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Theoretical modelling of post - buckling contact interaction of a drill string with inclined bore-hole surface

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Abstract. At present, the time of easy oil and gas is over. Now, the largest part of fossil fuels is concentrated in the deepest levels of tectonic structures and in the sea shelves. One of the most cumbersome operations of their extraction is the bore-hole drilling. In connection with austere tectonic and climate conditions, their drivage every so often is associated with great and diversified technological difficulties causing emergencies on frequent occasions. As a rule, they are linked with drill string accidents. A key role in prediction of these situations should play methods of theoretical modelling. For this reason, there is a growing need for development and implementation of new numerical methods for computer simulation of critical and post-critical behavior of drill strings (DSs). In this paper, the processes of non-linear deforming of a DS in cylindrical cavity of a deep bore-hole are considered. On the basis of the theory of curvilinear flexible rods, non-linear constitutive differential equations are deduced. The effects of the longitudinal non-uniform preloading, action of torque and interaction between the DS and the bore-hole surface are taken into account. Owing to the use of curvilinear coordinates in the constraining cylindrical surface and a specially chosen concomitant reference frame, it became possible to separate the desired variables and to reduce the total order of the equation system. To solve it, the method of continuation the solution by parameter and the transfer matrix technique are applied. As a result of the completed numerical analysis, the critical states of the DS loading in the cylindrical channels of inclined bore-holes are found. It is shown that the modes of the post-critical deforming of the DS are associated with its irregular spiral curving prevailing in the zone of bottom-hole-assembly. The possibility of invariant state generation during post-critical deforming is established, condition of its bifurcation is formulated. It is shown that infinite variety of loads can correspond to one geometrical configuration of the DS. They differ each from other by contact force functions.

Keywords: drill string; bore-hole cavity; post-critical buckling; contact forces; invariant state

1. Introduction

Improvement of technology and techniques for drilling deep oil and gas bore-holes is one of the most important problems of the modern mining industry. In this technology, the dominant position is acquired by the rotor method. Even now depths of the bores drilled by this way achieve several kilometers, but the problem of extraction of oil and gas from deeper tectonic levels continues to be urgent.

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In its use for drivage of vertical deep bores, when a rotational moment TOB (torque on bit) is applied to the top edge of the drill string (DS) and a force on the drill bit WOB (weight on bit) is created by its weight, the system functioning can be attended by occurrence of a series of mechanical phenomena exerting a detrimental affect on the whole working process. Among these are

(1) the DS stability loss at its lower part following the buckling mode typical for a rod stretched, compressed and twisted simultaneously (Lubinski 1987, Kwon 1988, Mitchell 2007, Mitchell and Miska 2006);

(2) excitation of longitudinal and bending vibrations under action of non-stationary perturbations of technological character (Gulyayev and Borshch 2011);

(3) excitation of the DS whirling vibration conditioned by the bit structure, geometrical imperfections, and imbalance of the whole system and its parts;

(4) initiation of the DS torsion (stick slip) vibration at its accelerating and braking, as well as a result of their self-excitation by non-linear friction forces of discontinuous interaction between the bit and processed rock (Gulyayev and Glushakova 2011).

Similar effects occur in drill strings operating in curvilinear bores (Gulyayev *et al*. 2011, Huang and Pattillo 2000).

The noted phenomena can lead to emergencies accompanied by the DS tube rupture, the bit capture or penetration of the tube segments into the rock in the zones of their contact and rubbing, deflection of the bore axis from vertical and its distortion.

The efforts to simulate theoretically static and dynamic behavior of the DS in the drilling process are associated with the necessity of integrating differential equations of their equilibrium and vibrations in the range of large length of the DS. These problems possess essential analytical and computational difficulties stemming from the system dependence on complicated combination of static and dynamic force factors acting on the DS in its working.

The most typical critical phenomena of the DS loading is its stability loss. There are two dissimilar approaches to its analysis, which differ by targets, statements, methods of solutions and results.

The first one consists in study of Eulerian stability of the DS and it is associated with the statement of the Sturm-Liouville boundary value problem for the equations of the rotary DS quasi-static equilibrium Gulyayev *et al*. (2009, 2010) solved this problem via the use of the transfer matrix technique. As a result, the eigen-values were calculated and eigen-modes were built for the whole length of the DS. Previously, it was not stated and solved owing to essential theoretical and calculation difficulties.

The principal source of these difficulties is the non-uniform field of internal longitudinal forces in the DS formed by gravity forces of the bit, stabilizers and the DS itself, as well as the vertical reaction of the bit contact interaction with the destructed rock medium (WOB). These forces make an essential impact on the DS stability and vibrations as it occurs in elongated structures.

Of essential importance is also the DS rotation with the resulting generation of centrifugal and Coriolis' inertia forces (Gulyayev *et al*. 2009), but in the DS they are realized in more complicated forms because proceed in combination with other mechanical effects.

In analysis of the DS behavior, less attention is generally focused on action of a torque bringing the DS into rotation and serving to produce the moment cutting the rock (TOB). However, notwithstanding the fact that it is uniform along the DS length, its influence on the DS deformation is one of the most essential, because the most conspicuous change of the buckling and vibrating modes are connected with its presence. These modes acquire shapes of irregular spatial spirals with varying diameters and pitches. Besides, as shown by Gulyayev *et al*. (2009), Gulyayev *et al*. (2010), wavelets of very small lengths and amplitudes are superimposed on these spirals. Taking into account the TOB action leads to essential complicating the problem statement, since the DS element displacements in the two relatively perpendicular planes appear to be interconnected and the total order of the differential equations doubles. In reference (Ziegler 1968), the problems about stability of twisted rods prestressed by constant internal longitudinal forces are stated and solved analytically. If the forces change along the DS, the analytical solution cannot be constructed.

In service, washing liquid (mud) required to remove the crushed particles of the destructed rock moves down inside the DS. It is known, that internal flows of liquids in tubes can lead to their divergent instability. But if liquid runs out from a free end of a tube, the forces generated by it are non-conservative and they generate flutter vibrations of the tube (as it sometimes occurs with free ends of hoses). Taking into consideration that the mud moves with moderate velocity but its density achieves 2 g/sm^3, it can be concluded that its influence on stability and vibrations of the DS is perceptible too. Notice, that stability and vibrations of rectilinear and curvilinear (spiral) tubes under action of heterogeneous flows of liquids are considered in reference (Gulyayev and Tolbatov 2004).

The second approach is related to consideration of a post-critical equilibrium of the DS inside the bore-hole. It is based on the assumption that the post-critical shape of the DS segment represents some regular or irregular cylindrical spiral with the prescribed radius, which is equal to the clearance between the DS and the bore-hole. Owing to this, it becomes possible to simulate the post-critical behavior of the DS through the statement of an inverse problem of the flexible curvilinear rod theory and to calculate internal and external (contact and friction) forces acting on the DS. It is generally assumed that only homogeneous external axial force, torque and wall contact force (WCF) are the dominant factors influencing on this phenomenon. Main peculiarities of the DS post-critical behavior are studied by Kwon (1988), Mitchell (2007), Mitchell and Miska (2006), Sun and Lukasiewicz (2006), Van der Heijden *et al*. (2002), Yongping and Youhong (2012), Zhao *et al*. (2010), and Fang *et al*. (2013). As a rule, their problem statements are based on assumption that in the post-critical states, the DS acquires regular spiral shape with constant pitch and then the main aim of these researches consists in acquisition of simple analytic solutions that are suitable for hand calculations. Similar approaches are also used in references (Krauberger *et al*. 2007, Singh and Kumar 2008). In paper (Thompson *et al*. 2012), critical values of axial forces and torques are found, the corresponding contact forces are calculated. Good agreement with experimental data is noted.

It should be remarked that in the process of the DS work all the considered factors act simultaneously with different values of parameters determining their magnitudes, for this reason the phenomena generated are characterized by wide variety and severity. Because of this, even general regularities of the proceeding of these phenomena have not been thoroughly studied until now.

One of the causes of this fact is conditioned by large lengths of the DSs which change in operation. In this connection the fields of strains and stresses formed in the DSs in the course of their vibrations and stability loss have the form of short boundary effects where the most complicated and dangerous processes occur.

As shown below, the differential equations describing these phenomena belong to the so called singularly perturbed type. To obtain their correct solutions special high precision numerical methods should be used.

In this paper, the second approach is used for investigation of post-critical deforming of elongated drill strings in cylindrical cavities of vertical and inclined bore-holes. The history-dependent process of a drill string transforming is traced on the basis of non-linear analysis methods permitting to take into account the possibility to achieve the invariant (locked up) states of the system. In these states, an infinite variety of loads can correspond to one geometrical configuration of the DS. They differ each from other by contact force functions. The examples of bore-holes with ideal rectilinear axial lines are considered, the possibility of variable clearance between the DS and bore-hole and influence of friction forces are discussed. The analysis results are given.

2. Singularly perturbed problem of a long DS bending

In mathematical physics, many differential equations simulating real phenomena contain different parameters. So, solutions of these equations are subject to wide variations with varying of their parameters. A. Poincare and A. Lyapunov appear to be the first ones who analysed the regular types of these equations including the ε parameter in their right-hand sides

$$x'' = F(t, x, x', \varepsilon) \qquad 0 \le t \le 1 \tag{1}$$

Here, the F function is differentiable with respect to t, x, x' and ε in some neighbourhood of the value ε=0 inside the interval $0 \le t \le 1$. Calculus of this type equations did not met with any difficulties and they were studied comprehensively.

But Chang and Howes (1984) showed that the situation changes substantially, if the small parameter $0<\varepsilon<<1$ is before the senior derivative

$$\varepsilon x'' = f(t, x, x') \tag{2}$$

because in this event Eq. (2) defines an intermediate group of phenomena located between solutions of two different equations

$$x'' = f(t, x, x') \qquad \text{and} \qquad 0 = f(t, x, x')$$

The problems described by this type of equations are named singularly perturbed. Chang and Howes (1984) studied them proceeding from the positions of applied mathematics. They demonstrated, considering two-point boundary value problem, that the singularly perturbed problems are poorly conditioned and have solutions in the forms of boundary effects, possessing poor convergence of calculations.

Studying the phenomenon of torsion autovibration of a drill string, Gulyayev and Glushakova (2011) found that Eq. (2) can determine nearly discontinuous relaxation oscillations with fast and slow motions. But singular character of differential equations does not need to be necessarily associated with the small multiplier before the senior derivative. As often happens, the singular perturbation can be implicit, for example, if it is caused by large size of the x function definition. Now, turn to the discussion of equations

$$EI\frac{d^4u}{dz^4}-\frac{d}{dz}\left(T\frac{du}{dz}\right)-\frac{d}{dz}\left(M_z\frac{d^2v}{dz^2}\right)=0$$
$$EI\frac{d^4v}{dz^4}-\frac{d}{dz}\left(T\frac{dv}{dz}\right)+\frac{d}{dz}\left(M_z\frac{d^2u}{dz^2}\right)=0 \qquad (0\le z\le L) \tag{3}$$

which determine critical states of a DS in a vertical bore-hole (Gulyayev *et al*. 2009). Here z is the coordinate axis coinciding with the bore-hole axis; $u(z)$, $v(z)$ are the functions of the DS elastic displacements in the planes xOz, yOz, correspondingly; EI is the bending stiffness; $T(z)$, M_z(zare the axial internal force and torque, correspondingly.

These equations are specified in the domain $0\le z\le L$ (L>>1). Through high order, their severity considerably excesses complications of other singularly perturbed equations known in scientific literature. To appreciate the plausibility of this statement, introduce new independent variable Z with the equality $Z=z/L$. Then, Eq. (3) will acquire the form

$$\frac{1}{L^4}EI\frac{d^4u}{dZ^4}-\frac{1}{L^2}\frac{d}{dZ}\left(T\frac{du}{dZ}\right)-\frac{1}{L^3}\frac{d}{dZ}\left(M_z\frac{d^2v}{dZ^2}\right)=0$$
$$\frac{1}{L^4}EI\frac{d^4v}{dZ^4}-\frac{1}{L^2}\frac{d}{dZ}\left(T\frac{dv}{dZ}\right)+\frac{1}{L^3}\frac{d}{dZ}\left(M_z\frac{d^2u}{dZ^2}\right)=0 \qquad (0\le Z\le 1) \tag{4}$$

By way of example, if L=5000 m, then $\varepsilon=\frac{1}{L^4}=0.16\cdot 10^{-14}$ and the first terms in (4) become negligibly small. For this reason, Eq. (4) are singularly perturbed in ($0\le Z\le 1$), as well as Eq. (3) in ($0\le z\le L$).

An effort to regularize Eq. (4) with small parameter $\varepsilon=1/L^4$ by discarding the terms $(EI/L^4)d^4u/dz^4$ and $(EI/L^4)d^4v/dz^4$ leads to the more regular equations

$$\frac{1}{L^2}\frac{d}{dZ}\left(T\frac{du}{dZ}\right)+\frac{1}{L^3}\frac{d}{dZ}\left(M_z\frac{d^2v}{dZ^2}\right)=0$$
$$\frac{1}{L^2}\frac{d}{dZ}\left(T\frac{dv}{dZ}\right)-\frac{1}{L^3}\frac{d}{dZ}\left(M_z\frac{d^2u}{dZ^2}\right)=0 \qquad (0\le Z\le 1) \tag{5}$$

which, nevertheless, are of lower order and, for this reason, lose the basic features of the original Eqs. (3) and (4). In this case, simplified consideration of the DS deforming achieved by ignoring the equation terms with small parameter leads to the degeneration of the initial equations and to distortion of the simulation results. Indeed, now Eq. (5) describe equilibrium of a thread (or a string and thereby justify the drill string term) and they cannot be used for analysis of local bending of a DS or for modelling boundary effects. For this reason, the stability investigation of the long DSs and their post-critical buckling should be performed on the basis of complete systems of differential equations, though they are singularly perturbed and their solutions have poor convergence (Chang and Howes 1984).

These conclusions were corroborated for eigen value problems of critical buckling of long vertical DSs (Gulyayev *et al*. 2009, 2010) and their free vibrations (Gulyayev and Borshch 2011).

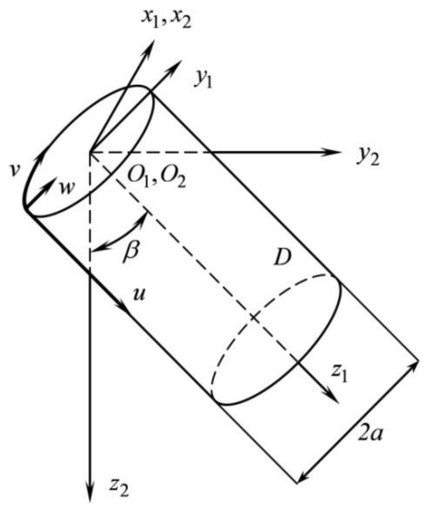


Fig. 1 Schematic of cylindrical surface *D* constraining displacements of the DS axis line in the inclined bore-hole

It was shown, that stability loss of a long DS is accompanied by its spiral curving prevailing in lower part, while free vibrations proceed with formation of running spiral waves. In what follows, the post-critical evolution of the DS shape will be considered. The non-linear theory of curvilinear flexible rods will be used.

3. Statement of the problem about post-critical evolution of a DS shape

Consider non-linear post-critical bending of a DS inside cylindrical cavity of a rectilinear bore-hole. Assume that in the considered cases the surfaces of the DS and bore-hole are in contact along entire length of the DS. The bore-hole axis line is inclined to the vertical under angle β. Then the DS axis line L can displace relative to the bore-hole in some cylindrical surface D of radius $a=r_2-r_1$, where r_1 is the DS radius and r_2 is the bore-hole radius (Fig. 1).

As the modern DSs located in deep bore-holes can be compared by condition of geometric similarity with a human hair, the following initial assumptions, typical for the theory of curvilinear flexible rods, can be accepted

- the dimensions of the rod cross-section are very small in comparison with the length and radii of curvature and torsion of its axial line;
- the displacements of the rod elements can be comparable with its length;
- in the process of the rod bending, the length of its axial line does not change;
- the function of internal axial force in the rod can be found from the conditions of equilibrium of all internal and external forces;
- notwithstanding the possible large displacements of the rod elements, the curvature radii of its axial line are so large in comparison with the dimensions of the rod cross-sections that the bending strains of the rod remain small and elastic;
- the ambient rock medium is absolutely rigid and is simulated as a constraint.

Introduce the Cartesian coordinate system $O_1x_1y_1z_1$ connected with the bore-hole and the reference frame $O_2x_2y_2z_2$. Axis O_2z_2 is vertical, axes O_1x_1 and O_2x_2 coincide. Draw coordinate lines u and v in the D surface, direct line u along the surface D generatrix and orient line v in the circumferential direction (Fig. 1).

Position of the L curve in the D surface can be specified by three projections of the radius-vector $\boldsymbol{\rho}(t)$ on the appropriate coordinate axes

$$x_1 = x_1(s), \qquad y_1 = y_1(s), \qquad z_1 = z_1(s) \tag{6}$$

with constraint

$$x_1^2 + y_1^2 = a^2 \tag{7}$$

Here s is the natural parameter measured as a line L length from some origin point till the considered one.

In 3D deforming, the DS is bent in two orthogonal planes, it is twisted and compressed (stretched) by axial force. So, the total order of constitutive equations based on presentations (6), (7) equals 12. However the problem statement can be essentially simplified if to analyze the deforming process in 2D space of the D surface and determine positions of the DS elements by two equalities

$$u = u(s), \qquad v = v(s) \tag{8}$$

without use of constraint (7). Below it will be shown that in this case, the total order of the system can be reduced to six (sometimes, to four) through the use of a special moving reference frame.

In modelling post-critical states of the DS, assume that it is no longer rotating, the centrifugal inertia forces equal zero and influence of washing liquid on the DS bending can be neglected. At the same time, as a result of its flowing and slight shaking of the DS in its small motion, the friction forces are alleviated and they can be also ignored. Then, the DS deforming is elastic and its stress-strain state is determined by the principal vectors of internal forces $\mathbf{F}(s)$, internal moment $\mathbf{M}(s)$ and the vector $\mathbf{f}(s)$ of external distributed forces, which consists of the gravity force vector $\mathbf{f}^{gr}(s)$ and contact force vector $\mathbf{f}^{c}(s)$. These forces and moments obey the equations of the DS element equilibrium

$$\frac{d\mathbf{F}}{ds} = -\mathbf{f}, \qquad \frac{d\mathbf{M}}{ds} = -\boldsymbol{\tau} \times \mathbf{F} \tag{9}$$

where $\boldsymbol{\tau}$ is the unit vector directed along the tangent to the L line.

Analyze general properties of these equations. Firstly, they are invariant relative to any coordinate system and to supplement them with geometric equations, different approaches can be used. As the rod elements are considered to experience large displacements and rotations, it seems to be natural to apply the notion of finite rotation for their description. These variables are discussed in the papers by Argiris (1982), Atluri and Cazzani (1995), Ibrahimbegovic (1997). To accomplish this task, the approaches based on application of quaternions, Rodrigues' parameters, and Euler's angles can be also used. Nevertheless, as shown in Gulyayev *et al*. (1992), Gulyayev and Tolbatov (2004), in the case of study of elastic constraint-free curvilinear rod bending, the system of direction cosines of a moving trihedron can be more attractive, despite the fact that its nine components are redundant in comparison with other systems. The advantages of direction cosines consist in their simplicity and obviousness, they are useful for determination of any orientation of a body and never degenerate.

If the inertia moments of the rod cross-sections do not depend on orientations of their principal inertia axis (annulus, circle, quadrate, etc.), then the Frenet trihedron is the most convenient reference frame. Its tangent unit vector $\boldsymbol{\tau}$, unit vector of principal normal **n** and unit vector of binormal **b** are calculated by the formulae

$$\boldsymbol{\tau} = \frac{d\boldsymbol{\rho}}{ds}, \qquad \mathbf{n} = R\frac{d\boldsymbol{\tau}}{ds}, \qquad \mathbf{b} = \boldsymbol{\tau}\times\mathbf{n} \tag{10}$$

Here, R is the curvature radius of the DS axis line.

These vectors completely determine orientations of the rod elements under conditions of their arbitrary rotations. Their properties are expounded in (Dubrovin *et al*. 1992)

Then, the Darboux vector can be introduced

$$\boldsymbol{\Omega} = k_R\mathbf{b} + k_T\boldsymbol{\tau} \tag{11}$$

which numerically equals the vector of angular velocity of the trihedron rotation at its motion with a unit linear velocity of its origin along the rod axis. Here k_R is the axis curvature, k_T is its torsion.

Nevertheless, usage of the moving reference frame connected with principal axes of the rod cross-section is more universal as it is applicable to the rods of arbitrary profile. In this case vector $\boldsymbol{\Omega}$ is replaced by the vector

$$\boldsymbol{\omega} = p\mathbf{i} + q\mathbf{j} + r\mathbf{k} \tag{12}$$

where **i**, **j**, **k** are the unit vectors of the introduced reference frame; p, q are the appropriate curvatures of the L line; r is its torsion.

Then, the absolute derivatives $d\mathbf{F}/ds$ and $d\mathbf{M}/ds$ are calculated through the Darboux rule

$$\frac{d\mathbf{F}}{ds} = \frac{\tilde{d}\mathbf{F}}{ds} + \boldsymbol{\omega}\times\mathbf{F}, \qquad \frac{d\mathbf{M}}{ds} = \frac{\tilde{d}\mathbf{M}}{ds} + \boldsymbol{\omega}\times\mathbf{M} \tag{13}$$

Here $\tilde{d}/ds$ is the symbol of local derivative.

Substituting (13) into (9) and projecting these vector equations onto the system *oxyz* axes, one obtains

$$\begin{cases} dF_x/ds = -qF_z + rF_y - f_x \\ dF_y/ds = -rF_x + pF_z - f_y \\ dF_z/ds = -pF_y + qF_x - f_z \end{cases} \tag{14}$$

$$\begin{cases} dM_x/ds = -qM_z + rM_y + F_y \\ dM_y/ds = -rM_x + pM_z - F_y \\ dM_z/ds = -pM_y + qM_x \end{cases} \tag{15}$$

In the framework of the formulated theory, the forces F_x, F_y, F_z are merely static factors and are determined by Eq. (14), while the moments M_x, M_y, M_z are represented in the form of elasticity equalities

$$M_x = Ap\,, \qquad M_y = Bq\,, \qquad M_z = Cr \tag{16}$$

where the bending stiffnesses A, B and torsion stiffness C are determined by the formulae

$$A = B = EI\,, \qquad C = GI_0 \tag{17}$$

Here E is the DS material elasticity module in tension; G is its module in shear; I and I_0 are axial and polar inertia moments of the DS cross-section area.

Eqs. (14)-(17) should be complemented by appropriate geometric correlations. Then, the total order of constitutive equations of the theory of curvilinear flexible rods will be equal to twelve (Gulyayev *et al*. 1992).

This system has a rather complicated structure which becomes harder if the rod movement is limited by a constraining surface. To overcome this difficulty, the approach used in classical mechanics for analogous equations will be used. Thus, insertion of relations (16) into Eq. (15) gives

$$A\frac{dp}{ds} = -(C-B)qr + F_y$$

$$B\frac{dq}{ds} = -(A-C)rp - F_x$$

$$C\frac{dr}{ds} = -(B-A)pq$$

These correlations are analogous to equations of a rigid body motion relative to an immovable point considered in classical mechanics by Goldstain *et al*. (2011) if to replace the parameter s, curvatures p, q, r and stiffnesses A, B, C by the time t, angular velocity components p, q, r and inertia moments A, B, C correspondingly. This analogy is not only a mere and curious feature of Eqs. (14), (15) but suggests a prompt how to simplify the problem about bending a rod constrained by a cylindrical surface. Indeed, in the problem about nonholonomic rolling a rigid body on a curvilinear surface, the simplest equations are arrived by consideration of the moments with respect to the point of contact (Neimark and Fufaev 1972). In this case, the equilibrium equations are formulated in the movable coordinate system with the origin situated at the contact point and one of the axes directed normally to the tangent plane.

Following this hint, analysis of the DS bending will be pursued in the moving right-handed coordinate system $Oxyz$ with axis Ox oriented along internal normal to the surface D and axis Oz directed along the tangent to the curve L (Fig. 2).

Choice of this reference frame permits to convert the problem difficulty associated with additional constraint (7) into advantage conditioned by the fact that the L curve lies in the D surface. In this case, it becomes possible to turn from three desired variables $x_1(s)$, $y_1(s)$, $z_1(s)$ in the 3D space to two variables $u(s)$, $v(s)$ in the D surface and to exclude constraint (7) from consideration.

The second advantage of this approach consists in simplification of the procedure of the L line curvatures calculation because in doing so, the values k_R, k_T and p, q, r become useless and new curvature parameters k_x, k_y, k_z are introduced. They are expressed through the known curvatures of the D surface.

Then, vector $\boldsymbol{\omega}$ will be determined by the equality

$$\boldsymbol{\omega} = k_x\mathbf{i} + k_y\mathbf{j} + k_z\mathbf{k} \tag{18}$$

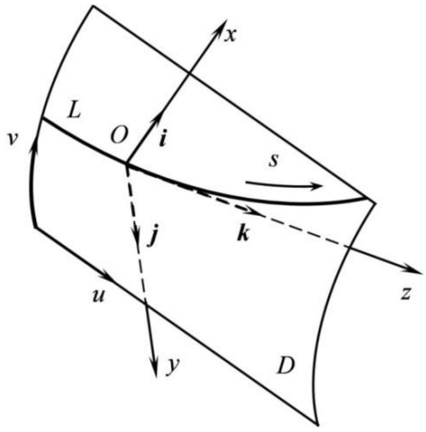


Fig. 2 Schematic of movable coordinate system *Oxyz*

where k_x and k_y are the curve L curvatures in the planes yOz, xOz correspondingly; k_z is its torsion.

Plane yOz is tangent to the D surface, because of this, curvature k_x coincides with the geodesical that for the L curve (Dubrovin *et al*. 1992). Its analytical presentation depends on the shape of the curve L projection on the tangent plane yOz at the considered point and on the surface D metrics which in the considered case is determined by the equality

$$(ds)^2 = (du)^2 + a^2 (dv)^2 \tag{19}$$

Then, the k_x curvature can be expressed through the desired functions $u(s)$, $v(s)$ as follows

$$k_x = -a(u''v' - v''u') \tag{20}$$

Here symbol prime denotes the procedure of differentiation with respect to s.

Inasmuch as plane xOz contains unit vector $\mathbf{i}$, normal to surface D, the k_y curvature of the L curve coincides with the normal curvature of the D surface in the direction of the L curve. It is expressed through the principal curvatures k_v=0, k_u=1/a of the cylindrical surface D with the use of the Euler formula

$$k_y = k_v \cos^2\theta + k_u \sin^2\theta \tag{21}$$

Here θ is the angle between the Oz axis and principal direction, appropriate to the k_v curvature.

In accordance with Eq. (21), one can gain

$$k_y = a(v')^2 \tag{22}$$

Torsion k_z for the cylindrical surface D is determined by the formula (Gulyayev *et al*. 1992)

$$k_z = u'v' \tag{23}$$

In the reference frame *Oxyz*, Eq. (9) take the form

$$\frac{\tilde{d}\mathbf{F}}{ds} = -\boldsymbol{\omega}\times\mathbf{F} - \mathbf{f}^{gr} - \mathbf{f}^{c}, \qquad \frac{\tilde{d}\mathbf{M}}{ds} = -\boldsymbol{\omega}\times\mathbf{M} - \mathbf{k}\times\mathbf{F} \tag{24}$$

Vectors f^{gr}, f^c used in Eq. (24) have the components

$$f_x^{gr} = -f^{gr} \sin\beta \cos v$$
$$f_y^{gr} = f^{gr}\left(\cos\beta \cdot av' + \sin\beta \cdot \sin v \cdot u'\right)$$
$$f_z^{gr} = f^{gr}\left(\cos\beta \cdot u' - \sin\beta \cdot \sin v \cdot av'\right)$$
$$f_x^c = f^c, \qquad f_y^c = 0, \qquad f_z^c = 0 \tag{25}$$

Here $f^c(s)$ is the additional required variable; f^{gr} is the intensity of the distributed gravity force

$$f^{gr} = g(\rho_t - \rho_l)F \tag{26}$$

where g=9.81 m/s^2; ρ_t, ρ_l are the densities of the tube material and mud, correspondingly; F is the area of the DS tube cross-section.

Based on Eqs. (18), (25) rewrite Eq. (24) in scalar form singly for the relations of force group

$$\begin{aligned} \frac{dF_x}{ds} &= -k_y F_z + k_z F_y - f_x^{gr} - f^c \\ \frac{dF_y}{ds} &= -k_z F_x + k_x F_z - f_y^{gr} \\ \frac{dF_z}{ds} &= -k_x F_y + k_y F_x - f_z^{gr} \end{aligned} \tag{27}$$

and moment group

$$\begin{aligned} \frac{dM_x}{ds} &= -k_y M_z + k_z M_y + F_y \\ \frac{dM_y}{ds} &= -k_z M_x + k_x M_z - F_x \\ \frac{dM_z}{ds} &= -k_x M_y + k_y M_x \end{aligned} \tag{28}$$

In Eqs. (28), moments M_x, M_y, M_z are calculated by the formulae

$$M_x = EIk_x, \qquad M_y = EIk_y \tag{29}$$

With the use of Eqs. (20), (22), (23) and two first equations of system (28) the equalities can be gained

$$\begin{aligned} F_x &= -2EIav'v'' - a\left(M_z - EIu'v'\right)\left(u''v' - v''u'\right) \\ F_y &= -EIa\frac{d}{ds}\left(u''v' - v''u'\right) + a\left(M_z - EIu'v'\right)\left(v'\right)^2 \end{aligned} \tag{30}$$

Stemming from system (29), the third equation of system (28) can be rewritten in the form

$$\frac{dM_z}{ds} = -k_x EIk_y + k_y EIk_x = 0 \tag{31}$$

Then, the torque preserves its constant value

$$M_z = const$$

which is determined by corresponding boundary condition.

Relations (20), (22), (23), (27)-(30) permit one to derive the constitutive system of six first order homogeneous differential equations

$$\frac{dF_y}{ds} = 2EI \cdot a \cdot u'(v')^2 v'' - (M_z - EIu'v')u'v'k_x + k_x F_z - f_y^{gr}$$

$$\frac{dF_z}{ds} = -k_x F_y - 2EIa^2 (v')^3 v'' + a(M_z - EIu'v')(v')^2 k_x - f_z^{gr}$$

$$\frac{dk_x}{ds} = -\frac{1}{EI} a(v')^2 M_z + au'(v')^3 + \frac{1}{EI} F_y$$

$$\frac{dv}{ds} = v'$$

$$\frac{d(v')}{ds} = \frac{k_x u'}{a}$$

$$\frac{du}{ds} = \sqrt{1 - a^2 (v')^2} \tag{32}$$

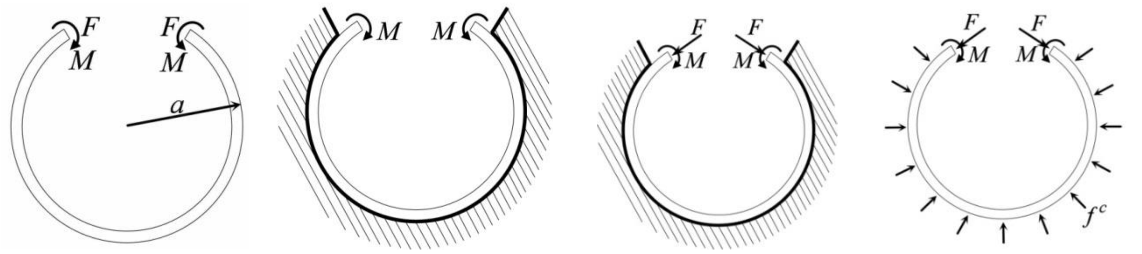
Owing to application of the special reference frame *Oxyz*, it became possible to express the variables F_x and M_y through geometry parameters of the *D* surface, to exclude them from consideration, and to reduce the total order of system (32) to six which is twice as less as the order of the general theory of curvilinear rods, in spite of the fact that one extra unknown function $f^c(s)$ is added to the required variables of the problem. It is not incorporated into system (32) owing to its orthogonality to the tangent plane *yOz* and can be calculated from the formula

$$f^c = -k_y F_z + u'v'F_y - f_x^{gr} - dF_x/ds \tag{33}$$

after solving the problem. Its values are also usable as a check upon adequacy of starting assumption concerning overall contact between the DS and the bore-hole surface. If function $f^c(s)$ is positive overall diapason $0 \le s \le S$ of the DS length, then the found solution is correct. If this condition is not satisfied throughout, the DS comes out the contact with the bore-hole surface and the obtained solution is not adequate.

The second feature of Eq. (32) is that variables *u* and *v* are not present in it. For this reason, the *v'* variable can be considered as desired one and the *u'* value can be excluded with the use of equality $u' = \sqrt{1 - a^2 (v')^2}$. Then, the fourth and sixth equations will drop out and the system order will be reduced to four. But this procedure is not performed for the sake of simplicity of the Runge-Kutta method application.

In realization of practical computations, system (36) is supplemented by six boundary conditions and initial conditions for every desired variable, corresponding, as a rule, to initially rectilinear shape of the DS.



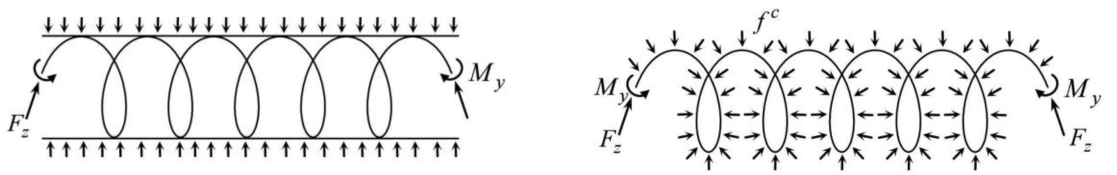
(a) A rod bent by a moment

(b) The bent rod inserted into a rigid clip

(c) The bent rod loaded by force F

(d) Scheme of active (F, M) and reaction f^c forces

Fig. 3 Invariant state of a rod in a circular clip



(a) Spiral rod in a cylindrical cavity

(b) Active (F_z, M_y) and reaction f^c forces acting on the rod

Fig. 4 Invariant state of a rod in a rigid cylindrical cavity

4. Invariant states of the drill strings and their stability

With the aid of Eq. (32), one of the important properties of post-critical deforming of the DS in the bore-hole channel has been gained. It consists in the existence of the so called invariant states of the DS wherein it becomes insensitive to a certain group of external actions. Inasmuch as such states are rare in occurrence in mechanics of deformable bodies and are not practically described in scientific literature, consider a simple example facilitating understanding of this phenomenon of the DS bending. Let bending moments M be applied to the ends of a rectilinear flexible rod. Under their action, the rod will take the shape of a circumference arc of radius a (Fig. 3(a)). Insert this rod with the moments into a rigid circular clip of radius a (Fig. 3(b)). In that event, contact interaction between the rod and clip is lacking and the distributed contact force f^c=0.

If thereafter external axial compressive forces F are applied to the ends of the deformed rod (Fig. 3(c)), then the same internal forces appear at every cross-section of the rod and, besides, the distributed forces $f^c=F/a$ of contact interaction come into being between the surfaces of the rod and the clip (Fig. 3(d)).Since, in this case, the stress-strain state of the bent rod remains unchanged, it is said to be invariant (insensitive) with respect to the external forces F.

The considered example has a three-dimension analogue. Take a rectilinear elastic rod, put it into a cylinder cavity of radius a and, in the absence of gravity forces, axial force F_z, and torque M_z, apply to its ends the bending moments M_y and shear forces F_y, transforming it into a cylindrical spiral of radius a (Fig. 4(a)). In this event, the equalities $v'=const$, $v''=0$, $k_x=0$, $M_y=EIa(v')^2$, $F_y=-EIa(v')^4[1-(av')^2]$ follow from Eqs. (32), (33) and a normal distributed contact force $f^c=-EIa(v')^4[1-(av')^2]$ is brought about between the cylinder surface and the spiral.

In line with Eq. (33), such elastic spiral with negative f^c can retain its shape only in the case, when it is wound up the cylindrical surface. But the most interesting feature of this system consists in the fact that its equilibrium will not be violated if to apply an axial compressive forces F_z to its ends. Indeed, as F_z is present only in the first equation of system (33) and has the multiplier k_x=0, then the spiral will not change its geometry for any F_z and only the contact force f^c will assume the new value

$$f^c = -EIa(v')^4[1-(av')^2] - k_y F_z \tag{34}$$

Since the first summand of Eq. (34) is negative and the second one is positive (for negative compressive force $F_z<0$), the states are possible when the contact between the spiral and cylinder surface is external ($f^c<0$), internal ($f^c>0$) and the contact interaction is absent ($f^c=0$).

Similar reasoning can be deduced for torque M_z. Really, as it follows from the third equation of system (32), if torque M_z is applied, only the force

$$F_y = M_z k_y - M_y u'v' \tag{35}$$

changes, while other equations are satisfied identically.

Thus, in the state under discussion, the considered system appears to be invariant (insensitive) with respect to constant axial force F_z and torque M_z. In the drilling practice, these situations are known as helical lockup. However, this does not mean that parameters F_z and M_z can acquire any values, because the elastic spiral can lose its equilibrium stability with their enlargement. In order to find critical values F_z^{cr} and M_z^{cr}, linearize Eq. (32) in the vicinity of the analyzed state of the spiral equilibrium under action of constant force F_z and moment M_z. Let the spiral shape be determined by parameter $v'=const$. Then, one has $u'=\sqrt{1-a^2(v')^2}=const$, k_x=0, $F_y=a(M_z-EIu'v')(v')^2=const$. These values satisfy Eq. (32). To check the system stability, give small perturbations δF_y, δF_z, δF_x, δv, $\delta v'$, δu to the considered functions and put variables $F_y+\delta F_y$, $F_z+\delta F_z$, $k_x+\delta k_x$, $v+\delta v$, $v'+\delta v'$, $u+\delta u$ into Eq. (32). After retaining the linear members relative to the perturbing variations in these correlations, the linearized equations

$$\begin{aligned}
&d\delta F_y/ds = 2EIau'(v')^2\delta v'' + [(EIu'v'-M_z)u'v' + F_z]\delta k_x\\
&d\delta F_z/ds = -2EIa^2(v')^3\delta v'' - [(EIu'v'-M_z)a(v')^2 + F_y]\delta k_x\\
&d\delta k_x/ds = av'[3u'v' - a^2(v')^3/u' - 2M_z/EI]\delta v' + (1/EI)\delta F_y\\
&d\delta v/ds = \delta v'\\
&d\delta v'/ds = (u'/a)\delta k_x\\
&d\delta u/ds = -[a^2v'/\sqrt{1-a^2/(v')^2}]\delta v'
\end{aligned} \tag{36}$$

are constructed.

Assume that the length of the spiral axis line equals S and it is pinned at its ends. It is necessary to determine the values of parameters F_z and M_z such that linear homogeneous system (36) has non-trivial solutions. Taking into account equalities $F_z=const$, $\delta k_x=(a/u')\delta v''$, transform Eq. (36) to the simpler form

$$\begin{aligned}
&d\delta F_y/ds = [3EIau'(v')^2 - av'M_z + aF_z/u']\delta v''\\
&d\delta v''/ds = [-2u'v'M_z/EI + 3(v')^2 - a^2(v')^4]\delta v' + (u'/aEI)\delta F_y
\end{aligned} \tag{37}$$

With allowance made for the chosen boundary conditions, solution of system (37) is built in the form

$$\delta F_y = C_1 \cos \gamma_n s\,, \qquad \delta v' = C_2 \cos \gamma_n s\,, \qquad \delta v'' = -C_2 \gamma_n \sin \gamma_n s \tag{38}$$

where $\gamma n = n\pi / L$, n=1, 2, 3, ...

Substituting (38) into (36) and cancelling sin $\gamma_n s$, cos $\gamma_n s$, one gains

$$\begin{aligned} &C_1 - [3EIau'(v')^2 - av'M_z + aF_z / u']C_z = 0 \\ &(u' / aEI)C_1 + \{[-2u'v'M_z / EI + 3(v')^2 - a^2(v')^4] + \gamma_n^2\}C_2 = 0 \end{aligned} \tag{39}$$

Equating the determinant of the matrix of system (39) to zero, one obtains the values of critical axial force

$$F_{z,n}^{cr} = -EI\gamma_n^2 + 3u'v'M_z + 2EI(v')^2[2a^2(v')^2 - 3]$$

It is minimal at n=1, so

$$F_z^{cr} = -\pi^2 EI / L^2 + 3u'v'M_z + 2EI(v')^2[2a^2(v')^2 - 3] \tag{40}$$

It is notable that the first term in the right-hand side of this equality represents the Eulerian critical value for the compressed pinned rod. Interestingly also, the torque M_z can enlarge or diminish the F_z^{cr} value depending on its orientation, while the last member always enlarges the F_z^{cr} module, because it is negative by the condition $a^2(v')^2<1$.

The performed analysis of invariant states of DSs in cylindrical cavities of bore-holes is performed under the assumption that the forces of gravity and friction are absent. Nevertheless, even with the use of such statement of the problem this analysis allowed to reveal one more important peculiarity of the DS behavior connected with the possibility of its sticking. In the neighborhood of this state, the problem of its computer simulation is also complicated as in this event indefinite assemblage of the right-hand members of the constitutive equations corresponds to its one solution and this solution becomes insensitive to changes of F_z and M_z.

5. Techniques of the non-linear problem solving

As indicated above, the severity of the stated problem on deforming of a curvilinear rod is caused by essential non-linearity of the constitutive equations and relatively small bending stiffness of the DS provided its length to be large. Therefore, the equations are singularly perturbed and their solutions may have appearances of boundary effects evolving with the change of external perturbations. To simulate these processes, the method of solution continuation by a parameter is used jointly with the Newton method.

Represent Eq. (32) in the vector form

$$\mathbf{q}' = \mathbf{f}(\mathbf{q}, s, \lambda) \tag{41}$$

where $\mathbf{q}(s) = [q_1(s), q_2(s), \ldots, q_6(s)]^T$ is the six-dimensional vector function of the required variables $q_1(s) = F_y(s)$, $q_2(s) = F_z(s)$, $q_3(s) = k_x(s)$, $q_4(s) = v(s)$, $q_5(s) = v'(s)$, $q_6(s) = u(s)$; $\mathbf{f}(\ldots)$ is the vector function of the right members of system (32); λ is the parameter of the external perturbation (load) intensity; prime symbol “ ’ ” denotes derivation procedure relative to s. Notice that introduced parameter λ may be both real and formal one, reflecting some quantitative characteristics of the deformation process.

At every boundary s=0 and s=S of the interval $0 \le s \le S$ of the variable s change, three boundary conditions are preset. Represent them in the vector form

$$\boldsymbol{\varphi}[\mathbf{q}(0),\lambda]=0\,, \qquad \boldsymbol{\psi}[\mathbf{q}(S),\lambda]=0 \tag{42}$$

Eqs. (41), (42) make up non-linear two-point boundary value problem depending on the λ parameter. Let solution $\mathbf{q}^{(0)}(s)$ of system (41), (42) be known for some initial value $\lambda=\lambda^{(0)}$. Selecting it as a basic one, increase parameter λ value by small increment $\delta\lambda^{(0)}$. Then corresponding variation $\delta\mathbf{q}^{(0)}(s)$ of the solution $\mathbf{q}^{(0)}(s)$ can be found following from the linearized equation

$$\frac{d}{ds}\delta\mathbf{q}^{(0)}=\frac{\partial\mathbf{f}}{\partial\mathbf{q}}\delta\mathbf{q}^{(0)}+\frac{\partial\mathbf{f}}{\partial\lambda}\delta\lambda^{(0)} \tag{43}$$

and boundary conditions

$$\frac{\partial\boldsymbol{\varphi}}{\partial\mathbf{q}}\delta\mathbf{q}^{(0)}(0)+\frac{\partial\boldsymbol{\varphi}}{\partial\lambda}\delta\lambda^{(0)}=0\,, \qquad \frac{\partial\boldsymbol{\psi}}{\partial\mathbf{q}}\delta\mathbf{q}^{(0)}(S)+\frac{\partial\boldsymbol{\psi}}{\partial\lambda}\delta\lambda^{(0)}=0 \tag{44}$$

Here Jacobians $\partial\mathbf{f}/\partial\mathbf{q}$, $\partial\boldsymbol{\varphi}/\partial\mathbf{q}$, $\partial\boldsymbol{\psi}/\partial\mathbf{q}$ and vectors $\partial\mathbf{f}/\partial\lambda$, $\partial\boldsymbol{\varphi}/\partial\lambda$, $\partial\boldsymbol{\psi}/\partial\lambda$ are constructed at the state $\mathbf{q}(s)=\mathbf{q}^{(0)}(s)$, $\lambda=\lambda^{(0)}$.

The $\delta\mathbf{q}^{(0)}(s)$ vector function can be found through the use of the transfer matrix method (Arici and Granata 2011, Gulyayev *et al*. 2009) in the form of superposition of particular solutions

$$\delta\mathbf{q}^{(0)}(s)=Y(s)\delta\mathbf{C}^{(0)} \tag{45}$$

where the $Y(s)$ matrix of fundamental solutions is calculated by the Runge-Kutta method, the $\delta\boldsymbol{C}^{(0)}$ vector is found from the conditions of satisfying Eq. (44). Thereafter, solution $\mathbf{q}^{(1)}(s)=\mathbf{q}^{(0)}(s)+\delta\mathbf{q}^{(0)}(s)$, corresponding to the value $\lambda^{(1)}=\lambda^{(0)}+\delta\lambda^{(0)}$, is built.

Continuing this process further, one can find the $\mathbf{q}^{(n)}(s)$ function for other values $\lambda^{(n)}$ of the λ parameter. But it is essential to note, that solutions $\mathbf{q}^{(n)}(s)$ satisfy Eq. (42) within the limits of calculation errors. To compensate these errors, the residuals of Eq. (42)

$$\mathbf{r}_{\varphi}^{(n)}=\boldsymbol{\varphi}\left[\mathbf{q}^{(n)}(0),\lambda^{(n)}\right], \qquad \mathbf{r}_{\psi}^{(n)}=\boldsymbol{\psi}\left[\mathbf{q}^{(n)}(S),\lambda^{(n)}\right] \tag{46}$$

are calculated at every n-th step, which are taken into account at the succeeding steps in the linearized boundary conditions

$$\frac{\partial\boldsymbol{\varphi}}{\partial\mathbf{q}}\delta\mathbf{q}^{(n+1)}(0)=-\frac{\partial\boldsymbol{\varphi}}{\partial\lambda}\delta\lambda^{(n+1)}-\mathbf{r}_{\varphi}^{(n)}\,, \qquad \frac{\partial\boldsymbol{\psi}}{\partial\mathbf{q}}\delta\mathbf{q}^{(n+1)}(S)=-\frac{\partial\boldsymbol{\psi}}{\partial\lambda}\delta\lambda^{(n+1)}-\mathbf{r}_{\psi}^{(n)} \tag{47}$$

In numerical simulation of the DS post-critical bending, the value $v(S)$ was selected as the leading parameter λ. To ensure calculation convergence its increment was chosen to equal $\delta\lambda^{(i)}=\pi/6000$. In doing so, the segment $0\le s\le S$ was divided into 5000 integration steps.

6. Critical and post-critical buckling of a DS

Of essential interest is also the problem on incipient buckling of a DS in an inclined cylindrical cavity of a bore-hole. Let the DS lie on the bottom v=0 of a cylindrical channel. The angle between the cylinder axis and vertical is β. The DS is subjected to action of the reaction force R, torque M_z and distributed gravity force $f^{gr}=g(\rho_t-\rho_l)e$, where g=9.81 m/s^2; ρ_t, ρ_l are the densities of the tube material and washing liquid, correspondingly; e is the area of the tube cross-section. Find the critical value of R and the mode of stability loss. With this aim in view, linearize Eq. (32) in the vicinity of the state $u'(s)=1$, $v(s)=0$, $k_x(s)=0$, $k_y(s)=0$, $k_z(s)=0$, $F_z(s)=g(\rho_t-\rho_l)e(L-s)\cos\beta-R$.

$$\begin{aligned} &d\delta F_y/ds = F_z\delta k_x - f^{gr}a\delta v', && d\delta F_z/ds = -f_z^{gr} \\ &d\delta k_x/ds = (1/EI)\delta F_y, && d\delta v/ds = \delta v' \\ &d\delta v'/ds = (1/a)\delta k_x, && d\delta u/ds = 0 \end{aligned} \tag{48}$$

As the first and sixth equations of this system are not connected with other ones, they can be considered separately. Then, taking into account equalities $\delta k_x=a\delta v''$, $\delta F_y=EIa\delta v''$ and denoting $a\delta v=\delta y$, one can gain

$$EI\delta y^{IV} - [f^{gr}\cos\beta(L-s)-R]\delta y'' + f^{gr}\cos\beta\delta y' + (f^{gr}\sin\beta/a)\delta y = 0 \tag{49}$$

It should be pointed out that according to the equation structure, critical values of reaction R do not depend on the M_z torque, if the DS is located inside inclined bore-hole.

Eq. (49) has not analytic solutions in closed form as the coefficient before $\delta y''$ is variable. But if $\beta=\pi/2$ (horizontal bore-hole), it attains the simplest form

$$EI\delta y^{IV} + R\delta y'' + (f^{gr}/a)\delta y = 0 \tag{50}$$

This equation is analogous to the equation of equilibrium of a beam on elastic foundation with elasticity coefficient $k=f^{gr}/a$, though the rock medium is assumed to be absolutely rigid and to play role of a constraint. The noticed analogy is due to the gravity force f^{gr} acting downward on the DS in its buckling and moving upward on the bore-hole surface.

In the case $L\to\infty$, the DS can lose its equilibrium stability with the sine mode

$$\delta y_\lambda = \delta c\cdot\sin(\pi s/\lambda)$$

where λ is the semi-wave length.

It corresponds to the critical value

$$R_\lambda^{cr} = \pi^2 EI/\lambda^2 + \lambda^2 f^{gr}/\pi^2 a \tag{51}$$

To find its minimal magnitude, it is necessary to minimize R_λ^{cr} with respect to λ. Then, one attains

$$F_z^{cr} = -R_{\min}^{cr} = -2\sqrt{EIf^{gr}/a}, \qquad l = \lambda_{\min} = \pi\sqrt[4]{EIa/f^{gr}}. \tag{52}$$

For the values $E=2.1\cdot10^{11}$ Pa, $\rho_t=7.8\cdot10^3$ kg/m^3, $\rho_l=1.3\cdot10^3$ kg/m^3, $I=1.555\cdot10^{-5}$ m^4, $e=4.9612\cdot10^{-3}$ m^2, this parameters are equal to l=25.25 m, $F_z^{cr}=-6.235\cdot10^5$ N .

Inasmuch as solutions (52) coincide with the results described by Fang *et al*. (2013), Mitchell (2008), Thompson *et al*. (2012), they can be assumed as one of confirmations of the elaborated approach validity.

If on the other hand, the bore-hole is inclined and has finite length, the mode of the DS stability loss acquires more complicated shape of a boundary effect at its lower end which is typical to singularly perturbed systems. To demonstrate this feature, Eq. (49) was solved with the use of the transfer matrix technique (Arici and Granata 2011, Gulyayev *et al*. 2009). The appropriate partial solutions of Eq. (49) were constructed by the Runge-Kutta method.

In Table 1, the critical magnitudes of the R^{cr} force found for different combinations of L, β and a are presented.

It is seen from these data that critical values of the compressive force R^{cr} are markedly inclination dependent and are practically unaffected by the L length of the DS. This is explained by the fact that buckling of the DS constrained by the bore-hole surface occurs inside a short segment of its length adjacent to the bore-hole bottom.

Table 1 Critical value R^{cr} for drill strings in cylindrical channels of inclined bore-holes

L, *m*	β, degree	*a*, *m*			
		0,075	0.10	0.15	0.20
50	0	31 729	31 729	31 729	31 729
	15	135 575	120 455	103 439	29 905
	30	184 155	160 440	133 575	118 835
	45	216 175	189 399	156 050	137 005
	60	236 415	208 085	171 879	150 025
	75	248 119	218 540	181 159	157 869
	90	252 005	221 949	184 219	160 505
100	0	15 899	15 899	15 899	15 899
	15	130 670	114 729	95 749	84 329
	30	177 305	155 119	128 620	112 969
	45	208 745	182 259	150 945	132 145
	60	229 869	200 529	165 870	145 139
	75	242 135	211 225	174 549	152 735
	90	246 205	214 785	177 459	155 295
250	0	12685	12685	12685	12685
	15	130 539	114 529	95 529	84 199
	30	177 129	154 880	128 489	112 750
	45	208 559	182 109	150 720	132 015
	60	229 649	200 375	165 649	144 949
	75	241 945	211 030	174 369	152 519
	90	246 035	214 589	177 299	155 079
500	0	11 629	11 629	11 629	11 629
	15	130 539	114 529	95 520	84 199
	30	177 129	154 880	128 480	112 750
	45	208 559	182 109	150 720	132 015
	60	229 649	200 375	165 649	144 949
	75	241 945	211 030	174 369	152 519
	90	246 029	214 589	177 299	155 079

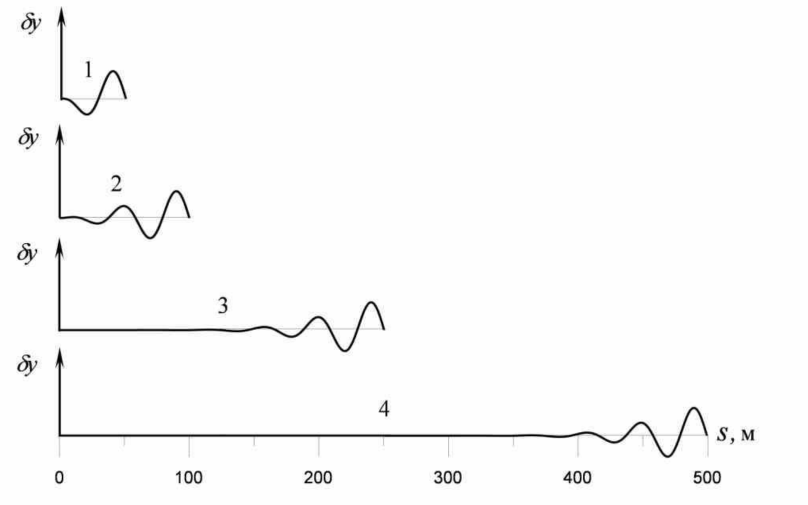


Fig. 5 Boundary effects in the modes of stability loss of drill strings in inclined bore-holes (1−*L*=50 m, 2−*L*=100 m, 3−*L*=250 m, 4−*L*=500 m)

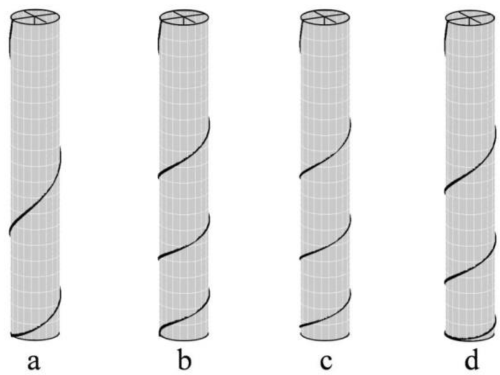
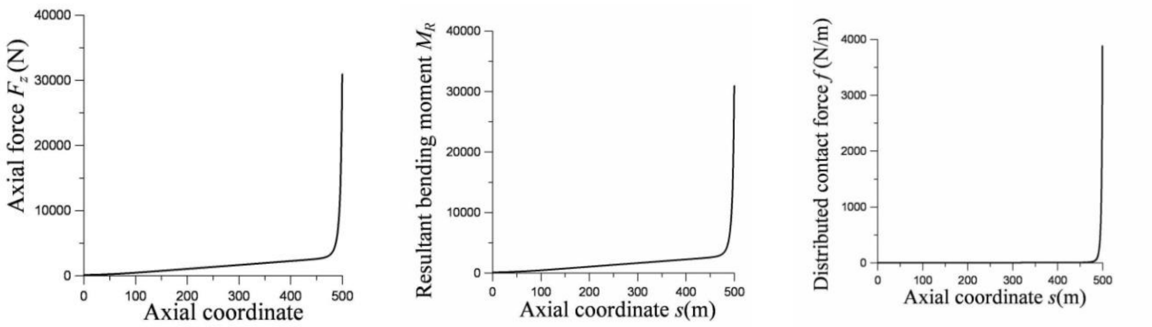


Fig. 6 A sequence of 3D modes of post-critical deforming of the DS (a- *R*=51,245 N; b- *R*=113,350 N; c- *R* =154,830 N; d- *R*=155,080 N)

This peculiarity derives from existence of the last term in Eq. (49) and singular perturbation of the problem connected with the large length of the DS and its prestressing by tensile longitudinal force in the upper zone. To ascertain this assertion consider the modes of stability loss of the DSs of lengths L=50 (curve 1), 100 (curve 2), 250 (curve 3), and 500 m (curve 4) presented in Fig. 5 for the case a=0.15 m, β=60°. All of them have the shape of a boundary effect, localized in the segment length l≈120 m. Though this effect completely covers the DS 50 m in length (the first curve in Fig. 5), the critical values R^{cr} do not essentially differ from each other (the R^{cr} values for these curves are framed in the Table 1).

The found regularity of the boundary effect appearance agrees well with the phenomena of boundary effect generation in singularly perturbed systems (Chang and Howes 1984). It can be estimated as additional qualitative corroboration of the reliability of the proposed theory.

The exception is the case when β=0 and the bore-hole is vertical. In this event the bore-hole



(a) Internal axial force diagram (b) Internal resultant bending moment diagram (c) Distributed contact force diagram

Fig. 7 Diagrams of elastic (F_z, M) and reaction (f^c) forces

surface does not constrain the DS buckling, the critical values R^{cr} are low (see the Table), the modes of stability loss do not contain boundary effects and are typical for the stretched-compressed rods.

If to continue enlarging the R force after the buckling incipience, the DS begins to coil up into an irregular spiral with diminishing pitches in its lower part. This process was simulated with the use of the technique outlined in Section 5 for solution of non-linear Eq. (28). In Fig. 6, (a)-(d), the sequence of 3D modes of post-critical deforming of the DS 500 m in length is exhibited for the appropriate R values. It demonstrates the elaborated approach advantage consisting in the possibility to track the DS transformation process with allowance made for its history.

The functions, characterizing the DS stress-strain state at R=133,440 N, are demonstrated in Fig. 7, (a)-(c). The $F_z(s)$ function (Fig. 7(a)) is represented by a rectilinear line with minimal value (maximal modulus) at the lower end $s=S$. The resultant bending moment $M_R = \sqrt{M_u^2 + M_v^2}$ is shown in Fig. 7(b). It has comparatively small values in the largest segment of the domain $0 \le s \le S$ and begins to enlarge steeply as the boundary $s=S$ is approached. The boundary effect is much more pronounced for the function $f^c(s)$ of contact interaction (Fig. 7(c)). It is positive throughout the whole diapason $0 \le s \le S$. Then, the unilateral contact constraint is stressed.

Nevertheless, the simulated process of non-linear post-critical deforming of the DS generates particular interest because of its nearing to the invariant state at $R \to 155{,}080$ N and deterioration of the calculation convergence. Indeed, from the start the DS displacements enlarged noticeably with the R increase. But thereupon, the displacement increments became smaller and smaller, until it came to a standstill.

In solving this non-linear problem, the questions of ensuring convergence of the step-type process described in Section 5 and calculation accuracy play important role. Notice, that in the joint use of the method of solution continuation by a parameter and Newton's procedure, the solution reliability check is realized automatically. Since at every computing step, inaccuracies (residuals) $r_\varphi^{(n)}$ and $r_\psi^{(n)}$ of Eq. (42) are calculated and thereupon they are compensated in linearized Eq. (47), the solution precision can be controlled by choice of the increments $\partial\lambda^{(n)}$ and Δs. In the considered case, they were selected proceeding from the condition of guaranteeing 5-6 correct ciphers in the largest calculated numbers.

7. Conclusions

• The problem about post-critical elastic bending of drill strings in cylindrical cavities of vertical and inclined oil and gas bore-holes is stated.

• On the basis of the theory of curvilinear rods the non-linear ordinary differential equations describing contact interaction of the drill string tube with the bore-hole surface are deduced. It is demonstrated that application of specially selected movable coordinate system as a concomitant reference frame makes it possible not only to simplify essentially the constitutive equations, but to halve their order as well.

• It has been found analytically that invariant states conditioned by the rigid constraint existence can be realized in the DS deforming when it is compressed to the bore-hole surface and is insensitive to increase of external forces. Possibility of their stability loss is established.

• Computer simulation of critical states and elastic non-linear post-critical bending of the DSs in cylindrical bore-hole cavities with different inclination angles is performed. The evolving modes of their deforming are constructed, stress-strain states are analyzed, and distributed contact forces are calculated.

• It is demonstrated that bending of the DS in a vertical well proceeds with shaping of a cylindrical spiral with variable pitch reducing as the lower boundary point is approached. In inclined bore-holes, the DS is compressed by the gravity forces to the bottom of the bore-hole channel and its axis line acquires the shape of a boundary effect with steeply enlarging functions of stress-strain state. An invariant state of the DS in respect to the axial force and torque can be achieved with their increase.

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