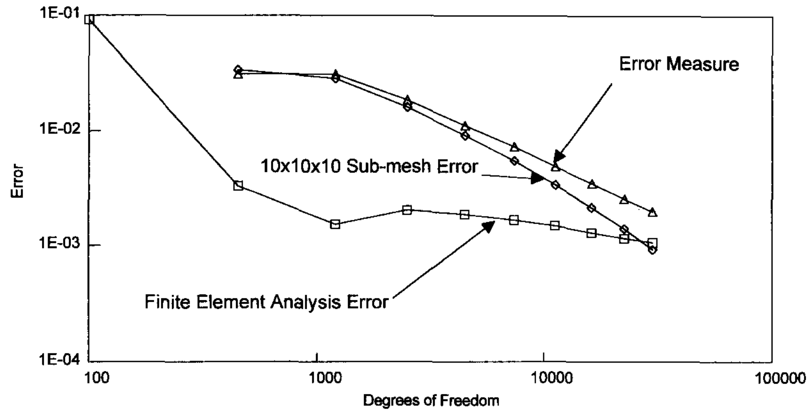

Fig. 10 Error plots for vibrating cantilever square column-Mode 4.

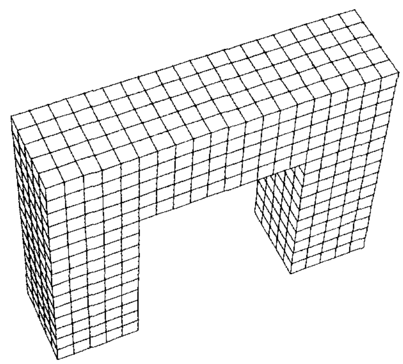

Fig. 11. Portal frame from brick elements.

a greatly improved eigenvalue and hence the error measure is not as accurate as the previous modes. This inaccuracy can be attributed to the eigenvector being less accurate due to the formulation of the finite elements using reduced integration techniques for shear type displacement and the eigenvalue/eigenvector extractor terminating when a sufficient eigenvalue tolerance has been reached and ignoring the eigenvector accuracy.

The finite element analysis converged from below the exact value due to a lumped mass matrix being used for the model. The improved eigenvalue converged to the exact solution from the opposite direction than the finite element analysis. This means that even if both the finite element solution and the improved eigenvalue had relative errors of similar magnitude, the error measure would still be quite accurate.

The second example was a portal frame of brick elements. This structure was proportioned so the the four vibrating modes examined were actually quite different in appearance. The portal frame structure is shown in Fig. 11.

The first mode was a sway type displacement of the legs of the frame out of the plane of the frame, similar to the cantilever displacement. The second more is the sway displacement of the portal frame. The third mode is the twisting displacement and the fourth mode is the vertical deformation of the horizontal member. These four modes are pictured in Figs. 12 to Fig. 15.

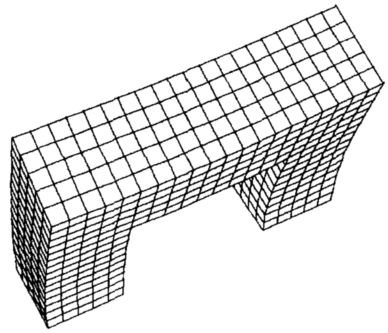

Fig. 12. Portal frame vibration-Mode 1.

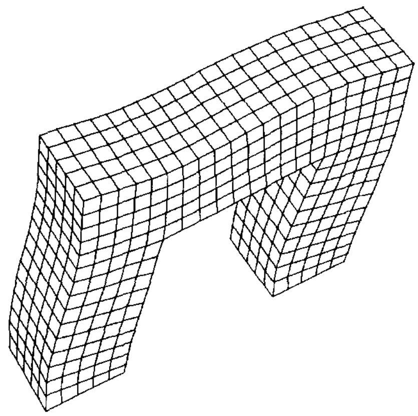

Fig. 13. Portal frame vibration-Mode 2.

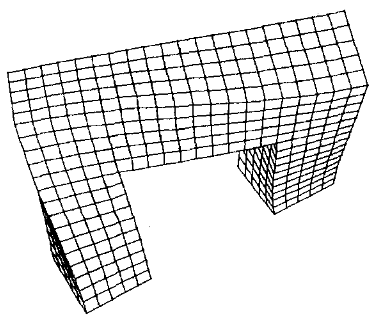

Fig. 14. Portal frame vibration-Mode 3.

The eigenvalue error for the finite element analysis and the improved value is plotted for each of the four modes on the Figs. 16 to 19. Also plotted is the error measure quantity attempting to predict the finite element analysis error.

The results obtained for this example were not as accurate at estimating the error in the first example. However as the improvd eigenvalue converged to the true solution from a direction opposite to the finite element analysis, the error measure was a conservative estimate of the finite element analysis error.

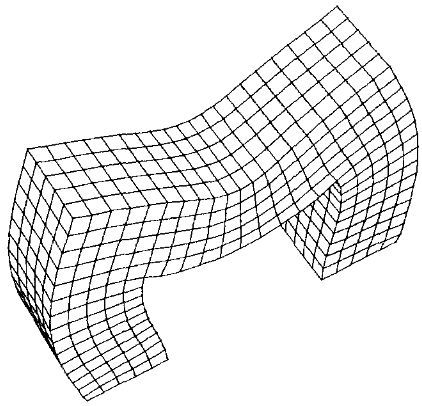

Fig. 15. Portal frame vibration-Mode 4.

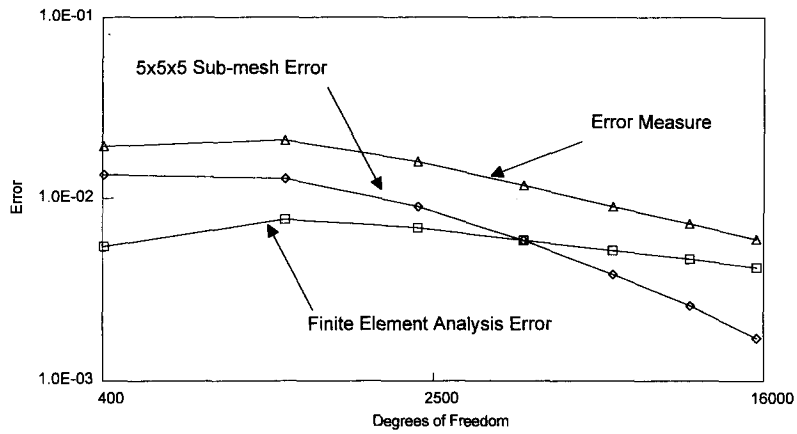

Fig. 16. Error plots for vibrating portal frame-Mode 1.

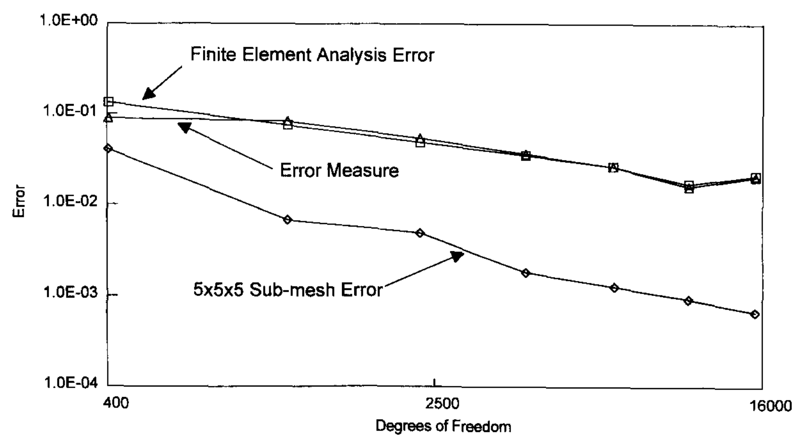

Fig. 17. Error plots for vibrating portal frame-Mode 2.

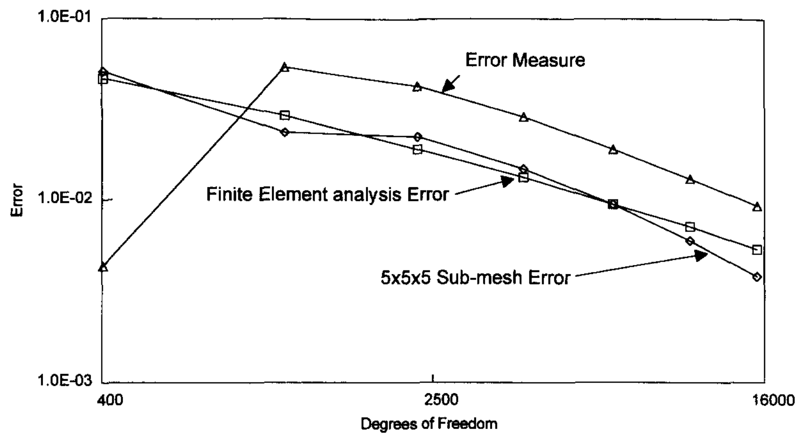

Fig. 18. Error plots for vibrating portal frame-Mode 3.

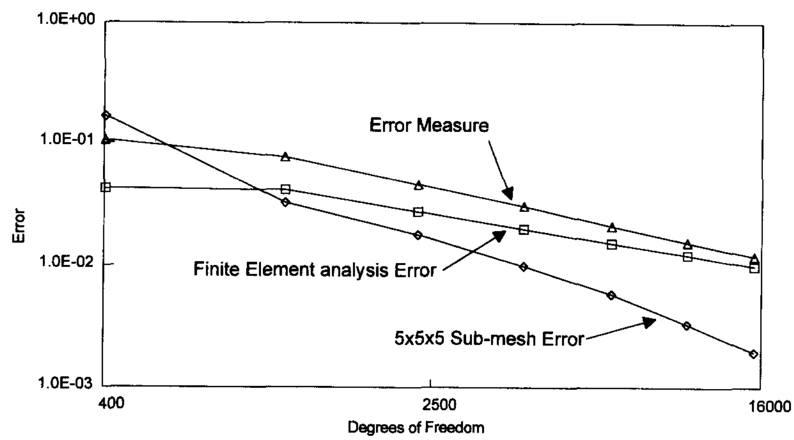

Fig. 19. Error plots for vibrating portal frame-Mode 4.

7. Conclusions

It has been made possible to estimate the finite element error for natural frequency of three dimensional brick elements. The theory of the calculation of an improved eigenvalue to use as a measure for the finite element analysis error is given along with two examples demonstrating the success of the principle.

Three dimensional finite element analysis is not yet highly used in most engineering applications due to the large number of equations to be solved with large band-widths involved. The results examined show adequate error measures for general engineering purposes with the error measure either being extremely close to the finite element analysis error or it produced a conservative estimate of the error.

This procedure is based upon the eigenvector being more accurate than the eigenvalue produced in a finite element analaysis, however the eigenvalue extraction algorithm used in the finite element package is an iterative technique based solely upon achieving an accurate eigenvalue. The eigenvector is only a bi-product of the extraction technique, and as such has no convergence checks upon the accuracy of the eigenvector as the solution progresses. The eigenvector develops inaccuracies which are transferred to the improved eigenvalue, which cause the improved eigenvalue to have a non uniform convergence for higher degrees of freedom and be a possible source of error in the improved eigenvalue.

An additional routine may be added to the finite element package to improve the numerical accuracy of the eigenvectors for models with high numbers of degrees of freedom to obtain improved eigenvalues and a subsequent error measures for the finite element analysis results.

Acknowledgements

The first author is supported financially by the Australian Postgraduate Research Award (Industry) in conjunction with G+D Computing of Sydney.

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