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Comprehensive evaluation of structural geometrical nonlinear solution techniques Part I: Formulation and characteristics of the methods

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Abstract. This paper consists of two parts, which broadly examines solution techniques abilities for the structures with geometrical nonlinear behavior. In part I of the article, formulations of several well-known approaches will be presented. These solution strategies include different groups, such as: residual load minimization, normal plane, updated normal plane, cylindrical arc length, work control, residual displacement minimization, generalized displacement control, modified normal flow, and three-parameter ellipsoidal, hyperbolic, and polynomial schemes. For better understanding and easier application of the solution techniques, a consistent mathematical notation is employed in all formulations for correction and predictor steps. Moreover, other features of these approaches and their algorithms will be investigated. Common methods of determining the amount and sign of load factor increment in the predictor step and choosing the correct root in predictor and corrector step will be reviewed. The way that these features are determined is very important for tracing of the structural equilibrium path. In the second part of article, robustness and efficiency of the solution schemes will be comprehensively evaluated by performing numerical analyses.

Keywords: nonlinear solution techniques; equilibrium path; condition equation

1. Introduction

To achieve the nonlinear structural behavior, there is a need of their equilibrium path in the space of load-displacement. This path specifies the critical and buckling points. Moreover, it identifies the amount of large displacements or the responses of structure to loadings. Nonlinear analysis is important in modeling the response of structures. Geometrical or material nonlinearity arises from many factors in structures. For instance, Gorgun and Yilmaz (2012) have assessed the effect of behavior of beam-to-column connections in the nonlinear analysis and design of steel structures. Nonlinear analysis of the structures is not as easy as the linear one, and it needs capable process. To overcome the difficulties, analysts have proposed different tactics for solving the structures with nonlinear behavior. In other words, researchers have formulated more practical and efficient methods to trace the complex equilibrium paths of structures. It should be added that the traditional schemes are not able to trace the equilibrium path of structures with complex nonlinear

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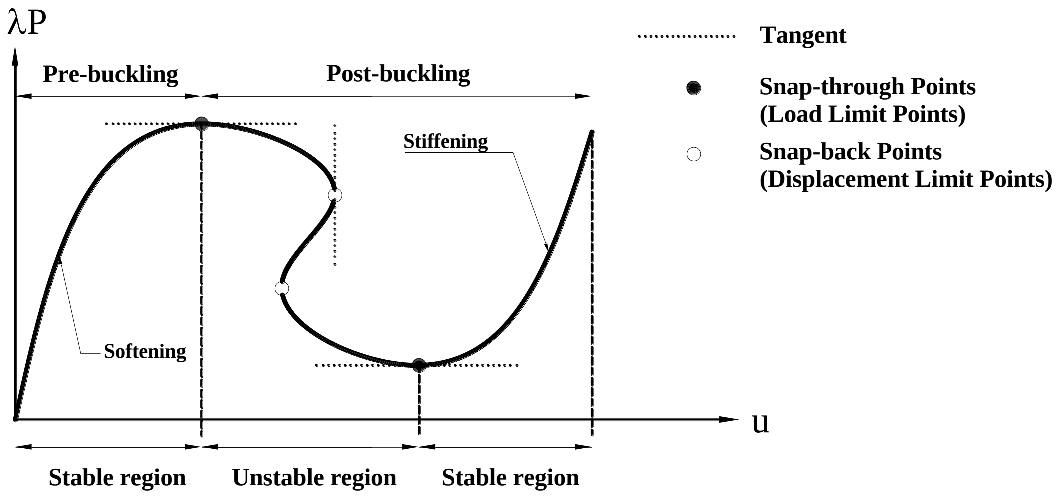

Fig. 1 General characteristics of the nonlinear equilibrium path

behavior, and it may diverge in passing the limit points. So far, the researchers have not found the desirable approach to trace all kinds of equilibrium paths. In fact, the most capable nonlinear solution techniques may fail in some cases. Therefore, analyst's interference is needed for correction of the analysis. Nonlinear behavior of the structures has different characteristics. Load limit and snap-back points, in stable and unstable paths, buckling points and post-buckling behavior of the structure and bifurcation points in equilibrium paths are very important. For example, if the structure behavior has a bifurcation point, the bifurcation paths should be detected, and the equilibrium path should be traced in the post-buckling area. Fujii and Ramm (1997) have proposed some solutions for these considerations. Ion Leahu-Aluas and Farid Abed-Meraim (2011) have assessed the advantages and drawbacks of nonlinear solution techniques by a set of buckling benchmark problems. Stable path of the structure may happen in a situation that the force and displacement increase simultaneously. Unstable path could occur when the force decreases and displacement increases. The general characteristics of the nonlinear behavior of structures are shown in Fig. 1.

The capabilities of nonlinear solution methods are different from each other. Choosing the solution processes is highly dependent on the analyst's experience. In other words, clear and comprehensive criteria are not available for selecting the nonlinear analysis strategy. Therefore, a broad comparison study between these methods and evaluating quantitative and qualitative of the merits and deficiencies of the solution process are very valuable. After considering the limitations and capabilities of the solution techniques, analysts can choose the most effective one.

In the present paper, some of the most well-known incremental-iterative nonlinear solution approaches will be studied. In the first part, characteristics, formulations, and steps of the algorithms will be explained. Then, a comprehensive evaluation, including quantitative and qualitative comparison, along with efficiency calculation of the techniques will be performed in the second part of paper. It is worth mentioning that previous researchers (Clarke and Hancock 1990, Carrera 1994, Ragon *et al*. 2002, Yang *et al*. 2003, Memon and Su 2004, Yang and Proverbs

2004, Torkamani and Sonmez 2008, Rezaiee-Pajand *et al*. 2009, Greco *et al*. 2012) have published valuable papers on evaluation of capabilities and deficiencies of different nonlinear schemes. They have also introduced the limitations and characteristics of these methods. Due to the importance of this issue, there is still a need of more comprehensive studies.

Researchers have proposed different strategies for structural nonlinear analysis, and currently this area of science is progressing. In the recent years, different schemes have been presented by analysts. Krenk and Hededal (1995) used orthogonal methods with quasi-Newton techniques for nonlinear analysis of the structures. Rezaiee-Pajand and Boroshaki (1999) proposed the variable arc length solution for analysis of the structures with nonlinear behavior. Kim and Kim (2001) performed the structural nonlinear analysis by using the neural network algorithm in predictor step and Newton-Raphson technique in corrector step. Toklu (2004) analyzed the trusses with nonlinear behavior by minimizing the total energy of structure used in the optimization process. Ligaro and Valvo (2006) studied the elastic pyramidal trusses having nonlinear behavior. Saffari and Mansouri (2011) proposed the two-point method for solving the nonlinear equations controlling the structural performance. This technique, which has been compared with the method of Newton-Raphson, cannot pass the load limit points. It should be mentioned that most researchers have not used similar symbols for writing the formulations. This leads to some difficulties in learning, comparison, and application of the solutions in algorithm and also computer programming. In the present study, all the procedures will be formulated with similar symbols. For instance, internal force vector of the structure, reference load, displacement, load factor, and residual load vector will be shown by F, P, u, λ, and R, respectively. Other factors, which will be used in the paper, are given in the Appendix. As it is shown in Fig. 2, some of the mentioned symbols are utilized to demonstrate the general process of an incremental-iterative scheme for nonlinear analysis.

2. Nonlinear analysis

The ability of nonlinear solvers in tracing the equilibrium path of structures can be investigated in the returning state to the equilibrium path after the first iteration. In this regard, the load factor and displacement increment at the first iteration of each solution step should move toward the equilibrium path. The reason is that residual load disturbs the equilibrium condition due to the first predictor step. By performing iterative process and utilizing the required accuracy, which should be defined by the investigator, the structural residual loads gradually decrease to zero. In advanced incremental-iterative methods, the load factor is calculated at the beginning of each analysis step. This step is called predictor step. Afterward, the iterative tactic achieves the equilibrium path of structure from the point obtained in the predictor step. This step which is shown in Fig. 2, is called the corrector step. As indicated in this figure, the symbol of increment factors in the predictor and corrector step are shown byΔ and δ, respectively. Superscripts n, and i demonstrate the number of incremental step and the performed iteration in that step, respectively. To simplify the formulations, the signs of vectors (→) and also matrices ([ ]) are avoided. It is worth mentioning that the difference between advanced incremental-iterative strategies is in the determination of process forming the iterative locus. In other words, the distinction between mentioned techniques is due to the diversity of proposed condition equation. As it is illustrated in the Fig. 2, this constraint is written in force-displacement space, which forms the analyses surface of corrector step.

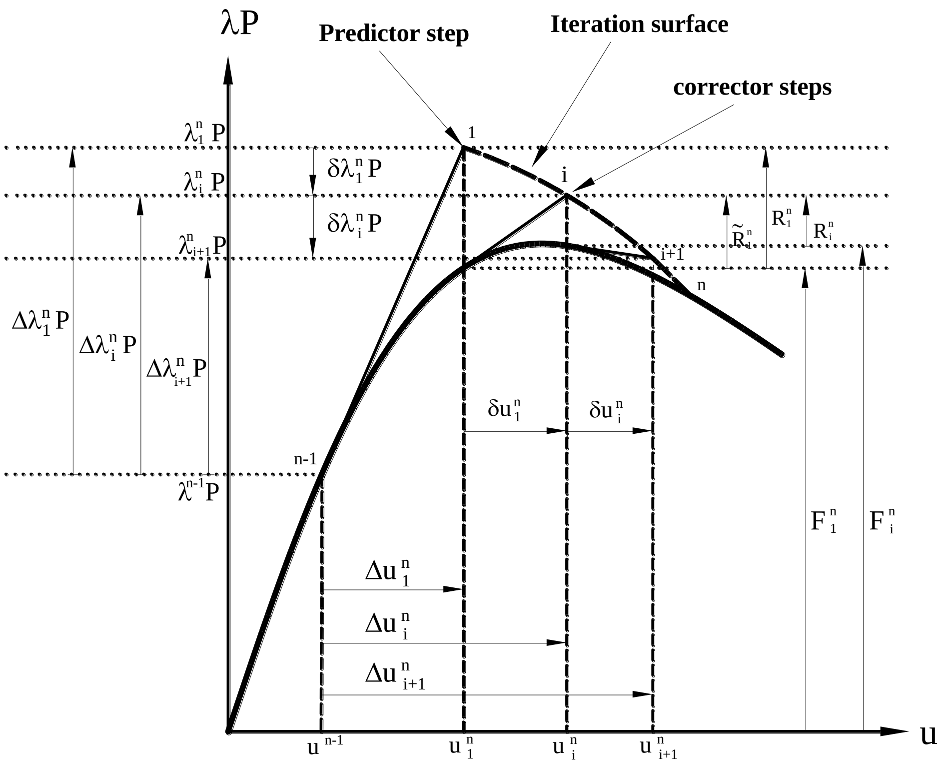

Fig. 2 General process of an incremental-iterative scheme of nonlinear analysis

2.1 Basic methods

Presenting comprehensive explanation and formulation of the traditional solution techniques are avoided because they are deficient and infrequently used. For example, algorithms like pure incremental (Waszczyszyn *et al*. 1995, Zienkiewicz *et al*. 2005), Newton-Raphson, modified Newton-Raphson, displacement Control (Argyris 1965), and its modified versions have a lesser solution ability than the advanced ones. Some of these shortages come from not passing the load limit points (for Newton-Raphson and modified Newton-Raphson methods), not passing the snap-back points (for displacement control schemes), long analysis time and slow convergence speed, error in the results, diverging from equilibrium path (for pure incremental technique), and other deficiencies. Although the pure incremental algorithm has the advantage of simple solution and low amount of analysis cost, it is not appropriate for analyzing structures with complex nonlinear behavior. Furthermore, this approach requires the tangent stiffness matrix at the beginning of each step. In the material nonlinearities, the response is dependent on the loading path, and incremental techniques can be used. It should be added that the incremental-iterative solutions have been proposed for analysis of the structures with nonlinear geometrical behavior, which are independent of the loading path (Felippa 1999). In spite of weakness of the well-known Newton-Raphson procedure and its modified version, they form the basis of many new and capable nonlinear solution methods. Based on this fact, a brief description of these two schemes will be discussed, before starting the advanced solution techniques.

2.1.1 Newton-Raphson method

This approach is one of the oldest and most applicable incremental-iterative procedures. It is also called load control technique, because loading level is constant during the iterative process. Based on this, the following condition equation can be written for Newton-Raphson scheme

$$\Delta\lambda_1^n = C \tag{1}$$

$$\delta\lambda_i^n = 0 \tag{2}$$

Where, C is a constant. Parameters $\Delta\lambda_1^n$ and $\delta\lambda_i^n$ are the load factor increments in the first and in i^{th} iteration of the n^{th} step, respectively. Displacement increment for the first and other iterations are specified by the next equations

$$\Delta\lambda_1^n P = K^{n-1}\Delta u_1^n \tag{3}$$

$$R_i^n = K_i^n \delta u_i^n \tag{4}$$

Here, P is the reference load vector, K is the tangent stiffness matrix of structure, and R_i^n is the residual load vector in i^{th} iteration. Referring to Fig. 2, the following load vector shows the difference between the external load of structure and the calculated internal force

$$R_i^n = \lambda_i^n P - F_i^n \tag{5}$$

In this relation, F_i^n is the internal force. This force for the nodal point is determined based on the structural internal stress by the following equation (Zienkiewicz 1977)

$$F_i^n = \iiint B_i^{nT} \sigma_i^n \, dV \tag{6}$$

Where, B_i^n is the strain matrix, and σ_i^n is the internal stress vector of the nodal point. Total displacements can be obtained in the below form

$$u_i^n = u^{n-1} + \Delta u_1^n + \sum_{i=1}^{n} \delta u_i^n \tag{7}$$

As it can be seen in Fig. 3, when the load level of first iteration goes higher than the load limit point in equilibrium path, divergence occurs. In this case, the deficiency of this method in passing the load limit points becomes clear (Bergan and Soreide 1978, Batoz and Dhatt 1979, Ramm 1981, Powell and Simons 1981). Other advanced techniques, which will be addressed in the following sections, are based on the Newton-Raphson method. In other words, those approaches employ Newton-Raphson in each iterative step.

2.1.2 Modified Newton-Raphson method

For analyzing the structures with a large number of degrees of freedom, using Modified Newton-Raphson method increases the time and cost of the analysis. This is due to the large number of updating the structural tangent stiffness matrix. Updating the stiffness matrix in each iteration is a very time-consuming process. In order to decrease the number of updating, the

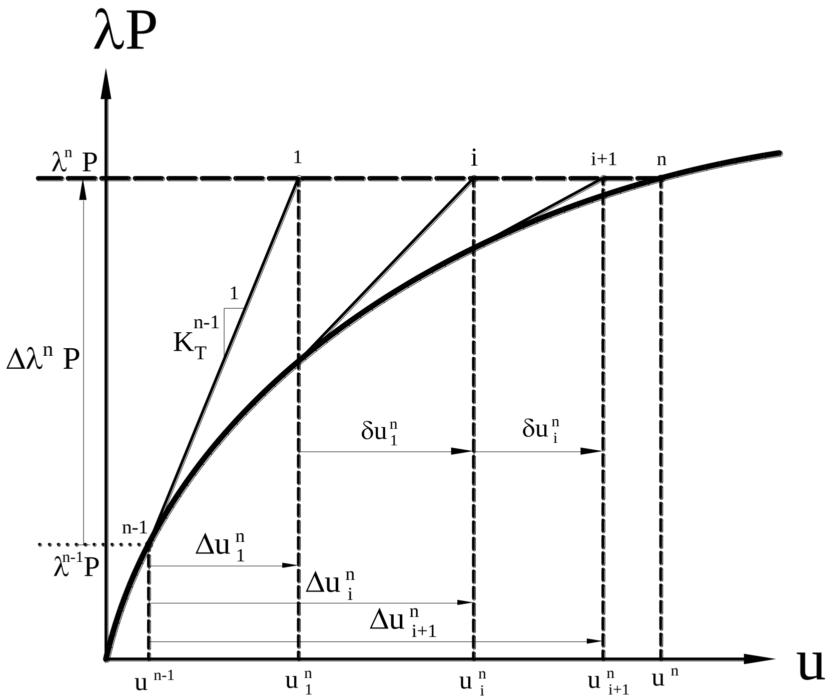

Fig. 3 Newton-Raphson method for a structure with one degree of freedom

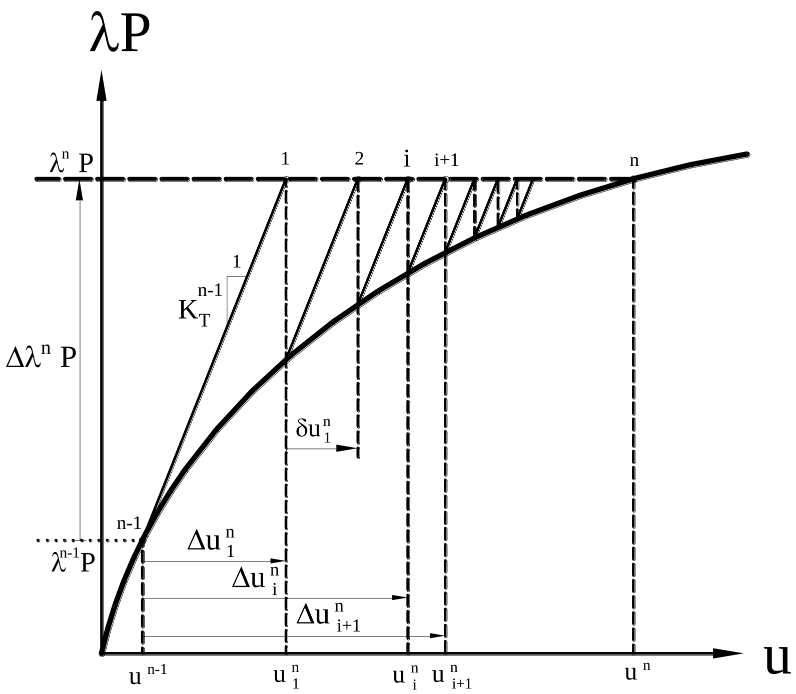

Fig. 4 Modified Newton-Raphson method for the structure with one degree of freedom

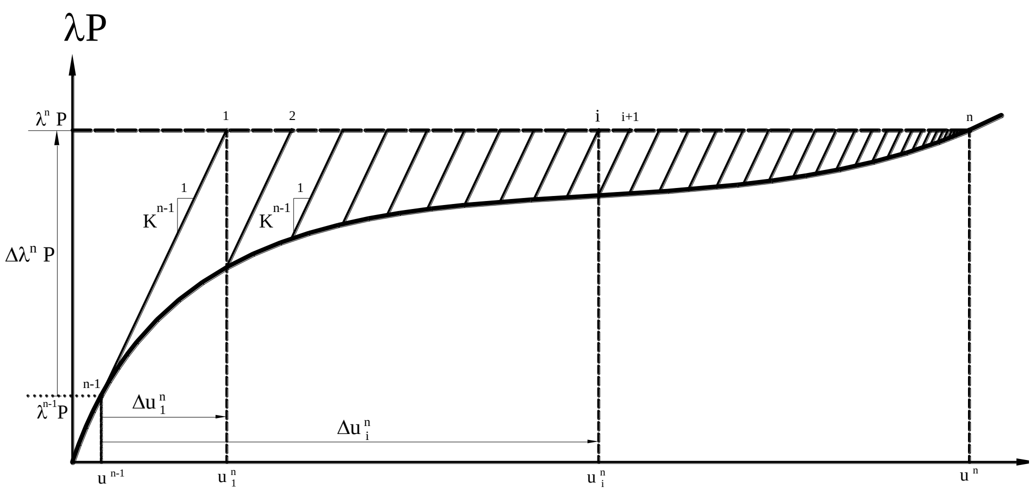

Fig. 5 Increasing the number of iterations in sudden softening behavior

modified Newton-Raphson scheme was proposed. Algorithm's steps of this technique are similar to regular Newton-Raphson. The only difference is that stiffness matrix is usually calculated once in each step. Fig. 4 illustrates this solution process. In modified Newton-Raphson, the number of updating tangent stiffness matrix depends on the nonlinear intensity of structure. In other words, by increasing the structural nonlinear intensity, the number of updating the stiffness matrix will be grown up. In case there is no information about structural behavior, stiffness matrix will be calculated only at the beginning of each step.

It is worth mentioning that the number of iterations for achieving convergence in Modified Newton-Raphson approach is more than the regular one, particularly, in a situation that structure has a sudden softening behavior in an increment. This characteristic is shown in Fig. 5. For improving this technique in problems with a slow convergence, acceleration procedures are used. One of the most common strategies is the Aitken (1937) solution. Furthermore, some researchers have proposed modified Aitken's scheme (Irons and Tuck 1969, Boyle and Jennings 1973).

2.2 Advanced solution techniques

As it was mentioned, traditional incremental-iterative tactics, like Newton-Raphson and modified Newton-Raphson are used in solving of the structures that do not have a buckling behavior. They are also utilized in structural analyses that their equilibrium path is traced before the load limit point. For instance, displacement control method cannot pass the snap-back point and fails (Ramm 1981, Crisfield 1981, Waszczyszyn 1983). To overcome these shortcomings, efficient incremental-iterative schemes are used. These techniques will be described later in the succeeding sections. Usually, Newton-Raphson is employed for the iterative steps of advanced tactics. In this study, methods of residual load minimization, normal plane, updated normal plane, cylindrical arc length, work control, residual displacement minimization, generalized displacement control, modified normal flow, three-parameter ellipsoidal, hyperbolic, and polynomial schemes are named advanced methods.

2.2.1 Residual load minimization method

Bergan (1980) calculated the load increment of each iteration by minimizing of the difference between external load of the structure and its internal force. Based on this, he minimized the square of reduced residual load by the following equation

$$\frac{\partial}{\partial \lambda}\left(\tilde{R}_i^{nT} \tilde{R}_i^n \right) = 0 \tag{8}$$

Here, the parameter $\tilde{R}_i^n$ is reduced residual load and is calculated by coming formula

$$\tilde{R}_i^n = R_i^n + \delta\lambda_i^n P \tag{9}$$

Fig. 1 shows this vector. By differentiating of the last equation of $\delta\lambda_i^n$, the next load increment is obtained

$$\delta\lambda_i^n = -\frac{P^T R_i^n}{P^T P} \tag{10}$$

In this technique, there is no need of finding displacement increments based on the reference or residual load vectors. In each iteration, the residual displacement is found only by solving the subsequent system of equation

$$K_i^n \delta u_i^n = \tilde{R}_i^n \tag{11}$$

2.2.2 Normal plane method

Nowadays, arc length approaches are well-known and are utilized extensively for tracing the nonlinear behavior of structures. The basic form of this scheme was first proposed by Wempner (1971), Riks (1972, 1979). These strategies were employed widely by other writers, and they were broadly expanded (Ramm 1981, Crisfield 1981, Rezaiee-Pajand and Akhaveysi 2000, Rezaiee-Pajand and Tatar 2005, Sousa and Pimenta 2010). In these formulations, condition equation is written based on the displacement and the force of structure. Actually, this constraint determines the distance from the last obtained equilibrium point to the iterative surface. This distance is shown by t_1^n and is called the arc length (Fig. 6). It is assumed that the distance in the first iteration is equal to a constant value, L_n. The difference of various arc length procedures is in determining the distance of obtained points from iterative analyses to prior equilibrium point. In normal plane strategy, locus of the obtained points from iterative analyses is perpendicular to the tangent at first equilibrium point of the current step (Riks 1975, Ramm 1981). Fig. 6 shows the scheme of normal plane for a structure with one degree of freedom. The subsequent condition equation is held for the normal plane tactic

$$t_1^n \cdot n_i^n = 0 \tag{12}$$

In this formulation, n_i^n is the passing vector from the iterative points of *i* and *i*+1. Based on the Fig. 6, vectors t_1^n and n_i^n can be formulated as follows

$$t_1^n = \Delta u_1^n e_1 + \Delta\lambda_1^n P e_2 \tag{13}$$

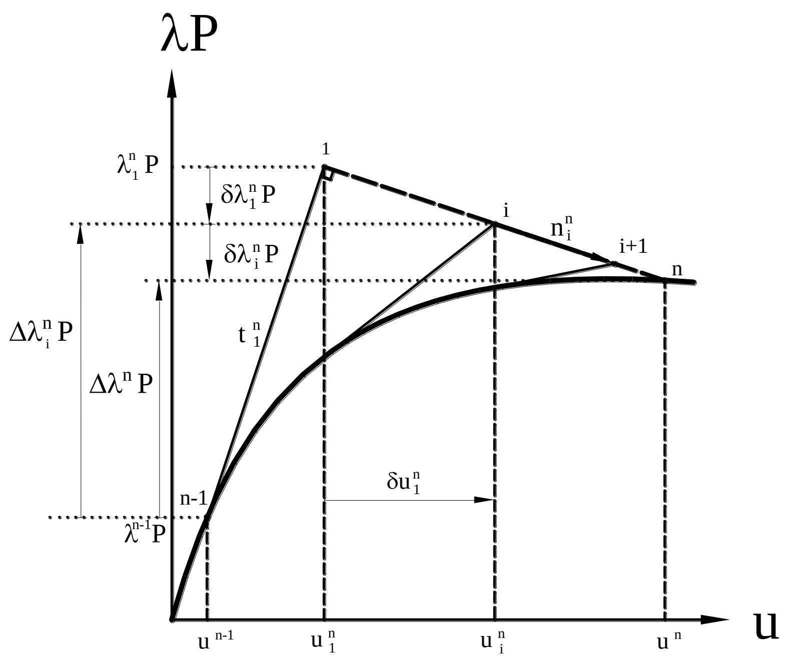

Fig. 6 Normal plane method for the structure with one degree of freedom

$$n_i^n = \delta u_i^n e_1 + \delta\lambda_i^n P e_2 \tag{14}$$

The unit vectors of the coordinate system are e_1 and e_2. According to $\left|t_1^n\right|^2 = \left(L_n\right)^2$ and Eq. (13), the load factor in predictor step can be calculated by coming relation

$$\Delta u_1^{nT} \Delta u_1^n + \left(\Delta\lambda_1^n\right)^2 P^T P = \left(L_n\right)^2 \tag{15}$$

Referring to the suggestion of Batoz and Dhatt (1979), for maintaining the symmetry of structural stiffness matrix, displacement increment can be obtained by the next linear combination

$$\delta u_i^n = \delta\lambda_i^n \delta u_i'^n + \delta u_i''^n \tag{16}$$

As it is shown in Fig. 7, the displacement increment is assumed as incremental linear combination of the displacements due to residual and reference load. In the present equation, $\delta u_i'^n$ is the displacement increment due to reference load. Displacement increment $\delta u_i''^n$ is resulted from the residual load of i^{th} iteration. Therefore, the following equations are obtained

$$\delta u_i'^n = \left(K_i^n\right)^{-1} P \tag{17}$$

$$\delta u_i''^n = \left(K_i^n\right)^{-1} R_i^n \tag{18}$$

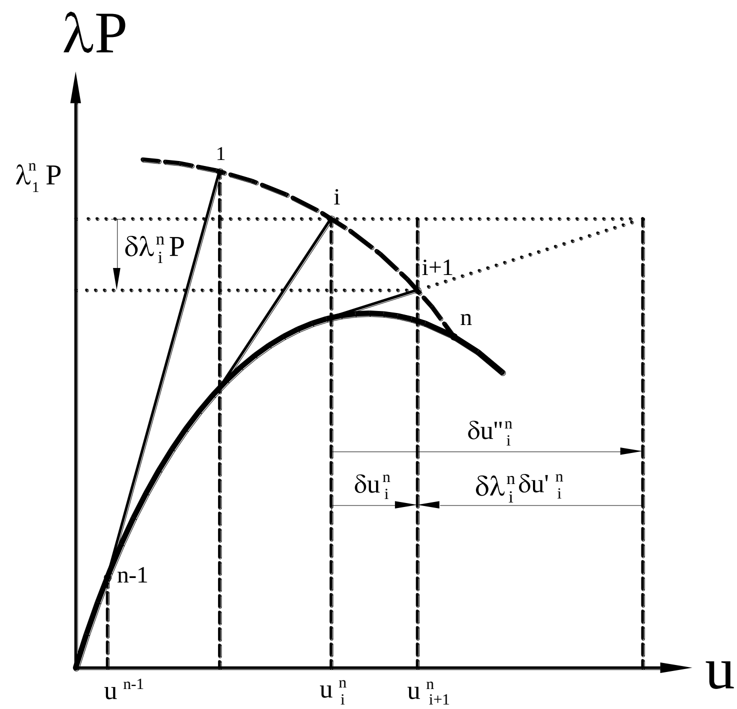

Fig. 7 Batoz and Dhatt's linear combination

Eq. (16) is written for the first iteration of each step as follows

$$\Delta u_1^n = \Delta\lambda_1^n \Delta u_1'^n \tag{19}$$

Where, parameter $\Delta u_1'^n$ is the displacement increment due to reference load, and it is in the first iteration of the n^{th} step. This factor can be obtained by coming formulation

$$\Delta u_1'^n = \left(K_1^n\right)^{-1} P \tag{20}$$

Displacement increment of the first iteration is zero due to residual load $\left(\Delta u_1''^n = 0\right)$. By replacing the Eq. (19) in Eq. (15), and solving the result, the next load increment of the first iteration (predictor step) has the following shape

$$\Delta\lambda_1^n = \pm \frac{L_n}{\sqrt{\Delta u_1'^{n^T} \Delta u_1'^n + P^T P}} \tag{21}$$

For the iterative steps, the following formula can be found by replacing the Eqs. (13) and (14) in condition Eq. (12) and using the Eq. (16)

$$\delta\lambda_i^n = -\frac{\Delta u_1^{n^T} \delta u_i''^n}{\Delta u_1^{n^T} \delta u_i'^n + \Delta\lambda_1^n P^T P} \tag{22}$$

2.2.3 Updated Normal Plane method

In contrary to the previous technique, in the updated normal plane, the vector passing the equilibrium points and iteration surface points $\left(t_i^n\right)$ is always perpendicular to the vector passing from its own iteration surface points $\left(n_i^n\right)$ (Ramm 1981, Hinton *et al*. 1982, Forde and Stiemer 1987). This strategy is illustrated in Fig. 8. The following constraint equation can be written for the updated normal plane

$$\vec{t}_i^{\,n} \cdot \vec{n}_i^{\,n} = 0 \tag{23}$$

In this formula, vector t_i^n connects equilibrium point $n-1$ to point i from the iteration surface. Parameters of this vector are similar to Eq. (13) and are connected by the below expression

$$t_i^n = \Delta u_i^n e_1 + \Delta\lambda_i^n P e_2 \tag{24}$$

In updated normal plane, load increment of the first iteration is similar to previous case, and it is calculated by the Eq. (21). To find the load factor increment in the corrector steps, Eqs. (24) and (14) are put in Eq. (23), and the obtained formula is solved for $\delta\lambda_i^n$. The result is given in the following form

$$\delta\lambda_i^n = -\frac{\Delta u_i^{n^T} \delta u_i''^n}{\Delta u_i^{n^T} \delta u_i'^n + \Delta\lambda_i^n P^T P} \tag{25}$$

Rezaiee-Pajand and Tatar (2006) have proposed several orthogonal methods for geometrical nonlinear analysis of the structures.

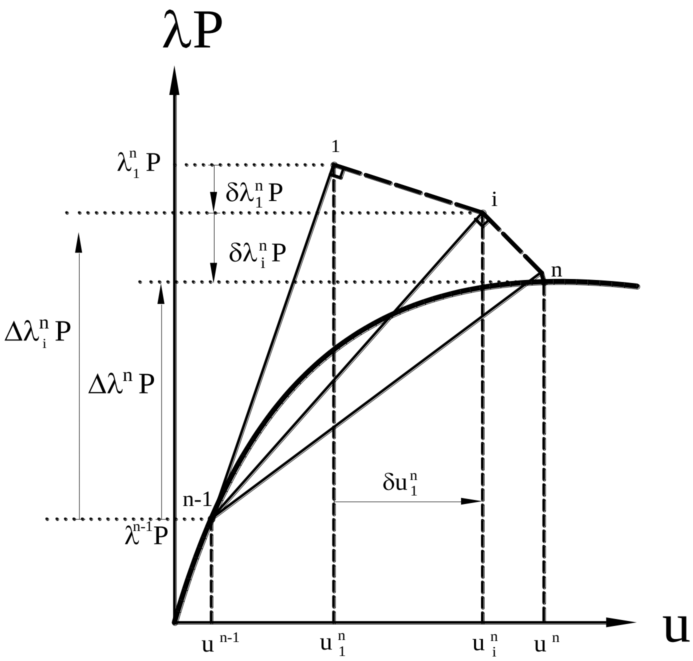

Fig. 8 Updated normal plane method for the structure with one degree of freedom

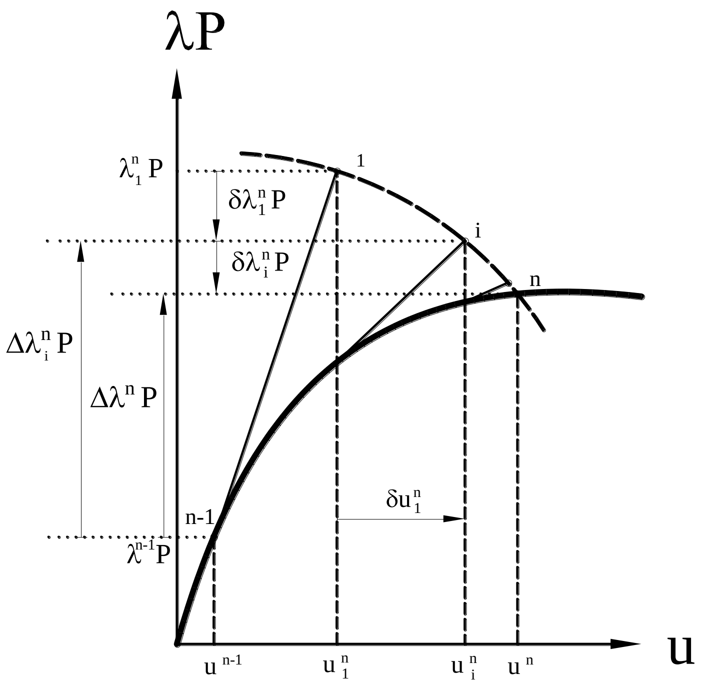

Fig. 9 Cylindrical arc length method for the structure with one degree of freedom

2.2.4 Cylindrical arc length method

One of the most applicable processes for nonlinear analysis of the structures is Crisfield cylindrical arc length (Crisfield 1981). In this algorithm, the distance of points obtained from the iterative analyses up to the previous equilibrium point is equal to a constant amount of L_n in all the step iterations. Fig. 9 shows the graphical representation of cylindrical arc length. Parameter t_i^n is the connector vector of previous equilibrium point ($n-1$) to i^{th} iterative point on the iterative surfaces. In addition, Crisfield ignored the force parameter forming the vector t_i^n and proposed his solution in a simpler way and called it modified Riks technique in 1981.

In cylindrical arc length method, vector t_{i+1}^n is obtained by the following equation

$$t_{i+1}^n \approx \Delta u_{i+1}^n e_1 \tag{26}$$

In this technique, arc length is characterized with the subsequent formulation and remains constant up to the end of the iterations of each step

$$(L_n)^2 = \Delta u_1^{n^T} \Delta u_1^n \tag{27}$$

It is clear that the difference between this formula and Eq. (15) is in deleting the force parameter. Furthermore, by using the expression $t_{i+1}^n \cdot t_{i+1}^n = (L_n)^2$, next constraint of the corrector iteration can be found in the below form

$$\Delta u_{i+1}^{n\ T} \Delta u_{i+1}^n = (L_n)^2 \tag{28}$$

Based on the Fig. 9, this formulation can be rewritten as the following

$$\left(\Delta u_i^n + \delta u_i^n\right)^T \left(\Delta u_i^n + \delta u_i^n\right) = (L_n)^2 \tag{29}$$

By putting Eq. (19) in Eq. (27), load increment of the first iteration is calculated by coming equation

$$\Delta\lambda_1^n = \pm \frac{L_n}{\sqrt{\Delta u_1'^{n^T} \Delta u_1'^n}} \tag{30}$$

In the corrector iterations, by inserting Eq. (16) into Eq. (29) and simplifying, the next second order equation for calculating the load increment is determined

$$a\left(\delta\lambda_i^n\right)^2 + b\left(\delta\lambda_i^n\right) + c = 0 \tag{31}$$

The constant coefficients of this equation are written in the following shapes:

$$a = \delta u_i'^{n^T} \delta u_i'^n \tag{32}$$

$$b = 2\left(\Delta u_i^n + \delta u_i''^n\right)^T \delta u_i'^n \tag{33}$$

$$c = \left(\Delta u_i^n + \delta u_i''^n\right)^T \left(\Delta u_i^n + \delta u_i''^n\right) - \left(L_n\right)^2 \tag{34}$$

After determining the related constant coefficients and solving Eq. (31), two later answers are available for load increment

$$\left(\delta\lambda\right)_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{35}$$

At this stage, the next three cases may occur for the roots of Eq. (31)

$$1) \quad b^2 - 4ac > 0 \tag{36}$$

$$2) \quad b^2 - 4ac = 0 \tag{37}$$

$$3) \quad b^2 - 4ac < 0 \tag{38}$$

In the first circumstance, the answers obtained from Eq. (35) are real roots. Selection procedure of the acceptable roots will be given later. In the second situation, there is only one acceptable root. The answers of Eq. (35) in third state are complex and are not suitable. To overcome this shortcoming, a common way is to split arc length into half. Then by solving these formulations again, new answers will be obtained. These roots should only satisfy either the Eqs. (36) or (37). Researchers have also proposed other processes to deal with this problem. For example, Zhou and Murray (1995) have suggested parallel correction method to avoid obtaining the complex root. It is worth adding that Ritto-Corrêa and Camotim (2008) have conducted a research on arc length methods and other second order techniques. Hrinda (2007) has used arc length solution for

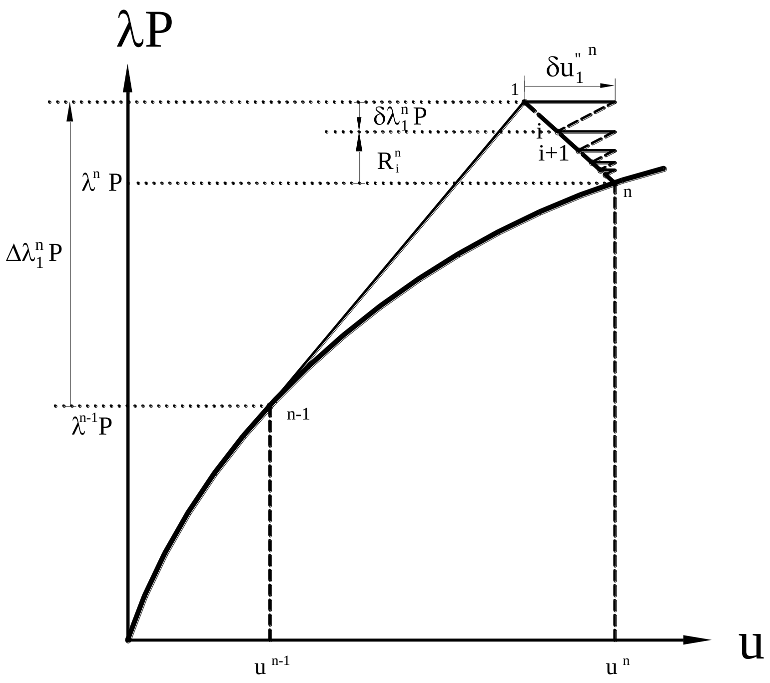

Fig. 10 Work control method for the structure with one degree of freedom

nonlinear analysis of the space truss structures. Torkamani and Shieh (2011) have also employed arc length approach to perform a nonlinear analysis of the plane truss structures with higher-order stiffness matrices.

2.2.5 Work control method

In 1981 and 1985, this technique was suggested by assuming work increment as a constant value in all analysis steps (Powell and Simons 1981, Yang and McGuire 1985). Fig. 10 shows the work control scheme for a problem with one degree of freedom. In this method, the work increment is assumed to be constant at the beginning of each step, and it is zero for other iterations.

Therefore, the work control strategy has the following condition equations

$$\Delta w_1^n = \Delta\lambda_1^n P^T \Delta u_1^n = C \tag{39}$$

$$dw_i^n = \delta\lambda_i^n P^T \delta u_i^n = 0 \tag{40}$$

Work increment in i^{th} iteration is dw_i^n. Parameter C is assumed to be constant and will be explained later. By solving Eq. (39) for $\Delta\lambda_1^n$ and inserting Eq. (19) in it, the load increment is determined for the first iteration

$$\Delta\lambda_1^n = \pm\sqrt{\frac{C}{P^T \Delta u_1'^n}} \tag{41}$$

By putting Eq. (16) in Eq. (40), the load increment in corrector iterations is obtained as

$$\delta\lambda_i^n = \pm \frac{P^T \delta u_i''^n}{P^T \delta u_i'^n} \tag{42}$$

2.2.6 Residual displacement minimization method

After the failure of the Newton-Raphson in passing the load limit points of the equilibrium path, many schemes were proposed to tackle this shortcoming. Solution strategies like arc length of Crisfield (1981), Riks (1972, 1979), Ramm (1981), displacement control (Zienkiewicz 1971, Pian and Tong 1971, Batoz and Dhatt 1979) and constant work of Powell and Simons (1981) and Yang and McGuire (1985) were not applicable in different conditions. These techniques do not trace the shortest path for obtaining the equilibrium point. In 1988, the residual displacement minimization approach was suggested by Chan (1988). This method traces the shortest path for achieving the convergence criteria by using Newton-Raphson process. In this way, displacement state is used in an incremental step for continuing or ending the iterations. By applying the condition equation and minimizing the second square of residual displacement, which is the analysis error, the following expression is written

$$\frac{\partial}{\partial \lambda_i^n} (\delta u_i^{n^T} \delta u_i^n) = 0 \tag{43}$$

By inserting Eq. (16) into Eq. (43), corrector load increment is easily calculated for each iteration

$$\delta\lambda_i^n = -\frac{\delta u_i''^{n^T} \delta u_i'^n}{\delta u_i'^{n^T} \delta u_i'^n} \tag{44}$$

The first corrector load increment can be determined by the user. Researchers have suggested that the amount of load increments should not be large in the first iteration. It is worth mentioning that in highly nonlinear structural behaviors, there is more need of this selection, because it plays a very important role in divergence or convergence of the analysis. Rezaiee-Pajand and Tatar (2009) have proposed several nonlinear methods with minimization of residual parameters for analyzing nonlinear structures.

2.2.7 Generalized displacement control method

This solution scheme is capable of passing load and displacement limit points, and it is known to be an efficient way of tracing the structural equilibrium path. According to the research of Young and Ku (1994), modified displacement control method, which has been proposed by Young and Shieh (1990), is more competent than arc length techniques due to the reasons of automatic compatibility with changing the direction of the loading in limit points, numerical stability in limit points and automatic adjustment of step size.

Generalized displacement control approach has been used successfully by Richard Liew *et al*. (1997) in advanced analysis and design of spatial structures. Moreover, Yang *et al*. (2008) have employed this process in inelastic post-buckling response of steel trusses under thermal loadings. Additionally, Cardoso and Fonseca (2007) and Thai and Kim (2009) have discussed this technique. Constraint equation of the present scheme is written as the following

$$(\Delta\lambda_1^n \Delta u_1'^{n-1})^T \delta u_i^n = H_i^n \tag{45}$$

In this equation, H_i^n is the indicator of the generalized displacement. By putting Eq. (16) in Eq. (45), the load factor increment in corrector iterations is determined as

$$\delta\lambda_i^n = \frac{H_i^n - \Delta\lambda_1^n (\Delta u_1'^{n-1})^T \delta u_i''^n}{\Delta\lambda_1^n (\Delta u_1'^{n-1})^T \delta u_i'^n} \tag{46}$$

By using the Eq. (19) and putting it in Eq. (45), load factor increment at the first iteration is formulated in the below form

$$\Delta\lambda_1^n = \sqrt{\left| \frac{H_1^n}{(\Delta u_1'^{n-1})^T \Delta u_1'^n} \right|} \tag{47}$$

The following result will be available by inserting *n*=1 in the preceding equation

$$H_1^1 = (\Delta\lambda_1^1)^2 (\Delta u_1'^1)^T \Delta u_1'^1 \tag{48}$$

For the first iteration, $\Delta u_1'^1$ is used instead of δu_i^n. Based on two previous formulations, the next load factor increment for the first iteration of each step has the coming shape

$$\Delta\lambda_1^n = \Delta\lambda_1^1 \sqrt{|GSP|} \tag{49}$$

Generalized stiffness parameter (GSP) is obtained by the following equation

$$GSP = \frac{(\Delta u_1'^1)^T \Delta u_1'^1}{(\Delta u_1'^{n-1})^T \Delta u_1'^n} \tag{50}$$

2.2.8 Modified normal flow method

This technique has been proposed by Saffari *et al*. (2008). It is said that the normal flow scheme (Watson *et al*. 1987, 1997) has not attracted the attention of researchers, in spite of the fact it has many capabilities. This algorithm is similar to the method of arc length. Fig. 11 demonstrates the modified normal flow scheme. The basis of this method is performing iterative analyses on the normal lines of Davidenko flow curves to achieve the equilibrium point (Allgower and Georg 1980). It is important to know that equations of Davidenko flow lines are determined by using the perturbation parameter η.

$$f(\lambda, u) = \eta \tag{51}$$

By changing the parameter η, a set of curves is obtained. These curves are called Davidenko flow lines (Allgower and Georg 1980). Notably, equilibrium equations of structures have infinite answers and modified normal flow method gives the unique and minimized answer of the equations (Watson *et al*. 1981). The reason is that iterative steps move in the shortest path to reach the equilibrium path (normal path). Fig. 11 shows these paths. It is important to know that in modified normal flow technique, the condition equation of residual displacement minimization algorithm is used. Therefore, the load factor increment is calculated by the next formula

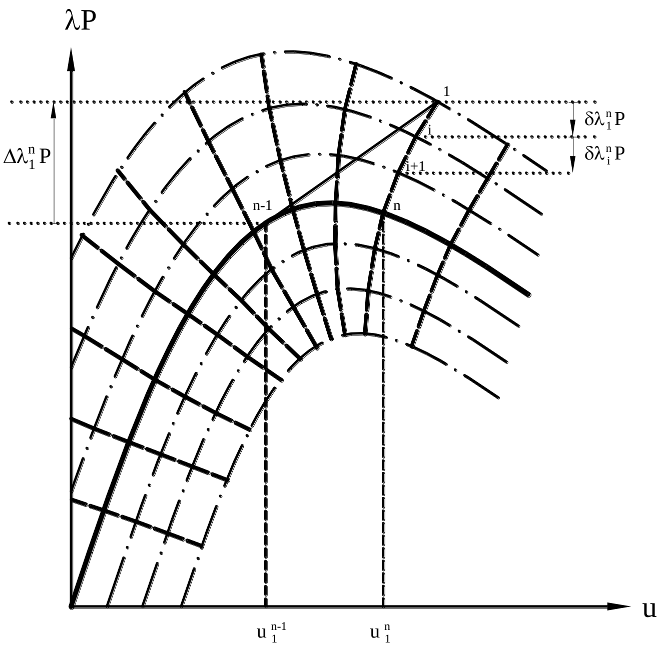

Fig. 11 Modified Normal Flow Method for the structure with one degree of freedom

$$\delta\lambda_i^n = -\frac{\delta u_i''^{nT} \delta u_i'^n}{\delta u_i'^{nT} \delta u_i'^n} \tag{52}$$

By selecting the following formulation and replacing the obtained load factor increment from Eq. (52) into it, the private answer *V* is specified

$$K_i^n V = \delta\lambda_i^n P + R_i^n \tag{53}$$

Finally, displacement increment is formulated by the following equation

$$\delta u_i^n = V - \frac{V^T \delta u_i'^n}{\left\| \delta u_i'^n \right\|} \delta u_i'^n \tag{54}$$

Where, $\left\| \delta u_i''^n \right\|$ shows the norm of displacement increment due to the residual load. Saffari *et al*. (2008) have compared this technique with method of Chan (1988) and the Crisfield (1983) arc length with linear search. In all the arc length solutions, not only the load factor increment, but also its sign should be characterized. The way of determining the sign of this parameter will be explained later.

2.2.9 Three-parameter methods

Some of the arc length techniques are formulated in two spaces of force and displacement. This leads to different numerical problems during the analysis. One of the methods used to overcome this shortcoming is called three-parameter arc length control scheme. This approach has been proposed by Krishnamoorthy *et al*. (1996). In this process, the constraint equations are dimensionless. In addition to specifying the primary load factor increment, three other parameters should also be determined (a_0, a_1, a_2). Here, three methods of Ellipsoidal, Hyperbolic, and Polynomial are studied.

2.2.9.1 Ellipsoidal method

Like the equation of an ellipse in two-dimensional space, the constraint of the ellipsoidal technique can be written in multi-dimensional space

$$\frac{\Delta u_{i+1}^{n\;T} \Delta u_{i+1}^{n}}{a_1^2} + \frac{\left(\Delta\lambda_{i+1}^{n}\right)^2 P^T P}{a_2^2} = a_0^2 \tag{55}$$

In this formula, a_1 and a_2 have the unit of displacement and force, respectively. Amount of a_2 is calculated by the following equation

$$a_2 = \sqrt{P^T P} \tag{56}$$

By putting Eq. (56) in Eq. (55), parameter a_2 is defined for the first step by the following expression

$$a_1 = \sqrt{\frac{\Delta u_1^{1\;T} \Delta u_1^1}{a_0^2 - \left(\Delta\lambda_1^1\right)^2}} \tag{57}$$

It is clear that in this formula, denominator should be always larger than zero. Therefore, the absolute value of a_0 ought to be greater than $\Delta\lambda_1^1$. Factor a_0 is determined by the analyst. The obtained amount for a_1 depends on the optional amount of $\Delta\lambda_1^1$. After determining parameters a_0, a_1 and a_2, the load factor increment at first iteration of each step is estimated by the following equation

$$\Delta\lambda_1^n = \pm \sqrt{\frac{a_0^2}{1 + \dfrac{\Delta u_1'^{n\;T} \Delta u_1'^{n}}{a_1^2}}} \tag{58}$$

Like cylindrical arc length method, by putting Δu_{i+1}^n and $\Delta\lambda_{i+1}^n$ into Eq. (55) and using Eq. (16), load factor increment is written in corrector steps as

$$a\left(\delta\lambda_i^n\right)^2 + b\left(\delta\lambda_i^n\right) + c = 0 \tag{59}$$

Constant parameters of a, b and c have the subsequent forms

$$a = \delta u_i'^{n\;T} \delta u_i'^{n} + a_1^2 \tag{60}$$

$$b = 2\delta u_i'^{n\;T}\left(\Delta u_i^n + \delta u_i''^{n}\right) + 2a_1^2 \Delta\lambda_i^n \tag{61}$$

$$c = \left(\Delta u_i^n + \delta u_i''^n\right)^T \left(\Delta u_i^n + \delta u_i''^n\right) + a_1^2 \left[\left(\Delta\lambda_i^n\right)^2 - a_0^2\right] \tag{62}$$

Rezaiee-Pajand and Akhaveysi (2000) have proposed the new ellipsoidal arc length method for analysis of the structures.

2.2.9.2 Hyperbolic method

Constraint equation of hyperbolic technique is written by the following equality

$$\frac{\Delta u_{i+1}^{n\ T} \Delta\lambda_{i+1}^n P}{a_1 a_2} = a_0^2 \tag{63}$$

Here, a_0, a_1 and a_2 are obtained similar to the previous section. After determining the sign of the a_0 by analyst, parameter a_1 for the first step is specified by coming relation

$$a_1 = \frac{\Delta u_1^{1^T} \Delta\lambda_1^1 P}{a_0^2 a_2} \tag{64}$$

Parameter a_2 is also found by Eq. (56). Appropriate selection of the $\Delta\lambda_1^1$ and P plays an important role in determining the sign of a_1. Load factor increment for first iteration of the analysis steps is computed by

$$\Delta\lambda_1^n = \pm\sqrt{\frac{a_0^2 a_1 a_2}{\Delta u_1'^{n^T} P}} \tag{65}$$

Similar to preceding technique, the amount of load factor increments in corrector steps is expressed by the following equation

$$a\left(\delta\lambda_i^n\right)^2 + b\left(\delta\lambda_i^n\right) + c = 0 \tag{66}$$

By the next formulas, a, b and c are specified

$$a = P^T \delta u_i'^n \tag{67}$$

$$b = P^T \left(\Delta u_i^n + \delta u_i''^n + \Delta\lambda_i^n \delta u_i'^n\right) \tag{68}$$

$$c = \Delta\lambda_i^n P^T \left(\Delta u_i^n + \delta u_i''^n\right)^T \left(\Delta u_i^n + \delta u_i''^n\right) - a_0^2 a_1 a_2 \tag{69}$$

2.2.9.3 Polynomial method

The condition equation for the polynomial solution approach is formulated from the combination of previous techniques. This equation is a second-order polynomial which is written in below shape

$$\frac{\Delta u_{i+1}^{n\ T} \Delta u_{i+1}^n}{a_1^2} + \frac{\Delta\lambda_{i+1}^n P^T \Delta u_{i+1}^n}{a_1 a_2} + \frac{\left(\Delta\lambda_{i+1}^n\right)^2 P^T P}{a_2^2} = a_0^2 \tag{70}$$

Parameters a_0, a_1 and a_2 are calculated in a similar way of the previous sections. It is worth mentioning that Eq. (70) has the following characteristics:

1- If the second and third terms are removed from the left side of this relation, the remaining parts will be Crisfield cylindrical arc length method.

2- If the second term is removed from the left side of equation, the ellipsoidal formulation is obtained.

3- By removing the first and third terms from the left side of this equation, the hyperbolic scheme is found.

By solving the Eq. (70) for a_1 at first iteration, the following second-order expression is obtained

$$a_2\left[a_0^2-\left(\Delta\lambda_1^1\right)^2\right]a_1^2-\left(\Delta\lambda_1^1\right)P^T\Delta u_1^1 a_1-a_2\Delta u_1^{1^T}\Delta u_1^1=0 \tag{71}$$

Parameter a_1 is determined by solving the preceding formula. The Delta equation after simplifying can be written as

$$Delta=\Delta u_1^{1^T}\Delta u_1^1 P^T P\left[4a_0^2-3\left(\Delta\lambda_1^1\right)^2\right] \tag{72}$$

In the case of $Delta>0$, by assuming $0<\Delta\lambda_1^1<1$, inequality $a_0>\sqrt{0.75}$ is obtained. By specifying three parameters of a_0, a_1 and a_2, the amount of load factor increment for first iteration of analysis steps is calculated by the next equation

$$\Delta\lambda_1^n=\pm\sqrt{\frac{a_0^2}{1+\dfrac{\Delta u_1'^{n^T}\Delta u_1'^n}{a_1^2}+\dfrac{P^T\Delta u_1'^n}{a_1 a_2}}} \tag{73}$$

Like two former techniques, the load factor increment for the corrector steps is obtained by the following formulation

$$a\left(\delta\lambda_i^n\right)^2+b\left(\delta\lambda_i^n\right)+c=0 \tag{74}$$

The parameters a, b and c are given by

$$a=a_2\delta u_i'^{n^T}\delta u_i'^n+a_1P^T\delta u_i'^n+a_1^2a_2 \tag{75}$$

$$b=2a_2\delta u_i'^{n^T}\left(\Delta u_i^n+\delta u_i''^n\right)+a_1\Delta\lambda_i^nP^T\delta u_i'^n+a_1P^T\left(\Delta u_i^n+\delta u_i''^n\right)+2\Delta\lambda_i^na_1^2a_2 \tag{76}$$

$$c=a_2\left(\Delta u_i^n+\delta u_i''^n\right)^T\left(\Delta u_i^n+\delta u_i''^n\right)+a_1\Delta\lambda_i^nP^T\left(\Delta u_i^n+\delta u_i''^n\right)-\left[\left(\Delta\lambda_i^n\right)^2-a_0^2\right]a_1^2a_2 \tag{77}$$

3. Methods characteristics

In the process of nonlinear structural analysis, determining the parameters such as the amount and sign of the load factor increment for the predictor step, choosing the correct root for the predictor and corrector steps (for second order methods) and convergence criterion are necessary. Researchers have used different ways of determining these characteristics. These methods are described in current section. It should be emphasized that the formula for determining these characteristics has an important role in the ability of nonlinear analysis strategies to trace the equilibrium path of structures.

3.1 Determining the load factor increment at predictor step

As it was mentioned, determining the load factor increment for the first iteration of each step plays a significant role in iteration processes of that step. This factor mainly depends on the intensity of nonlinear behavior of structure. Large selection of the first load factor increment may lead to the slowdown of convergence process or even may guide to divergence. Furthermore, small selection of this increment increases the analysis time. The reason is due to the large number of equilibrium points on the path of structure behavior. However, it may increase the analysis accuracy. Therefore, to improve the speed of analysis convergence, appropriate amounts of $\Delta\lambda_1^n$ and/or Δu_1^n should be determined at the beginning of iterations.

Usually, the amount of first load factor increment at the beginning of analysis $\left(\Delta\lambda_1^1\right)$ is assumed to be between 20 to 40 % of the predicted maximum load (Clarke and Hancock 1990). The current amount can also be utilized for other increments at the predictor step $\left(\Delta\lambda_1^n\right)$. However, later, researchers have proposed the automatic methods for calculating this increment for a more efficient analysis process.

Researchers have used different ways for determining the load factor increment at the predictor step. Choosing the load factor depends on the stiffness factor and the number of iterations of analysis. In the arc length algorithms, the load factor increment at the first iteration depends on the assumed arc length L_n. Increment of this factor in displacement control technique and constant work control is obtained based on the displacement increment and the assumed work increment, respectively. Therefore, the dependent increment should be specified and then predictor load factor should be calculated at the beginning of each step in these techniques. The required formulations for calculation of load factor increment at the predictor step have been given in the previous parts.

Crisfield (1981) and Ramm (1981) used J_d/J_{n-1} for estimation of the load factor increment. Parameter J_d is the number of modified iterations determined by the analyst for achieving convergence at $n-1^{\text{th}}$ step. This number is usually selected between 4 or 5. Parameter J_{n-1} is the actual number of available iterations at the step $n-1$ to achieve convergence. Based on this, the following formulation has been proposed for determination of the predictor load factor increment

$$\Delta\lambda_1^n = \pm\Delta\lambda_1^{n-1}\left(\frac{J_d}{J_{n-1}}\right)^{\gamma} \tag{78}$$

It should be noted that γ is chosen to be 0.5 by Crisfield (1982) and Ramm (1981), and 0.25 by Bellini and Chulya (1987). In displacement control method, the j^{th} component of displacement increment at the beginning of each step is calculated by coming relationship

$$\Delta u_{1j}^n = \Delta u_{1j}^{n-1}\left(\frac{J_d}{J_{n-1}}\right)^{\gamma} \tag{79}$$

To determine the arc length size at the beginning of each step, the following equation is used

$$L_n = L_{n-1}\left(\frac{J_d}{J_{n-1}}\right)^{\gamma} \tag{80}$$

In constant work control, the latter work increment at the beginning of all steps is determined by the next equality

$$\Delta W_1^n = \Delta W_1^{n-1}\left(\frac{J_d}{J_{n-1}}\right)^{\gamma} \tag{81}$$

Other strategies of determining the load factor increment for the first iteration are based on the stiffness and nonlinear intensity of structures. For instance, Bergan *et al*. (1978) suggested current stiffness parameter as a criterion for nonlinear intensity of structure behavior. Therefore, load increment for the first iteration is found by the following equation

$$\Delta\lambda_1^n = \pm\Delta\lambda_1^1 (S_P)^{\gamma} \tag{82}$$

In this equation, parameter γ is specified by the researcher. This parameter depends on the nonlinear intensity of structure behavior. For example, in the structures with quick geometrical change, like inelastic buckling of the columns, the amount of γ is assumed larger. The subsequent stiffness feature (S_P) is utilized

$$S_P = \frac{\Delta\lambda_1^n P^T \Delta u_1'^1}{\Delta\lambda_1^1 P^T \Delta u_1'^n} \tag{83}$$

To find the load factor increment, Chan's generalized stiffness parameter, which is obtained by Eq. (50), can also be used. Thus, this parameter for the predictor step is calculated by the following equation

$$\Delta\lambda_1^n = \pm\Delta\lambda_1^1 (GSP)^{\gamma} \tag{84}$$

It is clear that in structure with behavior close to the linear one, the predictor load factor increment can be chosen larger. In fact, this factor reaches its maximum for the linear behavior situation.

3.2 Determining the displacement increment at the predictor step

As it was discussed so far, the first iteration increments are called the predictor increment, and their related expressions are called predictor equations. When using displacement control method, it is necessary to determine a displacement increment on the predictor step. Most of the predictor equations are linear. The simplest one is Euler equation, which is given below:

$$\Delta u_1^n = \Delta\lambda_1^n \left(K^{n-1}\right)^{-1} P \tag{85}$$

In this equation, $K^{n-1}$is the tangent stiffness matrix, which is obtained at the end of $n-1^{\text{th}}$ step.

3.3 Determining the sign of load factor increment at the predictor step

In the process of incremental-iterative methods, after calculating the load factor increment, its sign should also be specified. In other words, it ought to be illustrated that this increment is increasing or decreasing. It is clear that the sign of load factor increment changes at the load limit points. Due to its importance, different solutions have been proposed for determining the sign of load factor increment. Actually, these schemes present the maximum and minimum points of the structure equilibrium paths for a specified degree of freedom.

Some analysts (Bergan *et al*. 1978, Bathe and Dvorkin 1983) have chosen the change of sign of external work increment as the criterion of determining the limit point. According to this scheme, sign of load factor increment does not change until the external work increment sign has not changed. In other words, change of the sign of external work increment alters the direction of the load factor increment. External work increment at the first iteration is calculated by the following formulation

$$\Delta W_1^n = \Delta\lambda_1^n P^T \Delta u_1'^n \tag{86}$$

The other approaches use the determinant of structural tangent stiffness matrix (Bergan and Soreide 1978, Ramm 1981, Crisfield 1981). In these techniques, the load factor increment sign changes when the sign of tangent stiffness matrix determinant changes. The reason is that, in the load limit point, the stiffness matrix determinant becomes zero and then its sign changes.

Bergan *et al*. (1978) have also proposed the sign of the current stiffness parameters as the criterion of changing the sign of load factor increment. This method has also been used by some other researchers (Yang and Shieh 1990, Kuo and Yang 1995). The current stiffness parameter of the structure is obtained by Eq. (83).

It should be noted that the last-mentioned techniques do not give an acceptable answer for determining the load factor increment sign in all situations. According to the research of Meek and Hoon Swee Tan (1984), in structures that their behavior has a large number of singular negative values, using the stiffness matrix determinant sign does not lead to an appropriate result. Inefficiency of the stated strategies has been examined by the researchers, and their shortcomings have been illustrated (Bellini and Chulya 1987, Clarke and Hancock 1990).

In the second part of this paper, another strong technique will be utilized for determining the load factor increment sign. This scheme has higher efficiency than other proposed solutions. By defining two new vectors of ΔQ^n and t^T, in the following formulations and their internal multiplying, it is easy to determine the sign of load factor increment

$$\Delta Q^n = \Delta u^n e_1 + \Delta\lambda^n P e_2 \tag{87}$$

$$t^T = \Delta u^T e_1 + P e_2 \tag{88}$$

Actually, ΔQ^n connects the obtained equilibrium point at $n-1^{\text{th}}$ step to point n. Vector t^T is also a tangent to the equilibrium path at point n. What is important in creating this vector is that its load factor increment is +1($\Delta\lambda$=1). These vectors are shown in Fig. 12. After internal multiplying of two vectors, there is following discussion about the sign: if the sign is positive ($t^T \bullet \Delta Q^n \geq 0$), the load factor increment is increasing and has a positive sign ($\Delta\lambda>0$). Otherwise, the load factor increment sign is negative ($\Delta\lambda<1$).

3.4 Selection of the correct root in predictor step

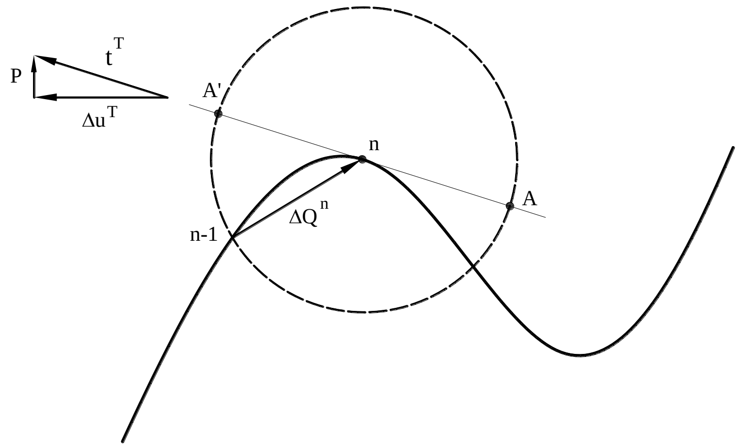

Fig. 12 Determining the sign of load factor increment

After determining the sign of load factor increment in predictor step, selection of the appropriate root of the load factor becomes easy. Based on Fig. 12, these roots are located on the constraint surface of analysis method. By calculating the roots of arc length equation, the load factor increment would be increasing or decreasing. In other words, the load factor increment sign determines the path for choosing the correct root of arc length. For example, Fig. 12 shows the negative load factor increment, because $t^T \bullet \Delta Q^n < 0$. Therefore, the structural equilibrium path goes to the down side, and finally, point *A* (instead of *A'* point) is selected as the correct root. Other processes have also been proposed for this issue (Feng *et al*. 1995, 1996, Souza Neto and Feng 1999).

3.5 Selection of the correct root at the corrector step

In the iterative analyses of the corrector step, the root which is closer to the obtained point in the last iteration is selected as the correct answer. For example, as point *O* is closer to the last iteration point (*A*) of Fig. 13, the acceptable root is identified, and point *O'* is not the correct answer. For selecting the correct root, the following criterion can be used. By internal multiplying of the two vectors of t^T and ΔQ^n, the accurate root is obtained

$$(\delta\lambda)_1 \xrightarrow{if} (t^T \cdot \Delta Q^n)(\delta\lambda)_1 > (\delta\lambda)_2 \tag{89}$$

$$(\delta\lambda)_2 \xrightarrow{if} (t^T \cdot \Delta Q^n)(\delta\lambda)_2 > (\delta\lambda)_1 \tag{90}$$

The values of $(\delta\lambda)_1$ and $(\delta\lambda)_2$ are the obtained roots in the corrector step. Other criteria have also been used for achieving the correct root. For example, the Carrera (1994) criterion can be mentioned. According to this method, the acceptable root is the one that is closer to the linear answer calculated by the method of Schweizerhof and Wriggers (1986). Hellweg and Crisfield (1998) have also proposed their own way for selection of the correct root in sharp snap-back points.

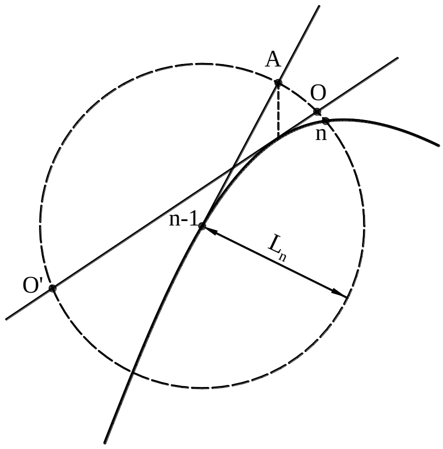

Fig. 13 Selection of the correct root

3.6 Convergence criterion

For limiting the iteration cycles and ending incremental-iterative calculations in each step, convergence criterion is used. Researchers have employed different criteria for achieving convergence. These methods are based on the displacement increment and work increment formed by residual forces (Bergan and Clough 1972, Chen and Blandford 1993). In the second part of this paper, the next convergence criterion will be utilized based on the residual load

$$R_i^{n^T} R_i^n < \varepsilon \tag{91}$$

Amount of ε is specified by the analyst, and depends on the structure. It is usually determined in the range of 10^{-5} to 10^{-2}. This amount will be considered to be 10^{-4}in the second part of paper. Based on this, iterative analysis of each step continues until Eq. (91) is fulfilled or the number of iterations exceeds the maximum number of iteration. The maximum number of iterations is assigned by the analyst.

4. Conclusions

In this paper, formulations and characteristics of the most well-known and applicable structural nonlinear solution techniques were investigated. In fact, general limitations, features, and capabilities of the 13 solution schemes were discussed. After studying the traditional methods of Newton-Raphson and modified Newton-Raphson, advanced analysis approaches were described. These algorithms are named residual load minimization, normal plane, updated normal plane, cylindrical arc length, work control, residual displacement minimization, generalized displacement control, modified normal flow, and three-parameter ellipsoidal, hyperbolic, and polynomial methods. Then, different ways of determining required parameters were presented. These characteristics include determining the amount and sign of load factor increment in the predictor step, selection of the correct root in the corrector and predictor steps (for second order methods), and convergence criterion. It should be noted that authors have written the formulations of all declared procedures with the similar symbols. As a result, some kinds of solution techniques with the wide range of abilities were investigated in this article.

Additionally, a comprehensive and accurate comparison of the mentioned methods for the analysis of 17 numerical problems will be given in the second part of paper. Two and three-dimensional truss structures, two and three-dimensional frames, and shells with simple and complex geometrical nonlinear behavior will be analyzed there. Based on the obtained results, capability of the solution techniques will be illustrated. In this regard, analysis running time, number of iterations, convergence speed, and capability to trace the equilibrium path, and other limitations and characteristics of these solution processes will be discussed.

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Notations

Δ	Increment of the factors at first iteration
δ	Increment of the factors at iterations except the first one
$\rightarrow$	Vector sign
$[\ ]$	Matrix sign
F	Internal force vector
λ	Load factor
P	Reference load
u	Displacement
K^n	Tangent stiffness matrix in step n
$\Delta\lambda_1^n$	Load factor increment in first iteration of step n
$\delta\lambda_i^n$	Load factor increment in iteration i of step n
R_i^n	Residual load vector in iteration i of step n
B_i^n	Strain matrix in iteration i of step n
σ_i^n	Internal strain vector in iteration i of step n
$\delta u_i'^n$	Displacement increment caused by reference load in iteration i of step n
$\delta u_i''^n$	Displacement increment caused by residual load in iteration i of step n
$\Delta u_1'^n$	Displacement increment caused by reference load in first iteration of step n

$\Delta u_1''^n$	Displacement increment caused by residual load in first iteration of step *n*
$\tilde{R}_i^n$	Reduced residual load vector
t_i^n	Connector vector of *i* and *n*
n_i^n	Connector vector of *i* and *i+1*
e_1	Unit vector in u direction
e_2	Unit vector in P direction
L_n	Arch length
a, b, c	Constant factors in the second-order equation of load factor increment
Δw_1^n	Work increment in first iteration of step *n*
dw_i^n	Work increment in iteration *i* of step *n*
H_i^n	Generalized displacement in iteration *i* of step *n*
GSP	Generalized stiffness parameter of Chan
η	Davidenko perturbation parameter
V	Particular solution of modified normal flow algorithm
$\lVert\ \rVert$	Norm sign
a_1, a_2, a_3	Factors of three-parameter methods
J_d	Optimum number of iterations
J_{n-1}	Number of iterations in step $n-1$
γ	Exponent of equation for determining the first load factor increment
S_P	Current stiffness parameter of Bergan
ΔQ^n	Connector vector of *n* and *n*-1
t^T	Tangent vector of equilibrium path with $\Delta\lambda = +1$
ε	Tolerance of convergence