Structural Engineering and Mechanics, *Vol. 49, No. 1 (2014) 111-128*
DOI: http://dx.doi.org/10.12989/sem.2014.49.1.111

Capabilities of stochastic response surface method and response surface method in reliability analysis

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(Received February 27, 2013, Revised November 8, 2013, Accepted December 10, 2013)

Abstract. The stochastic response surface method (SRSM) and the response surface method (RSM) are often used for structural reliability analysis, especially for reliability problems with implicit performance functions. This paper aims to compare these two methods in terms of fitting the performance function, accuracy and efficiency in estimating probability of failure as well as statistical moments of system output response. The computational procedures of two response surface methods are briefly introduced first. Then their capabilities are demonstrated and compared in detail through two examples. The results indicate that the probability of failure mainly reflects the accuracy of the response surface function (RSF) fitting the performance function in the vicinity of the design point, while the statistical moments of system output response reflect the accuracy of the RSF fitting the performance function in the entire space. In addition, the performance function can be well fitted by the SRSM with an optimal order polynomial chaos expansion both in the entire physical and in the independent standard normal spaces. However, it can be only well fitted by the RSM in the vicinity of the design point. For reliability problems involving random variables with approximate normal distributions, such as normal, lognormal, and Gumbel Max distributions, both the probability of failure and statistical moments of system output response can be accurately estimated by the SRSM, whereas the RSM can only produce the probability of failure with a reasonable accuracy.

Keywords: structural reliability; stochastic response surface method; response surface method; polynomial chaos expansion; performance function; probability of failure

1. Introduction

Numerous uncertainties exist in mechanical and geometrical properties of structural systems and applied loads, which should be considered in the structural design. It is widely accepted that reliability analysis can quantitatively measure these uncertainties (Melchers 1999, Ang and Tang

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http://www.techno-press.org/?journal=sem&subpage=8 ISSN: 1225-4568 (Print), 1598-6217 (Online)

2007). Many methods and algorithms have been developed to analyze complicated structural reliability problems, especially for reliability problems with implicit performance functions (Tang *et al*. 2012). However, some limitations (i.e., expensive computational efforts, difficulty in dealing with nonlinear and implicit performance functions) exist in the traditional methods so that some practical reliability problems at small probability levels and with implicit performance functions cannot be effectively solved. For example, Monte Carlo simulation (MCS) is usually employed to estimate the probability of failure (Ditlevsen and Madsen 1996), but an innate disadvantage of the MCS is its prohibitive computational costs for the cases with small probability of failure or computationally intensive deterministic finite element (FE) analysis. Both first order reliability method (FORM) and second order reliability method (SORM) are not directly applicable to reliability problems with implicit performance functions (Hasofer and Lind 1974, Der Kiureghian *et al*. 1987). In order to overcome these limitations, the response surface method (RSM) and stochastic response surface method (SRSM) emerge as alternatives and have been developed and widely applied to structural reliability problems with implicit performance functions.

The RSM is a statistical technique proposed by Box and Wilson (1951) to evaluate the operating conditions of a chemical process at which some response has been optimized. In the literature (Faravelli 1989, Bucher and Bourgund 1990, Gavin and Yau 2008, Nguyen *et al*. 2009, Li *et al*. 2010, Milani and Benasciutti 2010, Basaga *et al*. 2012, Roussouly *et al*. 2013), the RSM has been extensively applied to structural reliability problems. The SRSM based on a probabilistic collocation method (PCM) can be interpreted as an extension of the response surface method. The SRSM was originally applied to uncertainty analysis for ocean and geophysical models, and environmental and biological systems (Tatang *et al*. 1997, Isukapalli 1999). Recently, this method has been applied to reliability problems in structural and geotechnical engineering (Sudret 2008, Huang *et al*. 2009, Li *et al*. 2011, Mao *et al*. 2012, Li *et al*. 2013a, b). Later the SRSM based on a linearly independent PCM was proposed by Li and Zhang (2007) and applied to uncertainty analysis of groundwater and solute transport, which can greatly reduce the uncertainties of selecting collocation points and improve the calculation accuracy and efficiency. Sudret (2008) and Mao *et al*. (2012) also emphasized the linearly independent principle that the number of sample points selected should lead to an invertible information matrix in global sensitivity analysis and probabilistic analysis. Li *et al*. (2013b) further studied the calculation accuracy and efficiency of the SRSM based on the linearly independent PCM.

The basic ideas of these two methods are to employ a meta-model to approximate the actual system output response. Nonetheless, there still exist several differences between them such as basic principles, computational procedures, the methods of fitting the performance function, and accuracy and efficiency in estimating the probability of failure and statistical moments of system output response. To our best knowledge, only Lin *et al*. (2009) performed a preliminary comparison between the SRSM and RSM, focusing on the comparison of the basic principles and procedures. Furthermore, the capabilities of the SRSM and RSM in fitting the performance function, and estimating the probability of failure as well as statistical moments of system output response are not explored extensively. Finally, a systematic comparison between the SRSM and RSM may provide some guidance for engineers to select the reliability methods in a more reasonable way.

The objective of this study is to explore the capabilities and differences between the SRSM and RSM regarding fitting performance function, accuracy and efficiency in estimating the probability of failure as well as statistical moments of system output response for structural reliability problems. The applicability of the SRSM and SRM under several special cases is also discussed.

For comparison, two optimal response surface methods, namely the SRSM based on a linearly independent probabilistic collocation method and the RSM using a vector projection sampling technique are selected, otherwise the comparisons will not be meaningful. These two methods are introduced briefly in Sections 2 and 3. Then they are compared in detail in two typical examples in Section 4.

2. Stochastic response surface method based on a linearly independent probabilistic collocation method

The basic idea of the SRSM is to employ a meta-model to represent the system output response whose input uncertain parameters are modeled by random variables. Standard normal variables are usually chosen as input random variables due to the mathematical tractability. According to the Cameron-Martin Theorem (Cameron and Martin 1947), any elements from the Hilbert space $L^2(\mathbf{R}, \mu)$ can be well approximated using the multi-dimensional Hermite polynomial chaos expansion (PCE). The system output response $\boldsymbol{Y}$ is expanded on an orthogonal multi-dimensional Hermite polynomial basis as follows (Isukapalli 1999, Huang *et al*. 2009, Li *et al*. 2011, Jiang *et al*. 2013)

$$\begin{aligned}\boldsymbol{Y} = {} & a_0\Gamma_0 + \sum_{i_1=1}^{\infty} a_{i_1}\Gamma_1(U_{i_1}) + \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1} a_{i_1 i_2}\Gamma_2(U_{i_1}, U_{i_2}) + \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1}\sum_{i_3=1}^{i_2} a_{i_1 i_2 i_3}\Gamma_3(U_{i_1}, U_{i_2}, U_{i_3}) \\ & + \cdots + \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1}\sum_{i_3=1}^{i_2}\cdots\sum_{i_n=1}^{i_{n-1}} a_{i_1 i_2,\cdots,i_n}\Gamma_n(U_{i_1}, U_{i_2}, \cdots, U_{i_n})\end{aligned} \tag{1}$$

where $\boldsymbol{Y}$ is the vector of the model output response values; $\boldsymbol{a} = \left(a_0, a_{i_1}, \cdots, a_{i_1 i_2,\cdots,i_n}\right)$ are unknown coefficients to be determined, in which n is the number of random variables used to represent the uncertainty of the model inputs; $\boldsymbol{U_i} = \left(U_{i_1}, U_{i_2}, \cdots, U_{i_n}\right)$ is an independent standard normal random vector resulting from the transformation of the physical random vector $\boldsymbol{X_i}$; $\Gamma_n(U_{i_1}, U_{i_2}, \cdots, U_{i_n})$ are the multi-dimensional Hermite polynomials of degree n.

The PCM is based on the idea of a chaos transformation used in the polynomial chaos methods, which offers a computationally inexpensive alternative for uncertainty analysis of complex models (Tatang *et al*. 1997, Isukapalli 1999). The PCM is often used to determine the unknown coefficients of the Hermite PCE. A linearly independent PCM is adopted in this study (Li and Zhang 2007, Mao *et al*. 2012, Li *et al*. 2013b), which can satisfy the condition that the Hermite polynomial information matrix has a full rank with less probabilistic collocation points or Latin hypercube sampling points used (Roussouly *et al*. 2013, Jiang *et al*. 2013). In addition, a non-intrusive analysis is achieved where existing deterministic FE codes or commercial FE software can be used directly without modifications.

In this respect, the SRSM based on the linearly independent PCM consists of the following five steps:

(1) select probabilistic collocation points or Latin hypercube sampling points $\boldsymbol{U_i}$ in the independent standard normal space (**U** space), $\boldsymbol{U_i}=(u_{i,1}, u_{i,2}, \cdots, u_{i,n})$, in which i=1, 2, …, N_p, N_p is the number of samples, to satisfy the principle that the information matrix is linearly independent by rows (see references (Mao *et al*. 2012, Li *et al*. 2013b) for a more detailed presentation);

(2) convert the collocation point vector as selected $\boldsymbol{U_i}$ to the physical random sample vector $\boldsymbol{X_i}$ using the Nataf transformation which are then taken as input parameters for the deterministic FE

model (Nataf 1962, Li *et al*. 2012, Li *et al*. 2013c);

(3) calculate the model output response vector $\boldsymbol{Y}$, $\boldsymbol{Y}= (y_1, y_2, \ldots, y_{N\mathrm{p}})= [M(\boldsymbol{X}_1), M(\boldsymbol{X}_2), \ldots, M(\boldsymbol{X}_{N\mathrm{p}})]$, at the sample points selected using the FE codes or software, where $M(\cdot)$ relates the relationship between the model output responses and model inputs;

(4) back substitute $\boldsymbol{U}$ and $\boldsymbol{Y}$ into Eq. (1), establish a system of linear algebraic equations in terms of the unknown coefficients $\boldsymbol{a}$, and solve it to obtain the coefficients of the Hermite PCE using a singular-value matrix decomposition method;

(5) once the unknown coefficients $\boldsymbol{a}$ are determined, substitute the system output response by an analytical meta-model PCE, the probability of failure and statistical moments, probability distribution function (PDF) and cumulative distribution function (CDF) of system output response can be readily estimated by using the direct MCS on the explicit meta-model PCE.

3. Response surface method using a vector projection sampling technique

The basic idea of RSM is to employ a closed-form response surface function (RSF) based on a quadratic PCE to fit the performance function. By this way, a complicated implicit reliability problem can be changed to a simple explicit reliability problem. The accuracy and efficiency of RSM depend mainly on the shape of RSF and the position of experimental sample points. To avoid excessively complicated reliability analysis and reduce computational costs of deterministic FE analysis, the RSF should be as simple as possible under the premise of well fitting the performance function.

The RSM based on a quadratic PCE using a vector projection sampling technique can yield good convergence and accuracy, and less computational costs are required for response surface construction and the deterministic FE analysis. This method was first proposed by Kim and Na (1997) based on a linear polynomial chaos expansion. Das and Zheng (2002) subsequently improved this method and generated the response surfaces in a stepwise fashion. This RSM is employed in the present study. The RSF based on a quadratic PCE without cross-terms is often adopted and expressed as follows,

$$\boldsymbol{Y} = a + \sum_{i=1}^{n} b_i x_i + \sum_{i=1}^{n} c_i x_i^2 \tag{2}$$

where $\boldsymbol{X}=(x_1, \ldots, x_i,\ldots, x_n)^{\mathrm{T}}$ is the vector of input sample points in the physical space; $\boldsymbol{a}=(a, b_1, \ldots b_n, c_1,\ldots, c_n)^{\mathrm{T}}$ is the vector of unknown coefficients with a size of $2n+1$. The Bucher and Bourgund’s sample design method summarized in Bucher and Bourgund (1990) is often adopted to determine the unknown coefficients in Eq. (2). However, the RSM without cross-terms may not be sufficiently accurate for some reliability problems with performance functions including cross-terms (Bucher and Bourgund 1990). Therefore, the RSF based on a quadratic PCE with cross-terms is also adopted for comparison,

$$\boldsymbol{Y} = a + \sum_{i=1}^{n} b_i x_i + \sum_{i=1}^{n} c_i x_i^2 + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} d_{i,j} x_i x_j \tag{3}$$

where $\boldsymbol{a}=(a, b_1, \ldots b_n, c_1,\ldots, c_n, d_{1,2},\ldots, d_{n\text{-}1,n})^{\mathrm{T}}$ is the vector of the unknown coefficients. In this case the size of unknown coefficients is increased to $(n^2+3n+2)/2$, and an efficient D-optimum sample design method summarized in Cheng and Li (2009) is adopted here to determine the unknown coefficients in Eq. (3). In the subsequent sections, the RSM based on the quadratic PCE without cross-terms using the vector projection sampling technique is called RSMncross, while the RSM with cross-terms is called RSMcross unless stated otherwise.

In this regard, the RSM using the vector projection sampling technique mainly consists of the following five steps:

(1) select the sample points $\boldsymbol{X_i}$ in the physical space (**X** space) and determine the values of u_i, $u_i+f\sigma_i$, and $u_i-f\sigma_i$ for each basic variable during the first iteration, where u_i and σ_i are the mean and standard deviation of the *i*th variable, respectively; f is a sampling parameter, the value of which is determined according to a quantitative standard proposed by Kim and Na (1997);

(2) evaluate the values of the performance function at the sample points $\boldsymbol{X_i}$, determine the unknown coefficients $\boldsymbol{a}$, construct a trial RSF, and convert the physical random vector $\boldsymbol{X_i}$ to the independent standard normal random vector $\boldsymbol{U_i}$ using the Nataf transformation (Nataf 1962, Li *et al*. 2012, Li *et al*. 2013c) and search for the design point u^* using the FORM optimization method in Eq. (4) developed by Hasofer and Lind (1974),

$$\begin{cases} \text{minimize } \|u\| \\ \text{subject to } G(x)=G(u)=0 \end{cases} \tag{4}$$

(3) generate the new sample points based on the tentative design point value u^* by using the vector projection technique. The vector projection of a unit vector on the nonzero vector along the response surface obtained in the preceding iteration will be used herein, which is the orthogonal projection of the unit vector onto a straight line parallel to the response surface (Kim and Na 1997);

(4) repeat steps (2) and (3) and obtain the final RSF and design point until a convergence criterion on the probability of failure or design point is satisfied;

(5) estimate the probability of failure and the statistical moments, the PDF and CDF of the system output response on the final RSF and design point by using the direct MCS or importance sampling method (Melchers 1989).

Based on above computational procedures of two response surface methods, it can be observed that a non-intrusive analysis can be achieved on the genuine meaning, and both the probabilistic analysis and deterministic FE analysis are accomplished independently in the SRSM. This feature enables us to conveniently apply FE codes or commercial FE software to deal with complicated realistic reliability problems. Furthermore, the multi-dimensional Hermite PCE adopted in the SRSM to construct the RSF is expressed as independent standard normal random variables which can approximate any elements in the Hilbert space (Cameron and Martin 1947). Also the sample points selected can guarantee that the information matrix has a full rank and is well-conditioned when determining the unknown coefficients. In contrast, several iterative response surface adjustments are required for the RSM to determine unknown coefficients, and the previous iterative results including the tentative probability of failure, tentative RSF and design point are used in the next iteration. The RSF based on the quadratic PCE expressed as the physical random variables is adopted in the RSM. Thereby some ill-conditioned and convergence problems may appear when fitting the elements in the Hilbert space and iteratively constructing the approximate response surfaces.

However, the influences of these differences in the computational procedures between the SRSM and RSM on the fitting of the performance function, the accuracy and efficiency of estimating the probability of failure as well as statistical moments have not been investigated sufficiently.

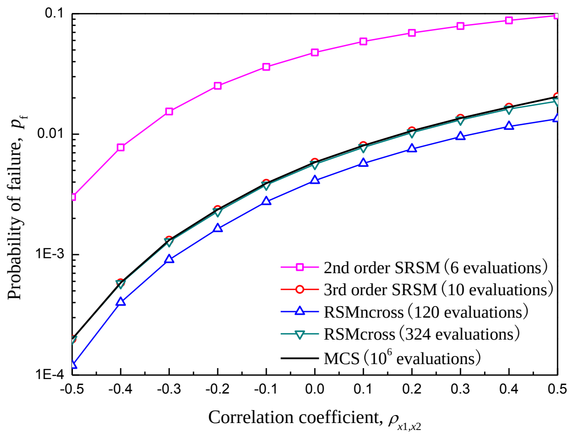


Fig. 1 Comparison of probabilities of failure obtained from different methods with x_1 and x_2 being normal random variables

Therefore it is necessary to perform an elaborate comparison between the SRSM and RSM, and explore their capabilities and differences through several examples covering a broad range of nonlinear structural reliability problems.

4. Numerical examples

4.1 Example # 1: a nonlinear performance function

A highly nonlinear performance function is considered in the first example. It has been studied by Kaymaz and McMahon (2005) and Cheng *et al*. (2008). The performance function is expressed as

$$g(x_1, x_2) = x_1^3 + x_1^2 x_2 + x_2^3 - 18 \tag{5}$$

To account for the effect of the tail distributions of the random variables, two cases are considered for random variables x_1 and x_2. The first case is that both random variables follow the normal distribution. The means of x_1 and x_2 are 10 and 9.9, respectively, and the standard deviation is 5 for both. The second case is that both random variables follow the exponential distribution, and their means are 10 and 9.9 for x_1 and x_2, respectively. It is also assumed that the correlation coefficient, $\rho_{x1,x2}$, between x_1 and x_2 ranges from -0.5 to 0.5.

For the first normal case, the probabilities of failure obtained from the 2nd and 3rd order SRSMs, RSMncross and RSMcross for various values of correlation coefficient $\rho_{x1,x2}$ and the corresponding numbers of performance function evaluations are shown in Fig. 1. For comparison, the results obtained from MCS with 10^6 samples are also provided in Fig. 1, which can be taken as the exact solutions because the MCS with 10^6 samples can accurately estimate the p_f exceeding 10^{-4} at an accuracy of COVp_f = 10% (Milani and Benasciutti 2010). It can be observed that both the 3rd order SRSM and RSMcross can produce satisfactory reliability results. On the contrary, the RSM based on a quadratic PCE without cross-terms using the vector projection sampling technique (referred to as RSMncross) leads to unsatisfactory reliability results. Such finding is significantly different from that reported in Gomes and Awruch (2004). The probabilities of failure obtained from the 2nd order SRSM differ greatly from the exact solutions.

As mentioned previously, the probability of failure is calculated based on the final RSF. Applying the SRSM and RSM when $\rho_{x1,x2}$= 0, the final approximate RSFs can be obtained. For consistency, the final approximate RSFs using different response surface methods in Eqs. (6)~(9) are all expressed as standard normal variables u_1 and u_2 that are transformed from the physical variables x_1 and x_2 in the performance function. They are given by

$$g_{SRSM}^{(\text{2nd order})}(u_1,u_2) = 4682.299 + 2865u_1 + 2345.15u_2 + 997.5\left(u_1^2 - 1\right) + 742.5\left(u_2^2 - 1\right) + 283.494u_1u_2 \tag{6}$$

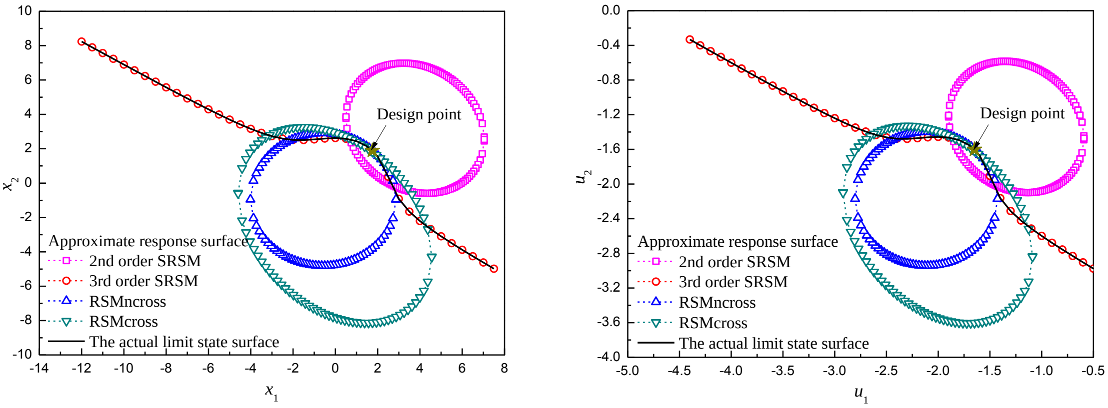
$$\begin{aligned} g_{SRSM}^{(\text{3rd order})}(u_1,u_2) = {} & 4682.3 + 2865u_1 + 2470.15u_2 + 997.5\left(u_1^2 - 1\right) + 742.5\left(u_2^2 - 1\right) \\ & + 500u_1u_2 + 125\left(u_1^3 - 3u_1\right) + 125\left(u_2^3 - 3u_2\right) + 125\left(u_1^2u_2 - u_2\right) \end{aligned} \tag{7}$$

$$g_{RSMncross}(u_1,u_2) = 1384.02 + 756.41u_1 + 616.43u_2 + 178.84u_1^2 + 142.01u_2^2 \tag{8}$$

$$g_{RSMcross}(u_1,u_2) = 1835.63 + 993.99u_1 + 794.59u_2 + 187.59u_1^2 + 121.17u_2^2 + 98.13u_1u_2 \tag{9}$$

The so-obtained final RSF in Eq. (7) using the 3rd order SRSM is just the same as the performance function in Eq. (5) through space transformation, whereas the final RSFs in Eqs. (6), (8) and (9) significantly differ from the performance function. By taking the values of the final approximate RSFs in Eqs. (6)~(9) as zeros, the final approximate response surfaces can be obtained from the SRSMs and RSMs and shown in Fig. 2. They are given both in the physical **X** space and independent standard normal **U** space in order to consider the effect of the tail distributions of random variables. The approximate response surfaces in these two spaces for this case are similar since only independent normal variables are involved. The actual limit state surface and the design point are also given in Fig. 2. It can be observed that the actual limit state surface cannot be well fitted by the RSMncross and RSMcross in the entire spaces as by the 3rd order SRSM, but it can only be well fitted in the vicinity of the design point, x*=(1.753, 1.893), through iteratively adjusting response surfaces. In contrast, the response surface obtained from the 2nd order SRSM significantly differs from the actual limit state surface in the entire spaces. This explains why the RSMncross and RSMcross can produce more accurate probabilities of failure than the 2nd order SRSM. It can also be concluded that the probability of failure mainly reflects the accuracy of the RSF fitting the performance function in the vicinity of the design point. Additionally, the number of performance function evaluations or deterministic FE runs is chosen in this study as a criterion to measure the efficiency of the SRSM and RSM (Duprat and Sellier 2006, Cheng and Li 2009). As seen from Fig. 1, the SRSM with an optimal order PCE (3rd order SRSM) based on the linearly independent PCM is much more efficient than the RSMncross and RSMcross. Only 10 performance function evaluations are needed, while 120 and 324 performance function evaluations are required for the RSMncorss and RSMcross to iteratively construct response surfaces, respectively.

The statistics of system output response (mean value, standard deviation, skewness and kurtosis) can also be obtained during reliability analysis when the SRSM and RSM are adopted, which are employed to further compare the performances of these two methods. Table 1 shows the first four statistical moments of the performance function with the correlation coefficient $\rho_{x1,x2}$= 0.



(a) **X** space (b) **U** space

Fig. 2 Comparison between approximate response surfaces and the actual limit state surface in the **X** and **U** spaces with x_1 and x_2 being normal random variables

Table 1 Comparison of statistical moments obtained from different methods with the correlation coefficient $\rho_{x1,x2}$= 0 for Example #1

Methods	Mean value	Standard deviation	Skewness	Kurtosis
2nd order SRSM	4682.1	4112.3	1.41	5.79
3rd order SRSM	4682.5	4232.0	1.89	8.81
RSMncross (Bucher and Bourgund's sample design)	1704.9	1028.3	0.93	4.2
RSMcross (D-optimum sample design)	2144.6	1315.6	0.94	4.21
MCS	4682.5	4232.0	1.89	8.81

Note that the results obtained from the 3rd order SRSM are also the same as those obtained from the MCS. In contrast, the results from the RSMncross and RSMcross are significantly different from those from the MCS. Their relative errors in standard deviation are up to 76% and 69%, respectively. It confirms that the statistical moments of the performance function reflect the accuracy of the RSF fitting the performance function in the entire space and they cannot be accurately estimated by the RSM.

In comparison with the RSM, another obvious advantage of the SRSM should be highlighted here. For some very complicated practical problems involving small probability of failure or computationally intensive deterministic FE analysis, as it is nearly impossible to obtain the exact solutions using the MCS, thereby there may be no reference standards to verify the quality of results. However, a convergence analysis is available for the SRSM to obtain the exact solutions. (Isukapalli 1999, Mao *et al*. 2012, Li *et al*. 2013a). This can be achieved by comparing the coefficients of the sequential order PCEs (Li *et al*. 2013a) or the CDFs of system output response obtained from the successive order SRSMs (Mao *et al*. 2012). Table 2 shows the coefficients of the 2nd, 3rd and 4th order Hermite PCEs used to judge whether the SRSM has achieved convergence. As seen from Table 2, not only the absolute differences between the coefficients of the common terms of the 3rd and 4th order PCEs are smaller than a prescribed tolerance ($\varepsilon \leq 10^{-3}$), but also the coefficients of the new added terms of the 4th order PCE all tend to zero (or a very small value).

Table 2 Comparison of coefficients of the 2nd, 3rd and 4th order Hermite PCEs for Example #1

Terms of the PCE		Coefficients		
Basis function	Polynomial chaos	2nd order PCE (6 evaluations)	3rd order PCE (10 evaluations)	4th order PCE (15 evaluations)
ψ_0	$\Gamma_{0,0}$	4682.299	4682.299	4682.299
ψ_1	$\Gamma_{1,0}\ (U_1)$	2865	2865	2865
ψ_2	$\Gamma_{0,1}\ (U_2)$	2345.15	2470.15	2470.15
ψ_3	$\Gamma_{2,0}\ (U_1)$	997.5	997.5	997.5
ψ_4	$\Gamma_{0,2}\ (U_2)$	742.5	742.5	742.5
ψ_5	$\Gamma_{1,1}\ (U_1,U_2)$	283.494	500	500
ψ_6	$\Gamma_{3,0}\ (U_1)$		125	125
ψ_7	$\Gamma_{0,3}\ (U_2)$		125	125
ψ_8	$\Gamma_{1,2}\ (U_1,U_2)$		0	0
ψ_9	$\Gamma_{2,1}\ (U_1,U_2)$		125	125
ψ_{10}	$\Gamma_{4,0}\ (U_1)$			0
ψ_{11}	$\Gamma_{0,4}\ (U_2)$			0
ψ_{12}	$\Gamma_{1,3}\ (U_1,U_2)$			0
ψ_{13}	$\Gamma_{3,1}\ (U_1,U_2)$			0
ψ_{14}	$\Gamma_{2,2}\ (U_1,U_2)$			0

However, the largest value in the coefficients of the new added terms of the 3rd order PCE is 125. This indicates that the 3rd order PCE has achieved convergence and its results can be regarded as exact solutions for reliability analysis, yet the 2nd order PCE not. The CDFs of the performance function as shown in Fig. 3 can be used to further determine the most optimal order PCE since the probability of failure is just calculated from the tail region of the CDF of the performance function. As seen from Fig. 3, it can further prove that it is reasonable to take the results of the 3rd order SRSM as exact solutions in this case, because there is no difference in the CDFs of the performance function associated with the 3rd and 4th order SRSMs.

The second special case that both random variables follow the exponential distribution is investigated to further demonstrate the capabilities of SRSM and RSM. Fig. 4 illustrates the probabilities of failure obtained from different methods and the corresponding numbers of performance function evaluations. Unlike the first case, even if the sixth order SRSM is employed, it still can not produce accurate probabilities of failure. This can also be explained by the relationships between the approximate response surfaces and the actual limit state surface shown in Fig. 5. Unlike Fig. 2, the actual limit state surface in this case must be bounded by the interval $x_1 \in (0, 2.62)$ as seen in the enlarged view in Fig. 5(a) due to the exponential variables involved (requiring both x_1 and x_2 greater than 0). Thereby the corresponding actual limit state surface in the **U** space is highly nonlinear as seen in Fig. 5(b) and it cannot be well fitted by the SRSM even when the 6th order SRSM is adopted. Compared with the first case, this limitation of the SRSM is mainly because the Hermite PCE in Eq. (1) used by the SRSM is an optimal method only for dealing with reliability problems involving random variables of approximate normal distributions (such as normal, lognormal, Gumbel Max, Beta, Weibull, Rayleigh distributions and so on). The SRSM with other orthogonal polynomial chaos such as Laguerre or Legendre polynomials should be adopted for the highly nonlinear performance functions involving strongly non-normal variables (such as exponential, Gamma and uniform distributions) (Xiu and Karniadakis 2003, Eldred *et al*. 2008, Li *et al*. 2013b). Orthogonal polynomials are such that any two different polynomials in the sequence are orthogonal to each other under some inner product, which provide an optimal basis for different continuous probability distribution types (Eldred *et al*. 2008). In contrast, the actual limit state surface can still be well fitted by the RSMncross and RSMcross in the vicinity of the design point, x*=(1.801, 1.838) due to the strong local optimization performance of the RSM. Therefore, both the RSMncross and RSMcross can yield the probabilities of failure with a reasonable accuracy (Fig. 4). It should be pointed out that the results of the RSM can be used to test the accuracy and performance of the SRSM if the direct MCS with very large samples are not feasible in this case (e.g., large nonlinear FE models). It should be mentioned that there exist two branches in the final response surfaces for the RSM based on a quadratic PCE as seen from Figs. 2 and 5. It may lead to a false branch and convergence cannot be guaranteed. The RSM using the vector projection sampling technique used in the study can effectively avoid this drawback.

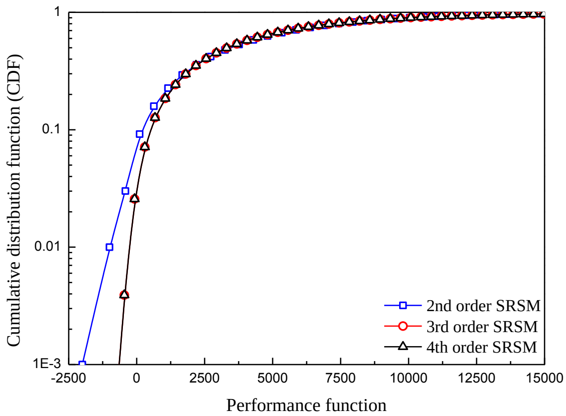


Fig. 3 Comparison of the CDFs of performance function obtained from the 2nd, 3rd and 4th order SRSMs with x_1 and x_2 being normal random variables

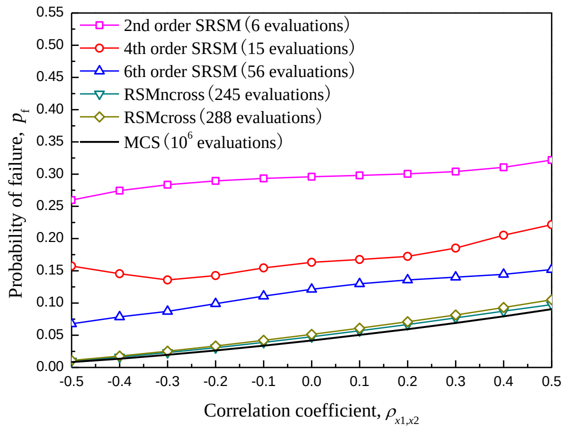
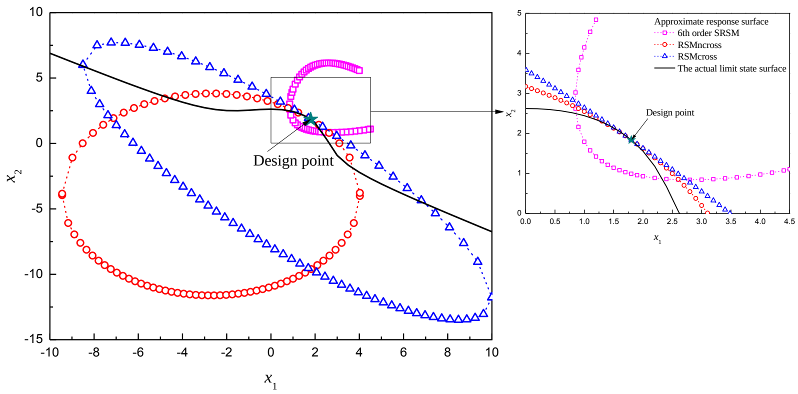
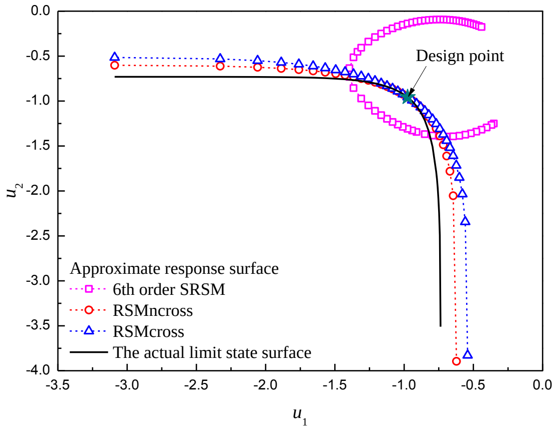


Fig. 4 Comparison of probabilities of failure obtained from different methods with x_1 and x_2 being exponential random variables



(a) **X** space



(b) **U** space

Fig. 5 Comparison between approximate response surfaces and the actual limit state surface in the **X** and **U** spaces with x_1 and x_2 being exponential random variables

4.2 Example # 2: Frame structure

Unlike the previous example that only involves two random variables, a 21- dimensional and implicit frame structural reliability problem is investigated to further explore the differences between these two methods in fitting the performance function and estimating the probability of failure and statistical moments. A three-bay five-storey rigid frame structure as shown in Fig. 6 is used. This structure was analyzed by Bucher and Bourgund (1990), Nguyen *et al*. (2009) and Roussouly *et al*. 2013. The structural properties associated with the beam elements are listed in Table 3. The statistical parameters of the basic random variables are summarized in Table 4. Note that lognormal distributions are used to avoid negative values of the geometrical and material properties. Some variables are assumed to be correlated. All loads are correlated with a correlation coefficient of ρ_{F_i,F_j} = 0.95. All the cross-sectional properties are correlated with correlation coefficients of $\rho_{A_i,A_j} = \rho_{I_i,I_j} = \rho_{A_i,I_j}$ = 0.13. Two different modulus of elasticity E_1 and E_2 are correlated with a correlation coefficient of ρ_{E_1,E_2} = 0.9. The other variables are assumed to be independent.

Table 3 Frame element properties for Example #2

Element	Modulus of elasticity	Moment of inertia	Cross section
1	E_1	I_5	A_5
2	E_1	I_6	A_6
3	E_1	I_7	A_7
4	E_1	I_8	A_8
5	E_2	I_1	A_1
6	E_2	I_2	A_2
7	E_2	I_3	A_3
8	E_2	I_4	A_4

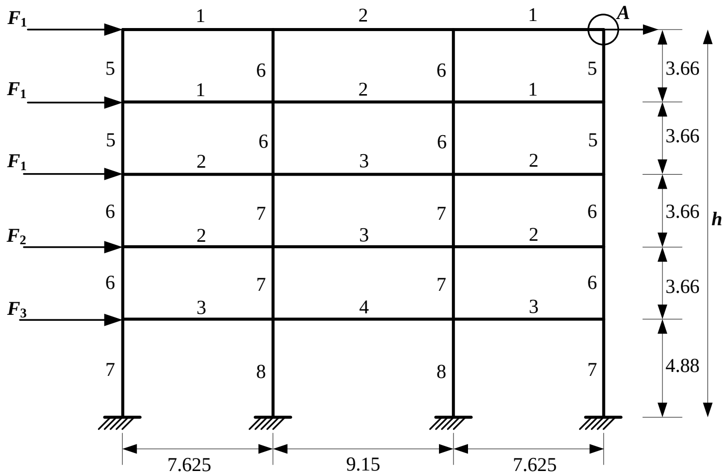


Fig. 6 Structural system for Example #2 (Unit: m) (after Bucher and Bourgund 1990)

Table 4 Statistical parameters of basic random variables for Example #2

Variable	Distribution	Unit	Mean value	Standard deviation
F_1	Gumbel max	kN	133.454	40.04
F_2	Gumbel max	kN	88.97	35.59
F_3	Gumbel max	kN	71.175	28.47
E_1	Lognormal	kN/m^2	2.173752×10^{7}	1.9152×10^{6}
E_2	Lognormal	kN/m^2	2.379636×10^{7}	1.9152×10^{6}
I_1	Lognormal	m^4	0.813443×10^{-2}	1.08344×10^{-3}
I_2	Lognormal	m^4	1.150936×10^{-2}	1.298048×10^{-3}
I_3	Lognormal	m^4	2.137452×10^{-2}	2.59609×10^{-3}
I_4	Lognormal	m^4	2.596095×10^{-2}	3.028778×10^{-3}
I_5	Lognormal	m^4	1.081076×10^{-2}	2.596095×10^{-3}
I_6	Lognormal	m^4	1.410545×10^{-2}	3.46146×10^{-3}
I_7	Lognormal	m^4	2.327853×10^{-2}	5.624873×10^{-3}
I_8	Lognormal	m^4	2.596095×10^{-2}	6.490238×10^{-3}
A_1	Lognormal	m^2	0.312564	0.055815
A_2	Lognormal	m^2	0.3721	0.07442
A_3	Lognormal	m^2	0.50606	0.093025

Table 4 Continued

A_4	Lognormal	m^2	0.55815	0.11163
A_5	Lognormal	m^2	0.253028	0.093025
A_6	Lognormal	m^2	0.29116825	0.10232275
A_7	Lognormal	m^2	0.37303	0.1209325
A_8	Lognormal	m^2	0.4186	0.195375

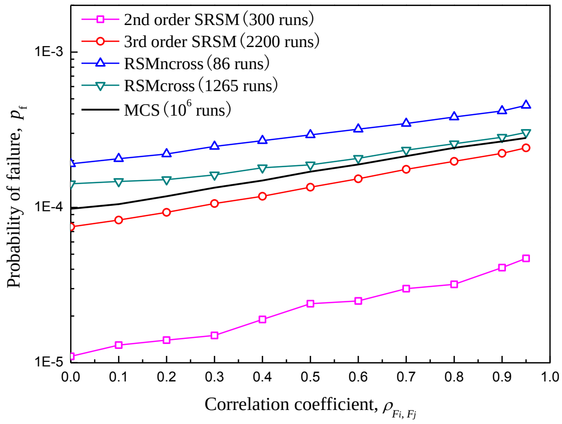


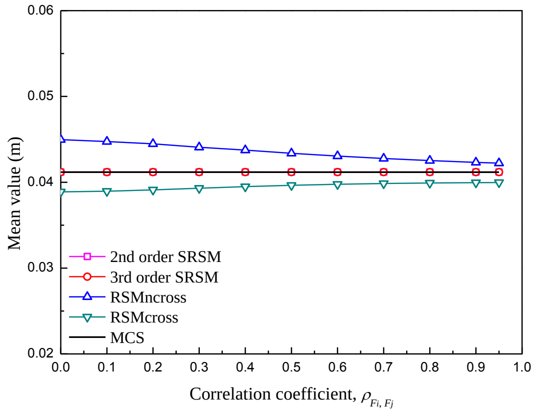
Fig. 7 Comparison of probabilities of failure obtained from different methods

The unsatisfactory performance underlying this problem is defined as the horizontal displacement at point A of the top floor exceeding the maximum allowable displacement h/320 =0.061 m (Bucher and Bourgund 1990). Thus the performance function regarding the structural safety margin is implicitly defined as

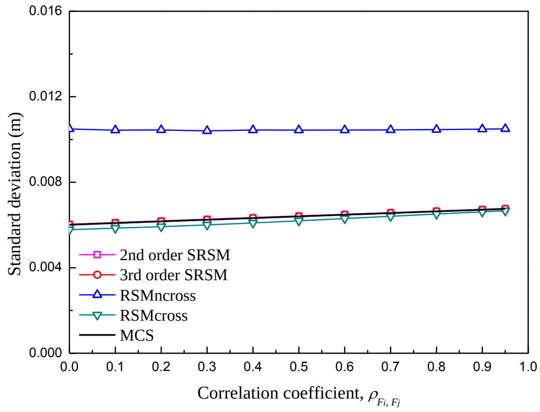
$$G(\boldsymbol{X}) = 0.061 - D(\boldsymbol{X}) \tag{10}$$

where $\boldsymbol{X}$ is a random vector representing the input random variables; $D(\boldsymbol{X})$ is the horizontal displacement of point A, which is calculated by structural FE analysis. It is obvious that the performance function shown in Eq. (10) cannot be explicitly expressed as a function of the physical random vector, $\boldsymbol{X}$, but the probability of failure and statistical moments associated with this implicit and high dimensional realistic problem can be readily evaluated by the SRSM and RSM.

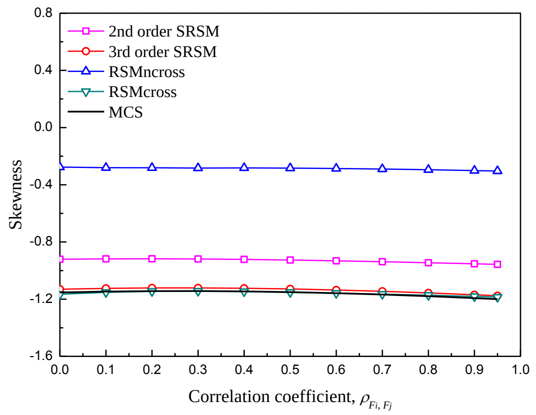
Fig. 7 shows that the probabilities of failure on log scale associated with the SRSMs, RSMs and MCS for various values of the correlation coefficient ρ_{F_i,F_j} among the loads. For this reliability problem involving random variables with approximate normal distributions (i.e., lognomal, Gumbel Max), the results from the 3rd order SRSM and RSMcross are in agreement with those from the MCS with 10^6 samples. For the reference case of the correlation coefficient ρ_{F_i,F_j} equal to 0.95, the probabilities of failure obtained from the 3rd order SRSM, RSMcross and MCS are 2.42×10^{-4}, 3.04×10^{-4} and 2.8×10^{-4}, respectively. In contrast, the results from the 2nd order SRSM and RSMncross are different from those from the MCS. The probabilities of failure obtained from the 2nd order SRSM and RSMncross are 4.7×10^{-5} and 4.54×10^{-4}. Compared with the MCS, the corresponding relative errors are 83.2% and 62.1%, respectively. The numbers of structural FE runs in Fig. 7 are used to compare the calculation efficiency of the SRSM and RSM. As discussed



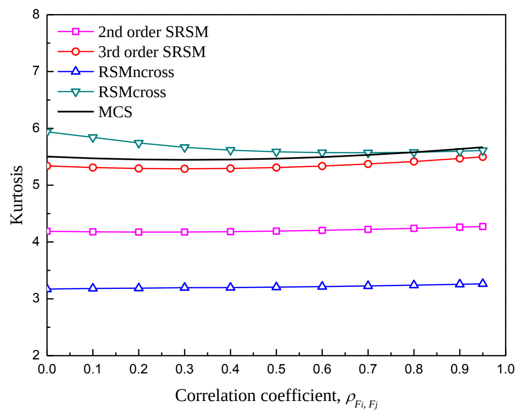
(a) Mean value



(b) Standard deviation



(c) Skewness



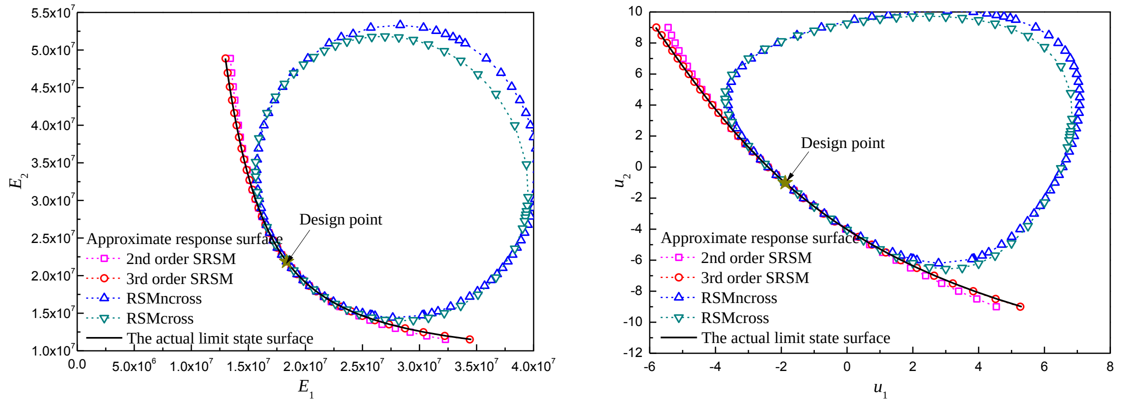
(d) Kurtosis

Fig. 8 Comparison of first four statistical moments of performance function obtained from different methods

previously, both the 3rd order SRSM and RSMcross can produce accurate probabilities of failure, but the former is less efficient than the later for such a high-dimensional reliability problem. As can be seen from Fig. 7, a total of 2200 runs of structural FE analysis are required for the 3rd order SRSM compared with 86 and 1265 runs for the RSMncross and RSMcross, respectively.

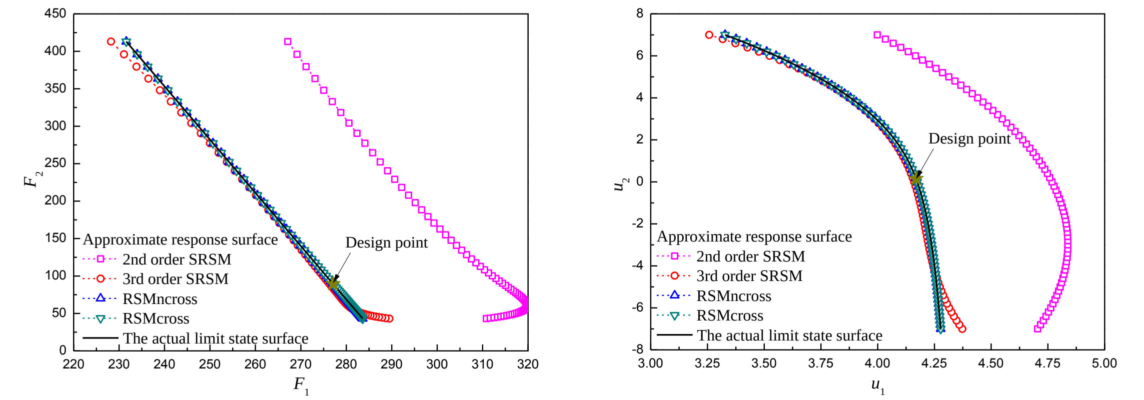
Fig. 8 shows the first four statistical moments of the performance function obtained from different methods. In Fig. 8, the results obtained from MCS with 10^6 samples are taken as exact solutions. The results obtained from the 3rd order SRSM match well with the exact solutions. The mean and standard deviation obtained from the 2nd order SRSM are consistent with the exact solutions. However, the RSMncross do not yield the first four statistical moments with a sufficient accuracy. The RSMcross can accurately estimate the probability of failure in Fig. 7, but it cannot produce an accurate mean value and kurtosis in comparison with the 3rd order SRSM. This can be explained in the following section.

For representing the actual limit state surface more intuitively, only E_1 and E_2 are considered as two independent random variables with the same statistical parameters given in Table 4. The value of F_1 is increased to 400 kN in order to make the limit state surface close to the origin and the



(a) **X** space (b) **U** space

Fig. 9 Comparison between approximate response surfaces and the actual limit state surface in the **X** and **U** spaces



(a) **X** space (b) **U** space

Fig. 10 Comparison between approximate response surfaces and the actual limit state surface in the **X** and **U** spaces

remaining 18 variables are taken as their mean values. Fig. 9 shows the corresponding approximate response surfaces and actual limit state surface both in the **X** and **U** spaces. The implicit performance function is a nonlinear function of E_1 and E_2 in the **X** space, while it seems an approximately linear function of u_1 and u_2 in the **U** space when the tail distributions of the random variables are incorporated. The actual limit state surface can be well fitted by the 2nd and 3rd order SRSMs in the entire **X** and **U** spaces, whereas it is only well fitted by the RSMncross and RSMcross in the vicinity of the design point, $x^* = (1.835\times10^7, 2.186\times10^7)$.

For completeness, Fig. 10 shows the approximate response surfaces and actual limit state surface in these two spaces when only F_1 and F_2 are considered as two independent random variables and the other 19 variables remain at their mean values. Contrary to Fig. 9, the implicit performance function seems a nearly linear function of F_1 and F_2, while it is a nonlinear function of u_1 and u_2. The actual limit state surface can still be well fitted by the 3rd order SRSM in both entire spaces. Unlike Fig. 9, the actual limit state surface can also be well fitted by the RSMncross and RSMcross in both entire spaces. It implies that the performance function is well fitted by the RSM in the entire space only when it is an approximate linear or quadratic function of variables in the **X** space. The same conclusion can also be drawn when only F_1 and F_3, or F_2 and F_3 are considered as random variables. These explain why the 3rd order SRSM can accurately yield both the probability of failure and statistical moments, whereas the RSMncross and RSMcross can only produce the probability of failure with a reasonable accuracy.

5. Conclusions

This paper has investigated the capabilities of two response surface methods in structural reliability analysis, namely the SRSM based on a linearly independent PCM and RSM using a vector projection sampling technique. Two numerical examples are investigated to compare the performances of these two methods in fitting the performance function and estimating the probability of failure as well as statistical moments of system output response. Several conclusions can be drawn from this study:

• The RSM using the vector projection sampling technique can efficiently avoid a false branch of final response surface and its convergence can be guaranteed, and several iterative response surface adjustments are required. In contrast, a non-intrusive analysis can be achieved using the SRSM where the probabilistic analysis and deterministic FE analysis are accomplished independently. Such feature enables us to apply structural FE codes or software for reliability analysis more conveniently. Additionally, a convergence analysis is available for the SRSM to obtain exact solutions in most cases when it is nearly impossible to use the MCS to provide the reference solutions.

• The probability of failure mainly reflects the accuracy of the RSF fitting the performance function in the vicinity of the design point, while the statistical moments reflect the accuracy of the RSF fitting the performance function in the entire space. The final response surfaces are compared both in the **X** and **U** spaces in order to account for the effect of the tail distributions of random variables. For normal cases with approximate normal variables, the performance function can be well fitted by the SRSM with an optimal order PCE in the entire spaces, while it can always be well fitted by the RSM in the vicinity of the design point. For a special case with strongly non-normal variables when the performance function is highly nonlinear and bounded in a narrow interval, the SRSM based on Hermite polynomials cannot fit it well, whereas the RSM can still accommodate it well in the vicinity of the design point.

• Both the SRSM and RSM can analyze complicated structural reliability problems with highly nonlinear and implicit performance functions involving multiple correlated non-normal random variables. The SRSM with an optimal order PCE can accurately estimate both the probability of failure and statistical moments, whereas the RSM can only produce the probability of failure with a reasonable accuracy. The SRSM is more preferable for dealing with low-dimensional reliability problems due to its high efficiency and accuracy. For very high-dimensional reliability problems, the computational costs for the SRSM will increase significantly since a large number of random variables are involved, and the RSM with cross-terms based on D-optimum sample design method is recommended if the probability of failure is of concern.

• For reliability problems involving non-normal random variables and highly nonlinear performance functions, the SRSM should adopt the optimal orthogonal polynomials for strongly non-normal random variables to achieve good accuracy and efficiency. Also, higher order SRSMs can be used to approximate highly nonlinear limit state functions.

Acknowledgments

This work was supported by the National Science Fund for Distinguished Young Scholars (Project No. 51225903), the National Natural Science Foundation of China (Project Nos. 51079112, 51329901), Doctoral Fund of Ministry of Education of China (Project No. 20120141110009) and the Academic Award for Excellent Ph.D. Candidates Funded by Ministry of Education of China (Project No. 5052012206001).

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