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Comparison of various refined nonlocal beam theories for bending, vibration and buckling analysis of nanobeams

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Abstract. In this paper, unified nonlocal shear deformation theory is proposed to study bending, buckling and free vibration of nanobeams. This theory is based on the assumption that the in-plane and transverse displacements consist of bending and shear components in which the bending components do not contribute toward shear forces and, likewise, the shear components do not contribute toward bending moments. In addition, this present model is capable of capturing both small scale effect and transverse shear deformation effects of nanobeams, and does not require shear correction factors. The equations of motion are derived from Hamilton's principle. Analytical solutions for the deflection, buckling load, and natural frequency are presented for a simply supported nanobeam, and the obtained results are compared with those predicted by the nonlocal Timoshenko beam theory and Reddy beam theories.

Keywords: nonlocal beam models; shear deformation; bending; buckling; vibration; nanobeam

1. Introduction

Structural elements such as beams, plates, and membranes in micro - or nano length scale are commonly used as components in micro - or nanoelectromechanical system devices (Lavrik *et al*. 2004, Ekinci and Roukes 2005). Two basically different approaches are available for theoretical modelling of nanostructured materials: the atomistic approaches and the continuum mechanics. The former includes the classical molecular dynamics (MD), tight-binding molecular dynamics (TBMD) and density functional theory (DFT) (Iijima *et al.* 1996, Yakobson *et al.* 1997, Hernandez *et al.* 1998, Sanchez-Portal *et al.* 1999, Qian *et al.* 2002). These approaches are often computationally expensive, especially for large-scale CNTs with high number of walls. Hence, the continuum mechanics is increasingly being viewed as an alternative way of modelling materials at the nanometer scale. Indeed, size effects are significant in the mechanical behavior of these structures in which dimensions are small and comparable to molecular distances. These effects can

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be captured using size-dependent continuum mechanics such as strain gradient theory (Nix and Gao 1998), modified couple stress theory (Ma *et al*. 2008), and nonlocal elasticity theory (Eringen 1972). Unlike classical theories, the nonlocal theories contain internal material length scale parameters that can capture size effects at the nano scale. A review of various nonlocal models can be found in Bazant and Jirasek (2002).

In this paper the nonlocal elasticity theory of Eringen (Eringen 1983, Eringen and Edelen 1972) is used to study the bending, buckling and free vibration of simply supported nanobeams. The nonlocal theory was developed by several authors as a response to the inability of local elasticity to handle elastic problems with sharp geometrical singularities (for example, a sharp crack-tip). The Eringen model was applied to Euler–Bernoulli micro and nanobeams by Peddieson *et al*. (2003), Sudak (2003), Amara *et al*. (2010) to the study of column buckling of carbon nanotubes and by Pisano and Fuschi (2003) for the study of an elastic bar in tension. Several authors extended the use of nonlocal theory to the study of free transverse vibrations (Reddy 2007, Xu 2006), bending (Reddy 2007, Wang *et al.* 2008) and buckling (Reddy 2007) of nanobeams. Zhang *et al*. (2010) a hybrid nonlocal Euler–Bernoulli beam model is proposed and nonlocal nonlinear formulation of beams is presented by Reddy (2010). Using the finite element method, Phadikar and Pradhan (2010) studied nanobeams with a linear nonlocal formulation. Tounsi and his co-workers (Heireche *et al.* 2008a, b, c, Tounsi *et al.* 2008) investigated the sound wave propagation in single- and double-walled CNTs taking into account the nonlocal effect, temperature and initial axial stress. Furthermore, Tounsi *et al.* (2009a, b) derived the consistent governing equation of motion for the free vibration of fluid- conveying CNTs with nonlocal effect, which is an important application of nonlocal elastic theory in CNTs. More recently, the effect of chirality on mechanical responses of CNTs is discussed by Maachou *et al.* (2011), Naceri *et al.* (2011), Besseghier *et al.* (2011), Zidour *et al.* (2012), Gafour *et al.* (2013), Benguediab *et al.* (2013), Semmah *et al.* (2013), Baghdadi *et al.* (2013). Tounsi *et al.* (2013) studied the nonlocal effect on thermal buckling properties of DWCNTs where the applicability of continuum beam model for CNTs is examined and the magnitude of the small-scale parameter (i.e., e_0a) is determined after comparing the results with those obtained from molecular dynamics (MD) simulations.

In the present study, various shear deformation theories are used for the bending, buckling and vibration of nanoscale beams using local and nonlocal elasticity. These theories are based on assumption that the in-plane and transverse displacements consist of bending and shear components, in which the bending components do not contribute toward shear forces and, likewise, the shear components do not contribute toward bending moments. The most interesting feature of this theory is that it accounts for a nonlinear variation of the transverse shear strains across the thickness and satisfies the zero traction boundary conditions on the top and bottom surfaces of the nanobeam without using shear correction factors. Based on the nonlocal constitutive relations of Eringen, equations of motion of nanobeams are derived using Hamilton’s principle. Analytical solutions for the deflection, buckling load, and natural frequency are presented for simply supported nanobeams, and the obtained results are compared with those predicted by the Euler–Bernoulli beam theory (EBT), Timoshenko beam theory (TBT), and Reddy’s beam theory (RBT).

2. Theoretical formulations

2.1 Basic assumptions

The displacement field of the proposed theory is chosen based on the following assumptions:

(i) The displacements are small in comparison with the nanobeam thickness and, therefore, strains involved are infinitesimal.

(ii) The transverse displacement $\boldsymbol{w}$ includes two components of bending $\boldsymbol{w_b}$, and shear $\boldsymbol{w_s}$. These components are functions of coordinate x only.

$$\boldsymbol{w}(\boldsymbol{x},\boldsymbol{z}) = \boldsymbol{w_b}(\boldsymbol{x}) + \boldsymbol{w_s}(\boldsymbol{x}) \tag{1}$$

(iii) The transverse normal stress $\boldsymbol{\sigma_z}$ is negligible in comparison with in-plane stresses $\boldsymbol{\sigma_x}$.

(iv) The displacement $\boldsymbol{u}$ in x-direction consists of bending, and shears components.

$$\boldsymbol{u} = \boldsymbol{u_b} + \boldsymbol{u_s} \tag{2}$$

The bending component $\boldsymbol{u_b}$ is assumed to be similar to the displacement given by the classical beam theory. Therefore, the expression for $\boldsymbol{u_b}$ can be given as

$$\boldsymbol{u_b} = -\boldsymbol{z}\frac{\partial \boldsymbol{w_b}}{\partial \boldsymbol{x}} \tag{3}$$

Displacement component due to shear deformation ($\boldsymbol{u_s}$) is assumed to be parabolic, sinusoidal, hyperbolic and exponential in nature with respect to thickness coordinate. Thus, the shear component $\boldsymbol{u_s}$ gives rise, in conjunction with $\boldsymbol{w_s}$, to a higher order variations of shear strain $\boldsymbol{\gamma_{xz}}$ and hence to shear stress $\boldsymbol{\tau_{xz}}$ through the thickness of the nanobeam in such a way that shear stress $\boldsymbol{\tau_{xz}}$ is zero at the top and bottom faces of the nanobeam. Consequently, the expression for $\boldsymbol{u_s}$ can be given as

$$\boldsymbol{u_s} = -\boldsymbol{f}(\boldsymbol{z})\frac{\partial \boldsymbol{w_s}}{\partial \boldsymbol{x}} \tag{4}$$

The functions $\boldsymbol{f(z)}$ assigned according to the shearing stress distribution through the thickness of the beam are given in Table 1.

2.2 Kinematics

Based on the assumptions made in the preceding section, the displacement field can be obtained using Eqs. (1)-(4) as

Table 1 Shape functions

Model	$\boldsymbol{f}(\boldsymbol{z})$
Model 1	$\boldsymbol{f}(\boldsymbol{z}) = \dfrac{4\boldsymbol{z}^3}{3\boldsymbol{h}^2}$
Model 2	$\boldsymbol{f}(\boldsymbol{z}) = \boldsymbol{z} - \dfrac{\boldsymbol{h}}{\boldsymbol{\pi}}\sin\left(\dfrac{\boldsymbol{\pi}\,\boldsymbol{z}}{\boldsymbol{h}}\right)$
Model 3	$\boldsymbol{f}(\boldsymbol{z}) = \boldsymbol{z} - \boldsymbol{h}\sinh\left(\dfrac{\boldsymbol{z}}{\boldsymbol{h}}\right) + \boldsymbol{z}\cosh\left(\dfrac{1}{2}\right)$
Model 4	$\boldsymbol{f}(\boldsymbol{z}) = \boldsymbol{z} - \boldsymbol{z}\boldsymbol{e}^{-2(\boldsymbol{z}/\boldsymbol{h})^2}$

$$u(x,z,t) = -z\frac{\partial w_b}{\partial x} - f(z)\frac{\partial w_s}{\partial x} \tag{5a}$$

$$w(x,z,t) = w_b(x,t) + w_s(x,t) \tag{5b}$$

The strains associated with the displacements in Eq. (5) are

$$\varepsilon_x = \varepsilon_x^0 + z\,k_x^b + f(z)\,k_x^s \text{ and } \gamma_{xz} = g(z)\,\gamma_{xz}^s \tag{6}$$

where

$$\varepsilon_x^0 = \frac{\partial u_0}{\partial x}, \quad k_x^b = -\frac{\partial^2 w_b}{\partial x^2}, \quad k_x^s = -\frac{\partial^2 w_s}{\partial x^2}$$

$$\gamma_{xz}^s = \frac{\partial w_s}{\partial x}, \quad g(z) = 1 - f'(z) \text{ and } f'(z) = \frac{df(z)}{dz} \tag{7}$$

2.3 Constitutive relations

Response of materials at the nanoscale is different from those of their bulk counterparts. Nonlocal elasticity is first considered by Eringen (1983). He assumed that the stress at a reference point is a functional of the strain field at every point of the continuum. Eringen (1983) proposed a differential form of the nonlocal constitutive relation as

$$\sigma_x - \mu\frac{d^2\sigma_x}{dx^2} = E\varepsilon_x \tag{8a}$$

$$\tau_{xz} - \mu\frac{d^2\tau_{xz}}{dx^2} = G\gamma_{xz} \tag{8b}$$

where E and G are the elastic modulus and shear modulus of the nanobeam, respectively; $\mu = (e_0 a)^2$ is the nonlocal parameter, e_0 is a constant appropriate to each material and a is an internal characteristic length. The nonlocal parameter depends on the boundary conditions, chirality, mode shapes, number of walls, and type of motion (Arash and Wang 2012). So far, there is no rigorous study made on estimating the value of the nonlocal parameter. It is suggested that the value of nonlocal parameter can be determined by conducting a comparison of dispersion curves from the nonlocal continuum mechanics and molecular dynamics simulation (Arash and Ansari 2010, Wang and Wang 2005, 2007). In general, a conservative estimate of the nonlocal parameter is $e_0 a < 2.0$ nm for a single wall carbon nanotube (Wang and Wang 2007).

2.4 Equations of motion

Hamilton’s principle is used herein to derive the equations of motion. The principle can be stated in analytical form as (Reddy 2002)

$$\delta\int_0^T (U + V - K)\,dt = 0 \tag{9}$$

where δU is the virtual variation of the strain energy; δV is the virtual variation of the potential energy; and δK is the virtual variation of the kinetic energy. The variation of the strain energy of the beam can be stated as

$$\begin{aligned}\delta U &= \int_0^L \int_A \left(\sigma_x \delta \varepsilon_x + \tau_{xz} \delta \gamma_{xz}\right) dA dx \\ &= \int_0^L \left(-M_b \frac{d^2 \delta w_b}{dx^2} - M_s \frac{d^2 \delta w_s}{dx^2} + Q \frac{d \delta w_s}{dx} \right) dx\end{aligned} \tag{10}$$

where M_b, M_s and Q are the stress resultants defined as

$$(M_b, M_s) = \int_A (z, f) \sigma_x dA \text{ and } Q = \int_A g \tau_{xz} dA \tag{11}$$

The variation of the potential energy by the applied loads can be written as

$$\delta V = -\int_0^L q \delta (w_b + w_s) dx - \int_0^L N_0 \frac{d(w_b + w_s)}{dx} \frac{d\delta(w_b + w_s)}{dx} dx \tag{12}$$

where q and N_0 are the transverse and axial loads, respectively.

The variation of the kinetic energy can be expressed as

$$\begin{aligned}\delta K &= \int_0^L \int_A \rho \left[\dot{u} \delta \dot{u} + \dot{w} \delta \dot{w} \right] dA dx \\ &= \int_0^L \left\{ I_0 (\dot{w}_b + \dot{w}_s)(\delta \dot{w}_b + \delta \dot{w}_s) + I_2 \left(\frac{d\dot{w}_b}{dx} \frac{d\delta \dot{w}_b}{dx} \right) + K_2 \left(\frac{d\dot{w}_s}{dx} \frac{d\delta \dot{w}_s}{dx} \right) \right. \\ &\quad \left. + J_2 \left(\frac{d\dot{w}_b}{dx} \frac{d\delta \dot{w}_s}{dx} + \frac{d\dot{w}_s}{dx} \frac{d\delta \dot{w}_b}{dx} \right) \right\} dx\end{aligned} \tag{13}$$

where dot-superscript convention indicates the differentiation with respect to the time variable t; ρ is the mass density; and (I_0, I_2, J_2, K_2) are the mass inertias defined as

$$(I_0, I_2, J_2, K_2) = \int_A \left(1, z^2, z f, f^2\right) \rho \, dA \tag{14}$$

Substituting the expressions for δU, δV, and δK from Eqs. (10), (12) and (13) into Eq. (9) and integrating by parts, and collecting the coefficients of δw_b, and δw_s, the following equations of motion of the proposed beam theory are obtained

$$\delta w_b : \frac{d^2 M_b}{dx^2} + q - N_0 \frac{d^2 (w_b + w_s)}{dx^2} = I_0 (\ddot{w}_b + \ddot{w}_s) - I_2 \frac{d^2 \ddot{w}_b}{dx^2} - J_2 \frac{d^2 \ddot{w}_s}{dx^2} \tag{15a}$$

$$\delta w_s : \frac{d^2 M_s}{dx^2} + \frac{dQ}{dx} + q - N_0 \frac{d^2 (w_b + w_s)}{dx^2} = I_0 (\ddot{w}_b + \ddot{w}_s) - J_2 \frac{d^2 \ddot{w}_b}{dx^2} - K_2 \frac{d^2 \ddot{w}_s}{dx^2} \tag{15b}$$

when the shear deformation effect is neglected ($w_s = 0$), the equilibrium equations in Eq. (15)

recover those derived from the Euler–Bernoulli beam theory.

By substituting Eq. (6) into Eq. (8) and the subsequent results into Eq. (11), the stress resultants are obtained as

$$M_b - \mu \frac{d^2 M_b}{dx^2} = -D \frac{d^2 w_b}{dx^2} - D_s \frac{d^2 w_s}{dx^2} \tag{16a}$$

$$M_s - \mu \frac{d^2 M_s}{dx^2} = -D_s \frac{d^2 w_b}{dx^2} - H_s \frac{d^2 w_s}{dx^2} \tag{16b}$$

$$Q - \mu \frac{d^2 Q}{dx^2} = A_s \frac{dw_s}{dx} \tag{16c}$$

where

$$\left(D, D_s, H_s\right) = \int_A \left(z^2, z\, f, f^2\right) E dA, \quad A_s = \int_A g^2 G dA \tag{17}$$

By substituting Eq. (16) into Eq. (15), the nonlocal equations of motion can be expressed in terms of displacements (w_b, w_s) as

$$\begin{aligned} & -D \frac{d^4 w_b}{dx^4} - D_s \frac{d^4 w_s}{dx^4} + q - \mu \frac{d^2 q}{dx^2} - N_0 \left(\frac{d^2 (w_b + w_s)}{dx^2} - \mu \frac{d^4 (w_b + w_s)}{dx^4} \right) \\ & = I_0 \left((\ddot{w}_b + \ddot{w}_s) - \mu \frac{d^2 (\ddot{w}_b + \ddot{w}_s)}{dx^2} \right) - I_2 \left(\frac{d^2 \ddot{w}_b}{dx^2} - \mu \frac{d^4 \ddot{w}_b}{dx^4} \right) - J_2 \left(\frac{d^2 \ddot{w}_s}{dx^2} - \mu \frac{d^4 \ddot{w}_s}{dx^4} \right) \end{aligned} \tag{18a}$$

$$\begin{aligned} & -D_s \frac{d^4 w_b}{dx^4} - H_s \frac{d^4 w_s}{dx^4} + A_s \frac{d^2 w_s}{dx^2} + q - \mu \frac{d^2 q}{dx^2} - N_0 \left(\frac{d^2 (w_b + w_s)}{dx^2} - \mu \frac{d^4 (w_b + w_s)}{dx^4} \right) \\ & = I_0 \left((\ddot{w}_b + \ddot{w}_s) - \mu \frac{d^2 (\ddot{w}_b + \ddot{w}_s)}{dx^2} \right) - J_2 \left(\frac{d^2 \ddot{w}_b}{dx^2} - \mu \frac{d^4 \ddot{w}_b}{dx^4} \right) - K_2 \left(\frac{d^2 \ddot{w}_s}{dx^2} - \mu \frac{d^4 \ddot{w}_s}{dx^4} \right) \end{aligned} \tag{18b}$$

The equations of motion of local beam theory can be obtained from Eq. (18) by setting the scale parameter μ equal to zero.

3. Analytical solution of simply supported nanobeam

In this study, analytical solutions are given for simply supported isotropic nanobeams for bending, buckling and free vibration.

The boundary conditions of simply supported nanobeams are

$$w_b = w_s = M_b = M_s = 0 \text{ at } x = 0, L \tag{19}$$

The following displacement field satisfies boundary conditions and governing equations.

$$\begin{Bmatrix} w_b \\ w_s \end{Bmatrix} = \sum_{n=1}^{\infty} \begin{Bmatrix} W_{bn} \sin(\alpha\, x)\, e^{i \omega t} \\ W_{sn} \sin(\alpha\, x)\, e^{i \omega t} \end{Bmatrix} \tag{20}$$

where W_{bn}, and W_{sn} are arbitrary parameters to be determined, ω is the eigenfrequency associated with *n*th eigenmode, and $\alpha = n\pi/L$. The transverse load q is also expanded in the Fourier sine series as

$$q(x) = \sum_{n=1}^{\infty} Q_n \sin \alpha\, x\,, \quad Q_n = \frac{2}{L}\int_0^L q(x)\sin(\alpha\, x)\,dx \tag{21}$$

The Fourier coefficients Q_n associated with some typical loads are given

$$Q_n = q_0\,, \quad n = 1 \quad \text{for sinusoidal load} \tag{22}$$

$$Q_n = \frac{4q_0}{n\pi}\,, \quad n = 1,3,5..... \ \text{for uniform load} \tag{22b}$$

$$Q_n = \frac{2q_0}{L}\sin\frac{n\pi}{2}\,, \ n = 1,2,3.... \ \text{for point load } Q_0 \text{ at the midspan} \tag{22c}$$

Substituting the expansions of w_b, w_s and q from Eqs. (20) and (21) into Eq. (18), the closed-form solutions can be obtained from the following equations

$$\left(\begin{bmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{bmatrix} - \lambda N_0 \alpha^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \lambda\omega^2 \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix}\right)\begin{Bmatrix} W_{bn} \\ W_{sn} \end{Bmatrix} = \begin{Bmatrix} \lambda Q_n \\ \lambda Q_n \end{Bmatrix} \tag{23}$$

where

$$S_{11} = D\alpha^4\,, \quad S_{12} = D_s\alpha^4\,, \quad S_{22} = H_s\alpha^4 + A_s\alpha^2\,, \quad \lambda = 1 + \mu\alpha^2$$

$$m_{11} = I_0 + I_2\alpha^2\,, \quad m_{12} = I_0 + J_2\alpha^2\,, \quad m_{22} = I_0 + K_2\alpha^2 \tag{24}$$

3.1 Bending

The static deflection is obtained from Eq. (23) by setting N_0 and all time derivatives to zero

$$w(x) = \sum_{n=1}^{\infty} \frac{(S_{11} + S_{22} - 2S_{12})}{(S_{11}S_{22} - S_{12}^2)}\lambda Q_n \sin\alpha\, x \tag{25}$$

3.2 Buckling

The buckling load is obtained from Eq. (23) by setting q and all time derivatives to zero

$$N_0 = \frac{S_{11}S_{22} - S_{12}^2}{\lambda\alpha^2\left(S_{11} - 2S_{12} + S_{22}\right)} \tag{26}$$

3.3 Vibration

By setting q and N_0 in Eq. (23) equal to zero, the natural frequency can be obtained from the following equation

$$\left(m_{11}m_{22} - m_{12}^2\right)\lambda^2\omega^4 + \left(2S_{12}m_{12} - S_{11}m_{22} - S_{22}m_{11}\right)\lambda\omega^2 + \left(S_{11}S_{22} - S_{12}^2\right) = 0 \tag{27}$$

4. Validity and applicability of continuum beam model for nanobeams

Applicability of continuum beam model for carbon nanotubes (CNTs) is examined by several authors (e.g., Wang and Hu (2005), Harik (2001, 2002)). Harik (2001, 2002) reported ranges of applicability for the continuum beam model in the mechanics of carbon nanotubes and nanorods. Wang and Hu (2005) present a rigorous study, in which they check the validity of the beam model in studying the flexural waves, simulated by the molecular dynamics (MD), in a single – walled carbon nanotube. In this study, Wang and Hu (2005) observed that when the wave number is getting very large, the microstructure of the carbon nanotubes plays an important role in the flexural wave dispersion and significantly decreases the phase velocity of the flexural waves of high frequency.

Recently Tounsi *et al.* (2013) presented numerical results for critical buckling strains obtained from the continuum mechanics theory and a comparison study is carried out with MD simulations (Silvestre *et al.* 2011). It is seen that the critical buckling strains obtained by Tounsi *et al.* (2013) are in good agreement as compared with the results obtained from MD simulations. Based on the MD simulation results, the value of nonlocal constant is determined for CNTs based on an averaging process. The best match between MD simulations and nonlocal formulations is achieved for a nonlocal constant value of e_0a=0.54 nm for CNT (5, 5) and e_0a=1.05 nm for CNT (7, 7) with good accuracy (the error is less than 10%).

5. Numerical results

In this section, numerical results are given for analytical solutions given in the previous sections. For all calculations, the Poisson's ratio is taken as 0.3. However, for calculation carried out using TBT, the shear correction factor is taken as 5/6. The side of nanobeam L is assumed to be 10 nm. For convenience, the following nondimensionalizations are used:

- $\bar{w} = 100\, w \dfrac{EI}{q_0 L^4}$ for uniform load;
- $\bar{w} = 100\, w \dfrac{EI}{Q_0 L^3}$ for point load;
- $\bar{\omega} = \omega L \sqrt{\dfrac{I_0}{EI}}$ frequency parameter;
- $\bar{N} = N_{cr} \dfrac{L^2}{EI}$ critical buckling load parameter:

The numerical results for bending under uniform load are given in Table 2. Results are obtained using 100 terms in the series Eq. (25). It can be seen that the results of various proposed shear deformation theories are in excellent agreement with those predicted by TBT and RBT for all values of thickness ratio L/h and scale parameter μ. According to these results, scale parameter is more obvious for lower thickness ratio L/h and it decreases with increasing L/h. Scale effects are more pronounced for point load. EBT underestimates deflections for lower thickness ratios L/h. With increasing L/h results are converging to a certain value. The difference between EBT and shear deformation theories (i.e., TBT, RBT, and present theories) is negligible for slender nanobeams and considerable for deep nanobeams. This is due to the fact that the EBT neglects the

Table 2 Comparison of dimensionless maximum center deflection under uniform load for simply supported nanobeams

L/h	$\mu(nm^2)$	EBT	TBT	RBT	Model 1	Model 2	Model 3	Model 4
	0	1.3021	1.4321	1.4320	1.4320	1.4317	1.4320	1.4311
	1	1.4271	1.5674	1.5673	1.5674	1.5671	1.5674	1.5665
5	2	1.5521	1.7028	1.7027	1.7028	1.7025	1.7028	1.7018
	3	1.6771	1.8381	1.8381	1.8382	1.8379	1.8382	1.8371
	4	1.8021	1.9734	1.9735	2.9736	2.9733	2.9736	2.9725
	0	1.3021	1.3346	1.3346	1.3346	1.3345	1.3346	1.3344
	1	1.4271	1.4622	1.4622	1.4622	1.4621	1.4622	1.4620
10	2	1.5521	1.5898	1.5898	1.5898	1.5897	1.5898	1.5896
	3	1.6771	1.7173	1.7174	1.7174	1.7173	1.7174	1.7171
	4	1.8021	1.8489	1.8450	1.8450	1.8449	1.8450	1.8447
	0	1.3021	1.3102	1.3102	1.3102	1.3102	1.3102	1.3102
	1	1.4271	1.4359	1.4359	1.4359	1.4359	1.4359	1.4358
20	2	1.5521	1.5615	1.5615	1.5615	1.5615	1.5615	1.5615
	3	1.6771	1.6871	1.6872	1.6872	1.6871	1.6872	1.6871
	4	1.8021	1.8128	1.8128	1.8128	1.8128	1.8128	1.8128
	0	1.3021	1.3024	1.3024	1.3024	1.3024	1.3024	1.3024
	1	1.4271	1.4274	1.4274	1.4274	1.4274	1.4274	1.4274
100	2	1.5521	1.5525	1.5525	1.5525	1.5525	1.5525	1.5525
	3	1.6771	1.6775	1.6775	1.6775	1.6775	1.6775	1.6775
	4	1.8021	1.8025	1.8025	1.8025	1.8025	1.8025	1.8025

Table 3 Comparison of dimensionless critical buckling load $\overline{N}$ for simply supported nanobeams

L/h	$\mu(nm^2)$	EBT	TBT	RBT	Model 1	Model 2	Model 3	Model 4
	0	9.8696	8.9509	8.9519	8.9519	8.9533	8.9519	8.9573
	1	8.9830	8.1468	8.1477	8.1477	8.1490	8.1477	8.1527
5	2	8.2426	7.4753	7.4761	7.4761	7.4773	7.4761	7.4807
	3	7.6149	6.9061	6.9068	6.9068	6.9080	6.9068	6.9111
	4	7.0761	6.4174	6.4181	6.4181	6.4191	6.4181	6.4220
	0	9.8696	9.6227	9.6228	9.6228	9.6231	9.6228	9.6242
	1	8.9830	8.7583	8.7583	8.7583	8.7587	8.7583	8.7597
10	2	8.2426	8.0364	8.0364	8.0364	8.0367	8.0364	8.0377
	3	7.6149	7.4244	7.4245	7.4245	7.4247	7.4245	7.4256
	4	7.0761	6.8990	6.8991	6.8991	6.8994	6.8991	6.9001
	0	9.8696	9.8067	9.8067	9.8067	9.8068	9.8067	9.8071
	1	8.9830	8.9258	8.9258	8.9258	8.9258	8.9258	8.9261
20	2	8.2426	8.1900	8.1900	8.1900	8.1901	8.1900	8.1904
	3	7.6149	7.5664	7.5664	7.5664	7.5665	7.5664	7.5667
	4	7.0761	7.0310	7.0310	7.0310	7.0310	7.0310	7.0313
	0	9.8696	9.8671	9.8671	9.8671	9.8671	9.8671	9.8671
	1	8.9830	8.9807	8.9807	8.9807	9.9807	8.9807	8.9807
100	2	8.2426	8.2405	8.2405	8.2405	8.2405	8.2405	8.2405
	3	7.6149	7.6130	7.6130	7.6130	7.6130	7.6130	7.6130
	4	7.0761	7.0743	7.0743	7.0743	7.0743	7.0743	7.0743

Table 4 Comparison of nondimensional fundamental frequency $\bar{\omega}$ of simply supported nanobeam

L/h	$\mu(nm^2)$	EBT	TBT	RBT	Model 1	Model 2	Model 3	Model 4
5	0	9.7112	9.2740	9.2745	9.2745	9.2752	9.2745	9.2781
	1	9.2647	8.8477	8.8482	8.8482	8.8488	8.8482	8.8515
	2	8.8747	8.4752	8.4757	8.4757	8.4763	8.4757	8.4789
	3	8.5301	8.1461	8.1466	8.1466	8.1472	8.1466	8.1497
	4	8.2228	7.8526	7.8530	7.8530	7.8536	7.8530	7.8561
10	0	9.8293	9.7075	9.7075	9.7075	9.7077	9.7075	9.7083
	1	9.3774	9.2612	9.2612	9.2612	9.2614	9.2612	9.2620
	2	8.9826	8.8713	8.8714	8.8714	8.8715	8.8713	8.8721
	3	8.6338	8.5269	8.5269	8.5269	8.5271	8.5269	8.5276
	4	8.3228	8.2196	8.2197	8.2197	8.2198	8.2196	8.2203
20	0	9.8595	9.8281	9.8281	9.8282	9.8283	9.8282	9.8281
	1	9.4062	9.3763	9.3763	9.3764	9.3764	9.3764	9.3763
	2	9.0102	8.9816	8.9816	8.9816	8.9817	8.9816	8.9816
	3	8.6604	8.6328	8.6328	8.6329	8.6330	8.6329	8.6328
	4	8.3483	8.3218	8.3218	8.3218	8.3220	8.3218	8.3218
100	0	9.8692	9.8679	9.8679	9.8723	9.8776	9.8722	9.8511
	1	9.4155	9.4143	9.4143	9.4184	9.4235	9.4184	9.3982
	2	9.0191	9.0180	9.0180	9.0219	9.0267	9.0219	9.0026
	3	8.6689	8.6678	8.6678	8.6716	8.6763	8.6716	8.6530
	4	8.3566	8.3555	8.3555	8.3592	8.3636	8.3592	8.3413

effects of shear deformation.

The nondimensional critical buckling loads are presented in Table 3. The results obtained using proposed theories are compared with those computed with TBT and RBT and good agreement is observed. It can be observed from Table 3 that buckling loads are reduced with increasing scale parameter $\boldsymbol{\mu}$. Critical buckling load parameters are insensitive to used theory. Similar to bending results, for lower thickness ratio ***L/h***, nonlocal effect is important and this effect is lost for higher ***L/h***.

The fundamental frequencies versus the nonlocal parameter $\boldsymbol{\mu}$ are presented in Table 4. It can be observed that the results of present theories are in excellent agreement with those predicted by TBT and RBT for all values of thickness ratio ***L/h*** and nonlocal parameter $\boldsymbol{\mu}$. According to these results, fundamental frequency is reduced with increasing nonlocal parameter $\boldsymbol{\mu}$. Furthermore, the effect of shear deformation on the fundamental frequency is significant as the value of $\boldsymbol{L/h}$ becomes low.

First three flexural nondimensional frequencies are presented in Tables 5 for different nonlocal parameter $\boldsymbol{\mu}$, thickness ratio ***L/h*** and for different theories. As the scale parameter $\boldsymbol{\mu}$ increases, the flexural nondimensional frequency obtained for the nonlocal beam theory becomes smaller than those for its local counterpart. This reduction is especially significant for higher values of the vibrational mode number ***n***, and thus the small-scale effect cannot be neglected. As is observed in the above examples, the proposed shear deformation theories are in excellent agreement with those predicted by TBT and RBT for all values of thickness ratio ***L/h***, scale parameter $\boldsymbol{\mu}$, and the vibrational mode number ***n***.

Table 5 Comparison of the first three nondimensional frequencies $\bar{\omega}$ of simply supported nanobeam (***L/h***=5)

Modes (***n***)	$\mu(nm^2)$	EBT	TBT	RBT	Model 1	Model 2	Model 3	Model 4
1	0	9.7112	9 .2740	9.2745	9.2745	9.2752	9.2746	9.2772
	1	9.2647	8.8477	8.8482	8.8482	8.8488	8.8483	8.8507
	2	8.8747	8.4752	8.4757	8.4757	8.4763	8.4758	8.4781
	3	8.5301	8.1461	8.1466	8.1466	8.1472	8.1467	8.1489
	4	8.2228	7.8526	7.8530	7.8530	7.8536	7.8531	7.8553
2	0	37.1120	32.1665	32.1847	32.1847	32.1948	32.1851	32.2180
	1	31.4239	27.2364	27.2519	27.2519	27.2604	27.2522	27.2800
	2	27.7422	24.0453	24.0589	24.0589	24.0664	24.0592	24.0837
	3	25.1104	21.7642	21.7765	21.7765	21.7833	21.7768	21.7990
	4	23.1088	20.0293	20.0407	20.0407	20.0470	20.0409	20.0614
3	0	78.0234	61.4581	61.5746	61.5746	61.6192	61.5718	61.7052
	1	56.7798	44.7247	44.8095	44.8095	44.8420	44.8075	44.9046
	2	46.8246	36.8831	36.9531	36.9531	36.9798	36.9514	37.0315
	3	40.7568	32.1036	32.1645	32.1645	32.1878	32.1631	32.2327
	4	36.5657	28.8023	28.8569	28.8569	28.8778	28.8556	28.9181

In general, it can be concluded from Tables 2-5 that the local theory underestimates the deflections and overestimates the buckling loads as well as natural frequencies of the nanobeams compared to the nonlocal one, and the difference between local and nonlocal theories is significant for high value of the scale parameter. This is due to the fact that the nonlocal theory is unable to capture the small scale effect of the nanobeams. In addition, the inclusion of the shear deformation and nonlocal effects increases the deflections and decreases the buckling loads and natural frequencies. In all cases, it can be observed that models 1 and 3 give almost identical results to those obtained by RBT.

The effect of shear deformation on the bending, buckling, and vibration responses of nanobeams is shown in Fig. 1 for a simply supported beam with μ = 1 nm^2. In this figure, the deflection, buckling load, and frequency ratios are defined as the ratios of those predicted by present theory to the correspondences predicted by EBT where the shear deformation effect is omitted. It is observed that, the inclusion of shear deformation effect makes the beam more flexible, and hence, leads to a reduction of buckling loads and natural frequencies and an increase of deflections. This indicates that the shear deformation effect results in a reduction of the beam stiffness.

The small scale effect on the bending, buckling, and vibration responses of nanobeams is clearly demonstrated in Fig. 2 for a simply supported beam with ***L/h***=10. The deflection, buckling load, and frequency ratios are defined as the ratios of those predicted by the nonlocal theory to the correspondences predicted by the local theory (i.e., μ=0). It can be observed that the deflection ratio is greater than unity, whereas the buckling load and frequency ratios are smaller than unity. It means that the local theory underestimates the deflections and overestimates the buckling loads as well as natural frequencies of the nanobeams compared to the nonlocal one. This is due to the fact that the local theory is unable to capture the small scale effect of the nanobeams.

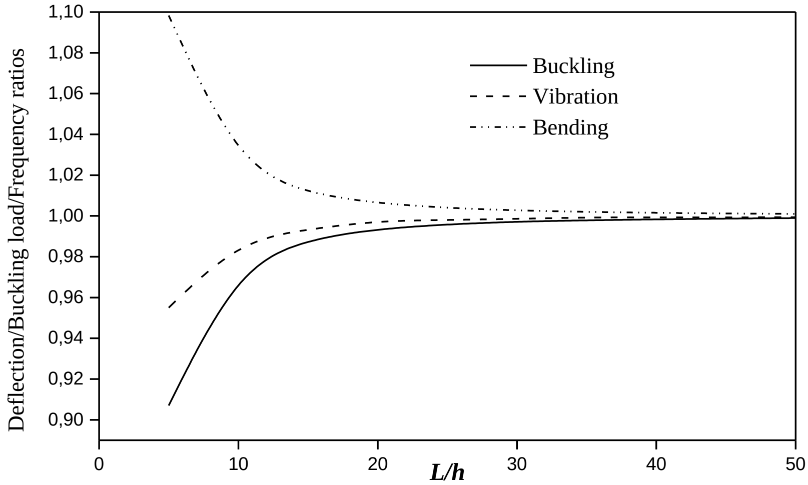

Fig. 1 Effect of transverse shear deformation on the deflection, critical buckling load, and fundamental frequency ratios for a simply supported nanobeam using model 1 with μ=1 nm^2

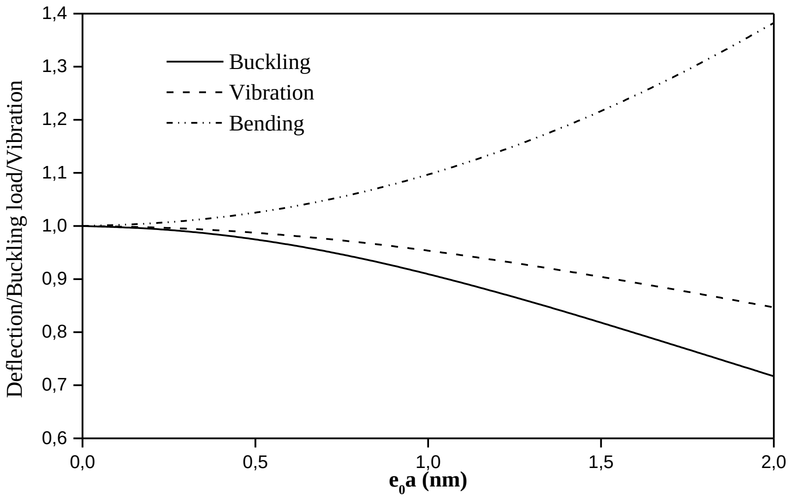

Fig. 2 Effect of small scale on the deflection, critical buckling load, and fundamental frequency ratios for a simply supported nanobeam using model 1 with L/h=10

6. Conclusions

Various nonlocal shear deformation beam theories are proposed for bending, buckling, and vibration of nanobeams. These theories are based on assumption that the in-plane and transverse displacements consist of bending and shear components, in which the bending components do not contribute toward shear forces and, likewise, the shear components do not contribute toward bending moments. In addition, the proposed models are capable of capturing both small scale and shear deformation effects of nanobeams, and doe not require shear correction factors. Nonlocal constitutive equations of Eringen are used in the formulations. The results of all proposed models are almost identical to each other, and agree well with TBT and RBT.

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