Structural Engineering and Mechanics, *Vol. 47, No. 3 (2013) 307-329*
DOI: http://dx.doi.org/10.12989/sem.2013.47.3.307

Exact stochastic solution of beams subjected to delta-correlated loads

G. Falsone* and D. Settineri

Dipartimento di Ingegneria Civile, Università di Messina, C.da Di Dio, 98166 Messina, Italy

(Received March 27, 2012, Revised July 16, 2013, Accepted July 17, 2013)

Abstract. The bending problem of Euler-Bernoulli discontinuous beams is dealt with, in which the discontinuities are due to the loads and eventually to essential constrains applied along the beam axis. In particular, the loads are modelled as random delta-correlated processes acting along the beam axis, while the ulterior eventual discontinuities are produced by the presence of external rollers applied along the beam axis. This kind of structural model can be considered in the static study of bridge beams. In the present work the exact expression of the response quantities are given in terms of means and variances, thanks to the use of the stochastic analysis rules and to the use of the generalized functions. The knowledge of the means and the variances of the internal forces implies the possibility of applying the reliability β-method for verifying the beam.

Keywords: Euler-Bernoulli beam; delta-correlated concentrated loads; generalized functions; reliability β -method

1. Introduction

The study of Euler-Bernoulli beams subjected to static concentrated forces can be of interest in many engineering applications. For example, it can be useful for defining the static behaviour of a bridge beam subjected to the vehicle actions, introducing the dynamic effects by means of meaningful coefficients. As confirmed by many codes in all the world, the static analysis is accepted because the dynamic characteristics of the vehicles and of the bridge are such that they have negligible effects on the response quantities. Neglecting the dynamic effects, the vehicle action can be represented by a point force applied to a certain abscissa of the beam axis. The solution of this simple problem can be obtained by solving the classical fourth order differential equation governing the response behaviour of the beam. This approach requires that the response quantities must be continuous. This means that, if the concentrated loads acting on the beams are n, then it is necessary dividing the beam into n+1 pieces, in each of which the response quantities are continuous. This implies the necessity of evaluating 4(n+1) integration constants by imposing the corresponding boundary essential and/or natural conditions (4 at the beam ends and n in correspondence of the point loads).

This approach can be made computationally lighter if the so-called generalized functions are

*Corresponding author, Professor, E-mail: gfalsone@unime.it

http://www.techno-press.org/?journal=sem&subpage=8 ISSN: 1225-4568 (Print), 1598-6217 (Online)

introduced for describing the concentrated loads, as made in some works (Macaulay 1919, Brungraber 1965, Falsone 2002, Biondi and Caddemi 2007, Colajanni *et al*. 2009). As shown in (Lighthill 1959) these functions can be all considered as derivatives or integrals, made in a generalized way, of the Dirac delta function (Dirac 1947). This last is a generalized function used in many field of the science for capturing the properties of some kinds of discontinuities, as those defined by the concentrated loads on a beam. As shown by Falsone (2002), the use of these functions, for the case under examination, allows to reduce the number of integration constants to 4, only those related to the end conditions.

When some essential constrains, as the rollers, act on the beam, the use of the generalized functions allows to reduce the number of the integration constants to be determined to r+4, against 4(r+n+1) necessary if the classical approach is used, r being the number of these constrains (Falsone 2002).

In the literature the generalized functions have been also used for solving the problem of discontinuous beams via the Green functions (Failla and Santini 2007, Failla 2011) or by applying the Finite Element method (Failla and Impollonia 2012).

Remaining in the field of the bridges, an accurate model of the concentrated loads simulating the presence of vehicles on the beam is that based on a distribution of random forces placed stochastically along the beam axis. In particular, a Poisson distribution along the beam axis with a given mean rate can give an accurate model of the traffic on the bridge (if the traffic level augments a bigger mean rate must be considered). In the field of random processes this kind of load is called delta-correlated process (Ross 1983, Lin and Cai 1995, Iwankiewicz and Nielsen 1999).

Aim of the present work is the application of the approach based on the use of the generalized functions for finding the exact response of beams subjected to delta-correlated processes.

2. Preliminary concepts

The differential equation governing the deflection $u(x)$ of a homogeneous elastic Euler-Bernoulli beam with constant bending stiffness subjected to a transversal continuous load $p(x)$ can be written as

$$u''''(x) = \frac{1}{EI} p(x) \tag{1}$$

where EI is the constant bending stiffness of the beam. The integrations of this equation and the consideration of the boundary conditions allow to find $u(x)$. Once that the deflection law is known, it is possible to obtain the other generalized quantities characterizing the beam from both a cinematic and a static point of view; they are the rotation, the bending moment and the shear force. In Appendix the sign convention about the cinematic and static quantities characterizing the beam is reported.

In some cases the load is not continuous, as for example when it acts only in a limited part of the beam, or, at the limit, when it is concentrated. However, as evidenced in some works (Macaulay 1919, Falsone 2002, Colajanni *et al*. 2009), even in these cases Eq. (1) can continue to be used if the load $p(x)$ is treated as a generalized function and all the integrations necessary for solving it are considered in generalized sense (Lighthill 1959).

Two of the most known generalized functions are the *Unit Step Function*, usually indicated with

$U(x-x_0)$, and the *Dirac Delta Function*, usually indicated with $\delta(x-x_0)$, having the following definitions

$$U(x-x_0)=\begin{cases}0 & \text{for } x<x_0\\ 1 & \text{for } x>x_0\end{cases}$$

$$\delta(x-x_0)=\begin{cases}0 & \text{for } x\neq x_0\\ \infty & \text{for } x=x_0\end{cases};\quad \int_{-\infty}^{+\infty}\delta(x-x_0)\mathrm{d}x=\lim_{\varepsilon\to 0}\left[\int_{x_0-\varepsilon}^{x_0+\varepsilon}\delta(x-x_0)\mathrm{d}x\right]=1 \tag{2a-c}$$

The Dirac Delta Function, that in the following will be indicated with $R_{-1}(x-x_0)$, can be considered as the generalized derivative of the Unit Step Function, that will be indicated with $R_0(x-x_0)$. Hence, the following relationships can be written

$$\delta(x-x_0)=R_{-1}(x-x_0)=R_0'(x-x_0)=U'(x-x_0)$$

$$U(x-x_0)=R_0(x-x_0)=\int_{-\infty}^{x}R_{-1}(x-x_0)\mathrm{d}x=\int_{-\infty}^{x}\delta(x-x_0)\mathrm{d}x \tag{3a, b}$$

where the notation $R_{i-1}(x-x_0)=R_i'(x-x_0)$ already introduced in (Falsone 2002) has been used. The further integrations of $R_0(x-x_0)$ bring to the following relationships

$$R_1(x-x_0)=\int_{-\infty}^{x}R_0(x-x_0)\mathrm{d}x=\begin{cases}0 & \text{for } x<x_0\\ (x-x_0) & \text{for } x>x_0\end{cases}$$

$$R_2(x-x_0)=\int_{-\infty}^{x}R_1(x-x_0)\mathrm{d}x=\begin{cases}0 & \text{for } x<x_0\\ \frac{1}{2}(x-x_0)^2 & \text{for } x>x_0\end{cases}$$

$$R_3(x-x_0)=\int_{-\infty}^{x}R_2(x-x_0)\mathrm{d}x=\begin{cases}0 & \text{for } x<x_0\\ \frac{1}{6}(x-x_0)^3 & \text{for } x>x_0\end{cases} \tag{4a-c}$$

where the generalized functions $R_i(x-x_0)$ ($i=1,2,3$) are defined as *Linear Ramp Function*, *Quadratic Ramp Function* and *Cubic Ramp Function*, respectively.

The Unit Step Function can be advantageously used in order to represent an uniformly distributed load acting between two axial abscissas x_1 and x_2 of the beam. In this case, the continuous (in a generalized sense) load $p(x)$ can be expressed as follows

$$p(x)=p\left[R_0(x-x_1)-R_0(x-x_2)\right] \tag{5}$$

p being the constant load intensity.

On the other hand, the Dirac Delta Function is used for representing a concentrated force, of intensity F and applied at the abscissa x_0, writing as follows

$$p(x)=F\delta(x-x_0) \tag{6}$$

In this work the load acting on the beam is considered to be represented by a sequence of concentrated loads having independent random intensities F_i characterized by the same probability distribution and acting at the abscissas x_i that are distributed along the beam axis following a Poisson distribution, that is

$$p(x) = \sum_{i=1}^{N(l)} F_i \delta\left(x - x_i\right) = \sum_{i=1}^{N(l)} F_i R_{-1}\left(x - x_i\right) \tag{7}$$

where $N(l)$ is a Poisson counting process with constant average rate λ and l is the beam length. The quantity here introduced is a stochastic process called *Poisson Delta-Correlated Process.* It is characterized by having the r-th correlation functions expressed as follows (Ross 1983, Lin and Cai 1995, Iwankiewicz and Nielsen 1999)

$$C_p^{(r)}\left(x_1, x_2, \ldots, x_r\right) = E\left[F_i^r\right] R_{-1}\left(x - x_1\right) R_{-1}\left(x - x_2\right) \ldots R_{-1}\left(x - x_r\right) \tag{8}$$

where $E[\square]$ indicates the mean of $(\square)$, so that $E\left[F_i^r\right]$ represents the r-th moment of the random variables F_i. For r = 1, 2, 3 these correlations coincide with some important statistical quantities of the stochastic process, that are

$$C_p^{(1)}\left(x_1\right) = E\left[p(x_1)\right]; \qquad C_p^{(2)}\left(x_1, x_2\right) = E\left[p(x_1) p(x_2)\right] - E\left[p(x_1)\right] E\left[p(x_2)\right] = \sigma_p\left(x_1, x_2\right);$$

$$C_p^{(3)}\left(x_1, x_2, x_3\right) = E\left[\left(p(x_1) - E\left[p(x_1)\right]\right)\left(p(x_2) - E\left[p(x_2)\right]\right)\left(p(x_3) - E\left[p(x_3)\right]\right)\right] \tag{9a-c}$$

These relationships show that the first correlation is the mean of the process, the second one is the covariance and the third one is the third order central moment of the process evaluated at different positions. When the abscissas are coincident, the second order correlation becomes the second cumulant that coincides with the variance of the process, while the third order correlation function degenerates to the third cumulant that coincides with the third central moment. For higher order correlation functions similar simple relationships cannot be evidenced.

In the following sections the exact probabilistic response, in terms of the first two order correlation functions, of some beams generically constrained and subjected to a delta-correlated process will be studied and obtained.

3. Beams with constrains at the ends

In this section some beams subjected to a delta-correlated load expressed as into Eq. (7) and characterized by different constrain conditions at their ends will be treated. Both the cases of statically determinate beams and redundantly constrained beams will be considered.

3.1 The hinged-hinged beam

Perhaps this is the most classical case of statistically determinate beam. The equation governing the problem is obtained by replacing Eq. (7) into Eq. (1), that is

$$u''''(x) = \frac{1}{EI} \sum_{i=1}^{N(l)} F_i R_{-1}\left(x - x_i\right) \tag{10}$$

whose solution requires the following four integrations, some of which have to be considered in generalized sense

$$u'''(x) = \frac{1}{EI}\sum_{i=1}^{N(l)} F_i R_0\left(x - x_i\right) + D_1;$$

$$u''(x) = \frac{1}{EI}\sum_{i=1}^{N(l)} F_i R_1\left(x - x_i\right) + D_1 x + D_2;$$

$$u'(x) = \frac{1}{EI}\sum_{i=1}^{N(l)} F_i R_2\left(x - x_i\right) + \frac{1}{2}D_1 x^2 + D_2 x + D_3;$$

$$u(x) = \frac{1}{EI}\sum_{i=1}^{N(l)} F_i R_3\left(x - x_i\right) + \frac{1}{6}D_1 x^3 + \frac{1}{2}D_2 x^2 + D_3 x + D_4 \tag{11a-d}$$

where D_i, with i = 1, …, 4, are the integration constants that must be evaluated by imposing the boundary conditions depending on the constrain conditions. It is not difficult to realize that they are random variables.

It is important to note that if the generalized functions are not considered for solving this problem, the use of continuous functions for solving Eq. (1) implies the necessity of dividing the beam axis into $N(l) + 1$ pieces and, as consequence, of finding $4(N(l) + 1)$ integration constants analogous to D_i, each of them being a random variable. As a consequence the application of this approach for finding the stochastic response appears to be very difficult.

In the field of random processes the summations appearing in the previous equations are stochastic quantities known as Filtered Poisson Processes (Lin and Cai 1995, Iwankiewicz and Nielsen 1999). Here they are indicated as follows

$$G_j(x) = \sum_{i=1}^{N(l)} F_i R_j\left(x - x_i\right); \quad j = 0,1,2,3 \tag{12}$$

and are characterized by the following correlation functions (Lin and Cai 1995, Iwankiewicz and Nielsen 1999)

$$C_{G_j}^{(r)}\left(x_1, x_2, \ldots, x_r\right) = \lambda E\left[F_i^r\right]\int_0^{\min(x_1, x_2, \ldots x_r)} R_j\left(x - x_1\right) R_j\left(x - x_2\right)\ldots R_j\left(x - x_r\right)\mathrm{d}x \tag{13}$$

For the constrains acting on the beam under consideration the boundary conditions impose

$$u(x)\,|_{x=0} = 0 \quad\Rightarrow\quad D_4 = 0; \qquad M(x)\,|_{x=0} = -EIu''(x)\,|_{x=0} = 0 \quad\Rightarrow\quad D_2 = 0;$$

$$u(x)\,|_{x=l} = 0 \quad\Rightarrow\quad \frac{1}{EI}G_3(l) + \frac{1}{6}D_1 l^3 + D_3 l = 0;$$

$$M(x)\,|_{x=l} = -EIu''(x)\,|_{x=l} = 0 \quad\Rightarrow\quad G_1(l) + EID_1 l = 0 \tag{14a-d}$$

where $M(x)$ indicates the bending moment. Eqs. (14a, b) imply that the constants D_2 and D_4 are deterministically zero, while the expressions of the random variables D_1 and D_3 are obtained by using Eqs. (14c, d). In particular, they are

$$D_1 = -\frac{1}{lEI} G_1(l); \quad D_3 = \frac{1}{EI}\left[\frac{l}{6} G_1(l) - \frac{1}{l} G_3(l)\right] \tag{15a, b}$$

Hence, the expression of the random processes defining the transverse deflection $u(x)$, the bending moment $M(x)$ and the shear force $T(x)$, obtained by using Eqs. (15), (12) and (11), are

$$u(x) = \frac{1}{EI}\left[G_3(x) - \frac{1}{6l} G_1(l) x^3 + \left(\frac{l}{6} G_1(l) - \frac{1}{l} G_3(l)\right) x\right];$$

$$M(x) = -G_1(x) + \frac{1}{l} G_1(l) x; \quad T(x) = -G_0(x) + \frac{1}{l} G_1(l) \tag{16a-c}$$

that are particular processes whose correlation functions can be exactly evaluated. For example, their mean values are

$$E[u(x)] = \frac{1}{EI}\left[E[G_3(x)] - \frac{1}{6l} E[G_1(l)] x^3 + \left(\frac{l}{6} E[G_1(l)] - \frac{1}{l} E[G_3(l)]\right) x\right];$$

$$E[M(x)] = -E[G_1(x)] + \frac{1}{l} E[G_1(l)] x; \quad E[T(x)] = -E[G_0(x)] + \frac{1}{l} E[G_1(l)] \tag{17a-c}$$

In order to make explicit the above reported quantities, it is necessary to evaluate the mean values of the filtered Poisson processes $G_i(x)$ (i = 0, 1, 3). This is obtained by particularizing Eq. (13) for r = 1, that are

$$E[G_0(x)] = C_{G_0}^{(1)}(x) = \lambda E[F_i]\int_0^x R_0(x-\rho)\,\mathrm{d}\rho = \lambda E[F_i]\int_0^x \mathrm{d}\rho = \lambda E[F_i] x;$$

$$E[G_1(x)] = C_{G_1}^{(1)}(x) = \lambda E[F_i]\int_0^x R_1(x-\rho)\,\mathrm{d}\rho = \lambda E[F_i]\int_0^x (x-\rho)\,\mathrm{d}\rho = \frac{1}{2}\lambda E[F_i] x^2;$$

$$E[G_3(x)] = C_{G_3}^{(1)}(x) = \lambda E[F_i]\int_0^x R_3(x-\rho)\,\mathrm{d}\rho = \frac{1}{6}\lambda E[F_i]\int_0^x (x-\rho)^3\,\mathrm{d}\rho = \frac{1}{24}\lambda E[F_i] x^4 \tag{18a-c}$$

that, replaced into Eqs. (17), give

$$E[u(x)] = \frac{\lambda E[F_i]}{12EI}\left[\frac{1}{2}x^4 - x^3 + \frac{l^3}{2}x\right]; \quad E[M(x)] = \frac{1}{2}\lambda E[F_i]\left(lx - x^2\right);$$

$$E[T(x)] = \lambda E[F_i]\left(\frac{l}{2} - x\right) \tag{19a-c}$$

By setting $x = 0$ and $x = l$ into Eq. (19c) it is possible to obtain the mean values of the two hinge reactions at the beam extremes as follows

$$E[V_A] = -E[T(0)] = -\frac{l}{2}\lambda E[F_i]; \quad E[V_B] = E[T(l)] = -\frac{l}{2}\lambda E[F_i] \tag{20a, b}$$

It is important to note that the laws of the displacement mean, the bending moment mean and the shear mean given into Eq. (19) coincide with the laws of the same corresponding deterministic quantities when the beam is loaded by a deterministic uniformly distributed force whose intensity is equal to the mean value of the delta correlated input, that is $p = \lambda E[F]$. The same consideration can be made for the restrain reaction mean values given into Eq. (20). These results are not surprising if the following consideration is made: by taking into account Eq. (11), it must be noted that their mean values, obtained by using Eq. (18), too, are coincident with the deterministic equations governing the derivatives up to the fourth order of the deflection of a beam subjected to a deterministic uniformly distributed load with intensity $p = \lambda E[F]$. As this consideration is not influenced by the boundary conditions, it can be considered true for any beam constrain condition.

The expressions of the second order correlation functions of the deflection and of the two internal forces $M(x)$ and $T(x)$ can be obtained by applying the relationships given into Eq. (16) and into Eq. (9b), that, after some algebra give

$$C_u^{(2)}(x_1,x_2)=\frac{1}{(EI)^2}\left[C_{G_3}^{(2)}(x_1,x_2)+\frac{1}{6}C_{G_3G_1}^{(2)}(x_1,l)x_2\left(l-\frac{1}{l}x_2^2\right)+\frac{1}{6}C_{G_3G_1}^{(2)}(x_2,l)x_1\left(l-\frac{1}{l}x_1^2\right)\right]$$
$$+\frac{1}{(EI)^2}\left[-\frac{1}{l}\left(C_{G_3}^{(2)}(x_1,l)x_2+C_{G_3}^{(2)}(x_2,l)x_1\right)+\frac{1}{36}C_{G_1}^{(2)}(l,l)x_1x_2\left(\frac{1}{l^2}x_1^2x_2^2-x_1^2-x_2^2+l^2\right)\right]$$
$$+\frac{1}{(EI)^2}\left[\frac{1}{3}C_{G_1G_3}^{(2)}(l,l)x_1x_2\left(\frac{1}{2l^2}x_1^2+\frac{1}{2l^2}x_2^2-1\right)+\frac{1}{l^2}C_{G_3}^{(2)}(l,l)x_1x_2\right];$$
$$C_M^{(2)}(x_1,x_2)=C_{G_1}^{(2)}(x_1,x_2)-\frac{1}{l}\left(C_{G_1}^{(2)}(x_1,l)x_2+C_{G_1}^{(2)}(x_2,l)x_1\right)+\frac{1}{l^2}C_{G_1}^{(2)}(l,l)x_1x_2;$$
$$C_T^{(2)}(x_1,x_2)=C_{G_0}^{(2)}(x_1,x_2)-\frac{1}{l}\left(C_{G_0G_1}^{(2)}(x_1,l)+C_{G_0G_1}^{(2)}(x_1,l)\right)+\frac{1}{l^2}C_{G_1}^{(2)}(l,l) \qquad \text{(21a-c)}$$

By setting $x_1 = x_2 = x$ and after some algebra, the previous equations give the corresponding values of the variances in the form

$$\sigma_u^2(x)=\frac{1}{(EI)^2}\left[\sigma_{G_3}^2(x)+\frac{1}{3}C_{G_3G_1}^{(2)}(x,l)x\left(l-\frac{1}{l}x^2\right)-\frac{2}{l}C_{G_3}^{(2)}(x,l)x+\frac{1}{36}\sigma_{G_1}^2(l)x^2\left(\frac{1}{l^2}x^4-2x^2+l^2\right)\right]$$
$$+\frac{1}{(EI)^2}\left[\frac{1}{3}\sigma_{G_1G_3}(l)x^2\left(\frac{x^2}{l^2}-1\right)+\frac{1}{l^2}\sigma_{G_3}^2(l)x^2\right];$$
$$\sigma_M^2(x)=\sigma_{G_1}^2(x)-\frac{2}{l}C_{G_1}^{(2)}(x,l)x+\sigma_{G_1}^2(l);\qquad \sigma_T^2(x)=\sigma_{G_0}^2(x)-\frac{2}{l}C_{G_0G_1}^{(2)}(x,l)+\frac{1}{l^2}\sigma_{G_1}^2(l) \qquad \text{(22a-c)}$$

The explicit values of the above response variances can be obtained once that the explicit values of the variances, covariances and cross-correlations appearing into the second members of the above equations are obtained by particularizing and/or extending Eq. (13), that are

$$C_{G_3}^{(2)}(x,l)=\lambda E\left[F^2\right]\int_0^x R_3(x-\rho)R_3(l-\rho)\mathrm{d}\rho=\frac{\lambda E\left[F^2\right]x^4}{144}\left(l^3-\frac{3}{5}xl^2+\frac{1}{5}x^2l-\frac{1}{35}x^3\right);$$

$$\sigma_{G_3}^2(x)=\frac{\lambda E\left[F^2\right]x^7}{252};$$

$$C_{G_1}^{(2)}(x,l)=\lambda E\left[F^2\right]\int_0^x R_1(x-\rho)R_1(l-\rho)\mathrm{d}\rho=\frac{\lambda E\left[F^2\right]x^2}{2}\left(l-\frac{1}{3}x\right);$$

$$\sigma_{G_1}^2(x)=\frac{\lambda E\left[F^2\right]x^3}{3};$$

$$C_{G_0}^{(2)}(x,l)=\lambda E\left[F^2\right]\int_0^x R_0(x-\rho)R_0(l-\rho)\mathrm{d}\rho=\lambda E\left[F^2\right]x=\sigma_{G_0}^2(x);$$

$$C_{G_3G_1}^{(2)}(x,l)=\lambda E\left[F^2\right]\int_0^x R_3(x-\rho)R_1(l-\rho)\mathrm{d}\rho=\frac{\lambda E\left[F^2\right]x^4}{24}\left(l-\frac{1}{5}x\right);$$

$$\sigma_{G_1G_3}(x)=\sigma_{G_3G_1}(x)=\frac{\lambda E\left[F^2\right]x^5}{30};$$

$$C_{G_0G_1}^{(2)}(x,l)=\lambda E\left[F^2\right]\int_0^x R_0(x-\rho)R_1(l-\rho)\mathrm{d}\rho=\lambda E\left[F^2\right]x\left(l-\frac{1}{2}x\right) \tag{23a-h}$$

At last, by replacing Eq. (23) into Eq. (22), the following results are obtained

$$\sigma_u^2(x)=\frac{\lambda E\left[F^2\right]}{45(EI)^2}\left(\frac{1}{7l}x^8-\frac{4}{7}x^7+\frac{2l}{3}x^6-\frac{l^3}{3}x^4+\frac{2l}{21}x^3\right);$$

$$\sigma_M^2(x)=\frac{\lambda E\left[F^2\right]}{3}x^2\left(\frac{1}{l}x^2-2x+l\right);\quad \sigma_T^2(x)=\lambda E\left[F^2\right]\left(\frac{1}{l}x^2-x+\frac{l}{3}\right) \tag{24a-c}$$

It is not difficult to verify that at the beam ends the variances of both the displacements and the bending moments are zero, as must be because these quantities are deterministically zero in correspondence of the constrains. On the contrary, the variances of the shear forces and, hence, of the constrain reactions are given by

$$\sigma_{V_A}^2=\sigma_T^2(0)=\frac{\lambda E\left[F^2\right]l}{3};\quad \sigma_{V_B}^2=\sigma_T^2(l)=\frac{\lambda E\left[F^2\right]l}{3} \tag{25a, b}$$

In Figs. 1(a)-(f) the diagrams related to the response mean values and variances are reported in a non-dimensional form. From the analysis of the results related to the bending moment and to the shear force, the critical section for the moment seems to be the middle one in which the moment mean value and the corresponding variance are maxima. In terms of shear force, the most critical sections seem to be the extreme ones with maximum mean value and variance. Hence, verifying the beam in these sections seems to be necessary. In the next sections a procedure to implement this kind of operations will be introduced.

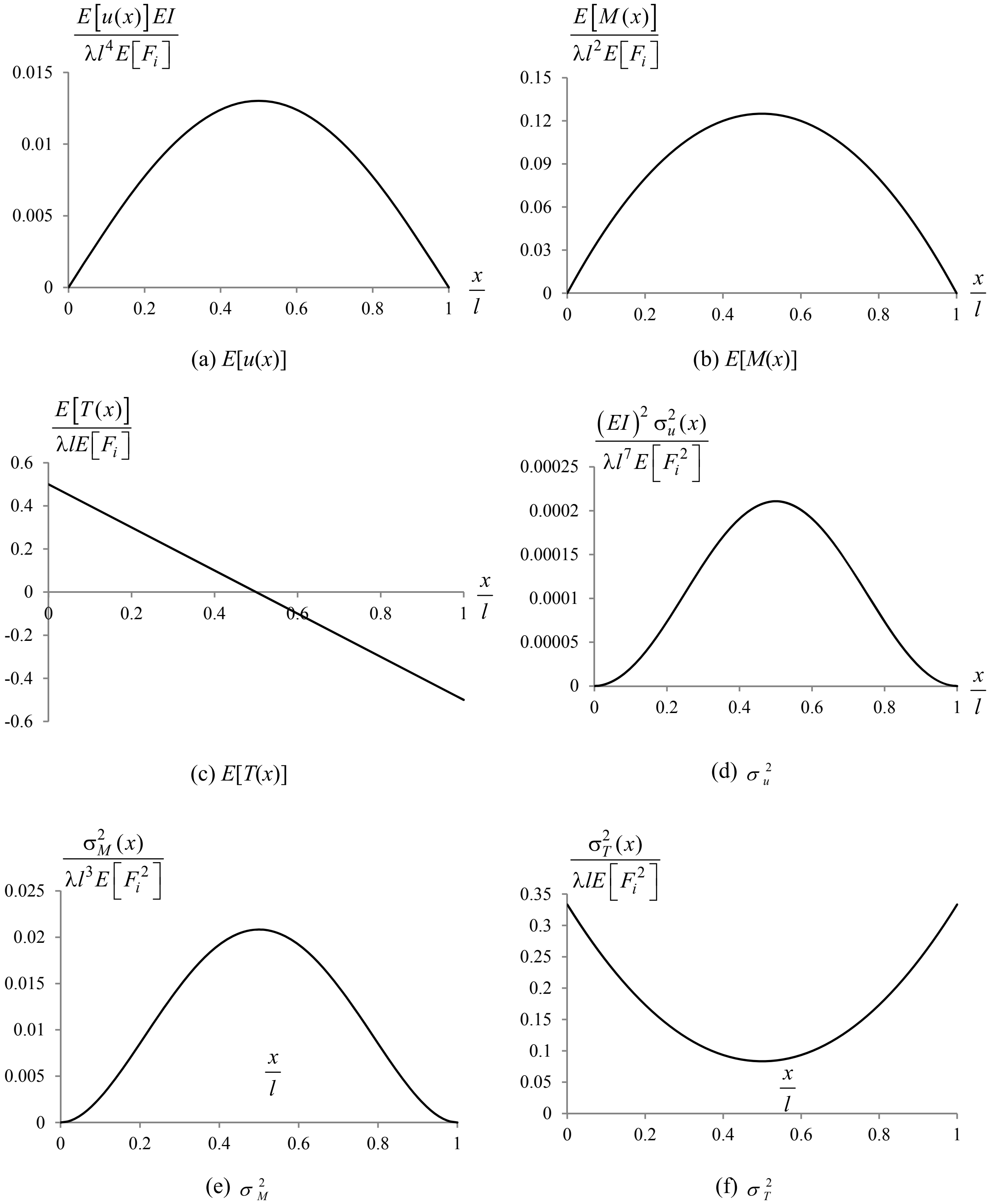


Fig. 1 Dimensionless means and variances for the response quantities of the hinged-hinged beam

3.2 The clamped-clamped beam

In this section the case of the clamped-clamped beam subjected to the same load defined into Eq. (7) is analyzed. Respect to the previous restrain conditions, the differences arise starting from Eq. (14) where the boundary conditions are imposed. For the case now under consideration, these conditions are

$$u(x)|_{x=0}=0 \quad \Rightarrow \quad D_4=0; \qquad \varphi(x)|_{x=0}=-u'(x)|_{x=0}=0 \quad \Rightarrow \quad D_3=0;$$
$$u(x)|_{x=l}=0 \quad \Rightarrow \quad \frac{1}{EI}G_3(l)+\frac{1}{6}D_1 l^3+\frac{1}{2}D_2 l^2=0;$$
$$\varphi(x)|_{x=l}=-u'(x)|_{x=l}=0 \quad \Rightarrow \quad \frac{1}{EI}G_2(l)+\frac{1}{2}D_1 l^2+D_2 l=0 \qquad \text{(26a-d)}$$

Eqs. (26a, b) evidence that the constants D_3 and D_4 are deterministically zeros, while Eqs. (26c,d) allows to find the following expressions for the other two constants

$$D_1=\frac{12}{l^3 EI}\left[G_3(l)-\frac{l}{2}G_2(l)\right]; \quad D_2=\frac{6}{l^2 EI}\left[-G_3(l)+\frac{l}{3}G_2(l)\right] \qquad \text{(27a, b)}$$

Hence, the expressions of the random processes defining the transverse deflection $u(x)$, the bending moment $M(x)$ and the shear force $T(x)$ for the case of the clamped-clamped beam are

$$u(x)=\frac{1}{EI}\left[G_3(x)+\left(\frac{2}{l^3}G_3(l)-\frac{1}{l^2}G_2(l)\right)x^3-\left(\frac{3}{l^2}G_3(l)-\frac{1}{l}G_2(l)\right)x^2\right];$$
$$M(x)=-G_1(x)-\frac{6}{l^2}\left(\frac{2}{l}G_3(l)-G_2(l)\right)x+\frac{2}{l}\left(\frac{3}{l}G_3(l)-G_2(l)\right);$$
$$T(x)=-G_0(x)-\frac{12}{l^3}\left(G_3(l)-\frac{l}{2}G_2(l)\right) \qquad \text{(28a-c)}$$

The mean values of these processes are obtained by simply applying the mean operator to each member of the previous equations

$$E\left[u(x)\right]=\frac{1}{EI}\left[E\left[G_3(x)\right]+\left(\frac{2}{l^3}E\left[G_3(l)\right]-\frac{1}{l^2}E\left[G_2(l)\right]\right)x^3-\left(\frac{3}{l^2}E\left[G_3(l)\right]-\frac{1}{l}E\left[G_2(l)\right]\right)x^2\right];$$
$$E\left[M(x)\right]=-E\left[G_1(x)\right]-\frac{6}{l^2}\left(\frac{2}{l}E\left[G_3(l)\right]-E\left[G_2(l)\right]\right)x+\frac{2}{l}\left(\frac{3}{l}E\left[G_3(l)\right]-E\left[G_2(l)\right]\right);$$
$$E\left[T(x)\right]=-E\left[G_0(x)\right]-\frac{12}{l^3}\left(E\left[G_3(l)\right]-\frac{l}{2}E\left[G_2(l)\right]\right) \qquad \text{(29a-c)}$$

that, besides of the mean value expressions already reported into Eq. (18), require the knowledge of the expression of $E[G_2(x)]$, that is

$$E\left[G_2(x)\right]=C_{G_2}^{(1)}(x)=\lambda E\left[F\right]\int_0^x R_2\left(x-\rho\right)\mathrm{d}\rho=\frac{1}{2}\lambda E\left[F\right]\int_0^x\left(x-\rho\right)^2\mathrm{d}\rho=\frac{1}{6}\lambda E\left[F\right]x^3 \qquad (30)$$

Taking into account Eqs. (18) and (30), the mean values given into Eqs. (29), after some algebra, can be rewritten in the following explicit form

$$E\left[u(x)\right]=\frac{\lambda E\left[F\right]}{24EI}\left(x-l\right)^{2}x^{2};\quad E\left[M(x)\right]=-\frac{1}{2}\lambda E\left[F\right]\left(x^{2}-lx+\frac{l^{2}}{6}\right);$$

$$E\left[T(x)\right]=-\lambda E\left[F\right]\left(x-\frac{l}{2}\right)\tag{31a-c}$$

By setting $x=0$ and $x=l$ into Eqs. (31b, c) it is possible to obtain the mean values of the reactions at the beam ends as follows

$$E\left[V_{A}\right]=-E\left[T(0)\right]=-\frac{l}{2}\lambda E\left[F\right];\quad E\left[V_{B}\right]=E\left[T(l)\right]=-\frac{l}{2}\lambda E\left[F\right];$$

$$E\left[M_{A}\right]=-E\left[M(0)\right]=\frac{1}{12}\lambda E\left[F\right]l^{2};\quad E\left[M_{B}\right]=E\left[M(l)\right]=-\frac{1}{12}\lambda E\left[F\right]l^{2}\tag{32a, d}$$

It is important to note that what before said about the fact that the laws of all the response quantity means coincide with the laws of the same corresponding deterministic quantities when the beam is loaded by a deterministic uniformly distributed load whose intensity is equal to the mean value of the delta correlated input, is confirmed for the present case of clamped-clamped beam, too.

The expressions of the second order correlation functions of the deflection and of the two internal forces $M(x)$ and $T(x)$ can be obtained by applying the relationships given into Eqs. (28) and (9b), that, after some algebra, give

$$\begin{aligned}
C_{M}^{(2)}\left(x_{1},x_{2}\right)=&\,C_{G_{1}}^{(2)}\left(x_{1},x_{2}\right)+\frac{4}{l^{4}}\sigma_{G_{2}}^{2}\left(l\right)\left(l-3x_{1}\right)\left(l-3x_{2}\right)+\frac{36}{l^{6}}\sigma_{G_{3}}^{2}\left(l\right)\left(2x_{1}-l\right)\left(2x_{2}-l\right)\\
&+\frac{2}{l^{2}}C_{G_{1}G_{2}}^{(2)}\left(x_{1},l\right)\left(l-3x_{2}\right)+\frac{2}{l^{2}}C_{G_{1}G_{2}}^{(2)}\left(x_{2},l\right)\left(l-3x_{1}\right)+\frac{6}{l^{3}}C_{G_{1}G_{3}}^{(2)}\left(x_{1},l\right)\left(2x_{2}-l\right)\\
&+\frac{6}{l^{3}}C_{G_{1}G_{3}}^{(2)}\left(x_{2},l\right)\left(2x_{1}-l\right)+\frac{12}{l^{5}}\sigma_{G_{3}G_{2}}\left(l\right)\left[\left(2x_{1}-l\right)\left(l-3x_{2}\right)+\left(2x_{2}-l\right)\left(l-3x_{1}\right)\right];
\end{aligned}$$

$$\begin{aligned}
C_{u}^{(2)}\left(x_{1},x_{2}\right)=&\,\frac{1}{\left(EI\right)^{2}}\left[C_{G_{3}}^{(2)}\left(x_{1},x_{2}\right)+\frac{1}{l^{6}}\sigma_{G_{3}}^{2}\left(l\right)x_{1}^{2}x_{2}^{2}\left(2x_{1}-3l\right)\left(2x_{2}-3l\right)\right]\\
&+\frac{1}{\left(EI\right)^{2}}\left[\frac{1}{l^{4}}\sigma_{G_{2}}^{2}\left(l\right)x_{1}^{2}x_{2}^{2}\left(x_{1}-l\right)\left(x_{2}-l\right)+\frac{1}{l^{3}}C_{G_{3}}^{(2)}\left(x_{1},l\right)x_{2}^{2}\left(2x_{2}-3l\right)\right]\\
&+\frac{1}{\left(EI\right)^{2}}\left[\frac{1}{l^{3}}C_{G_{3}}^{(2)}\left(x_{2},l\right)x_{1}^{2}\left(2x_{1}-3l\right)-\frac{1}{l^{2}}C_{G_{3}G_{2}}^{(2)}\left(x_{1},l\right)x_{2}^{2}\left(x_{2}-l\right)\right]\\
&-\frac{1}{\left(EI\right)^{2}}\left[\frac{1}{l^{2}}C_{G_{3}G_{2}}^{(2)}\left(x_{2},l\right)x_{1}^{2}\left(x_{1}-l\right)+\frac{1}{l^{5}}\sigma_{G_{3}G_{2}}\left(l\right)x_{1}^{2}x_{2}^{2}\left(2x_{1}-3l\right)\left(x_{2}-l\right)\right]\\
&-\frac{1}{\left(EI\right)^{2}}\left[\frac{1}{l^{5}}\sigma_{G_{3}G_{2}}\left(l\right)x_{1}^{2}x_{2}^{2}\left(2x_{2}-3l\right)\left(x_{1}-l\right)\right];
\end{aligned}$$

$$C_T^{(2)}(x_1,x_2)=C_{G_0}^{(2)}(x_1,x_2)+\frac{144}{l^6}\sigma_{G_3}^2(l)+\frac{36}{l^4}\sigma_{G_2}^2(l)+\frac{12}{l^3}C_{G_0G_3}^{(2)}(x_1,l)+\frac{12}{l^3}C_{G_0G_3}^{(2)}(x_2,l)$$
$$-\frac{6}{l^2}C_{G_0G_2}^{(2)}(x_1,l)-\frac{6}{l^2}C_{G_0G_2}^{(2)}(x_2,l)-\frac{144}{l^5}\sigma_{G_3G_2}(l) \quad \text{(33a-c)}$$

By setting $x_1 = x_2 = x$, the variances of the response quantities are obtained as

$$\sigma_u^2(x)=\frac{1}{(EI)^2}\left[\sigma_{G_3}^2(x)-\frac{2}{l^2}C_{G_3G_2}^{(2)}(x,l)x^2(x-l)+\frac{2}{l^3}C_{G_3}^{(2)}(x,l)x^2(2x-3l)+\frac{1}{l^4}\sigma_{G_2}^2(l)x^4(x-l)^2\right]$$
$$+\frac{1}{(EI)^2}\left[-\frac{2}{l^5}\sigma_{G_3G_2}(l)x^4(2x-3l)(x-l)+\frac{1}{l^6}\sigma_{G_3}^2(l)x^4(2x-3l)^2\right];$$
$$\sigma_M^2(x)=\sigma_{G_1}^2(x)+\frac{4}{l^4}\sigma_{G_2}^2(l)(l-3x)^2+\frac{36}{l^6}\sigma_{G_3}^2(l)(2x-l)^2+\frac{4}{l^2}C_{G_1G_2}^{(2)}(x,l)(l-3x)$$
$$+\frac{12}{l^3}C_{G_1G_3}^{(2)}(x,l)(2x-l)+\frac{24}{l^5}\sigma_{G_2G_3}(l)(2x-l)(l-3x);$$
$$\sigma_T^2(x)=\sigma_{G_0}^2(x)+\frac{144}{l^6}\sigma_{G_3}^2(l)+\frac{36}{l^4}\sigma_{G_2}^2(l)+\frac{24}{l^3}C_{G_0G_3}^{(2)}(x,l)-\frac{12}{l^2}C_{G_0G_2}^{(2)}(x,l)-\frac{144}{l^5}\sigma_{G_3G_2}(l) \quad \text{(34a-c)}$$

Besides of the quantities reported into Eq. (23), the explicit expressions of these variances require the evaluation of

$$\sigma_{G_2}^2(x)=\lambda E\left[F^2\right]\int_0^x R_2(x-\rho)R_2(l-\rho)\mathrm{d}\rho=\frac{\lambda E\left[F^2\right]x^5}{20};$$
$$C_{G_3G_2}^{(2)}(x,l)=\lambda E\left[F^2\right]\int_0^x R_3(x-\rho)R_2(l-\rho)\mathrm{d}\rho=\frac{\lambda E\left[F^2\right]x^4}{24}\left(\frac{1}{30}x^2-\frac{l}{5}x+\frac{l^2}{2}\right);$$
$$\sigma_{G_3G_2}(x)=\frac{\lambda E\left[F^2\right]x^6}{72}=\sigma_{G_2G_3}(x);$$
$$C_{G_1G_3}^{(2)}(x,l)=\lambda E\left[F^2\right]\int_0^x R_1(x-\rho)R_3(l-\rho)\mathrm{d}\rho=\frac{\lambda E\left[F^2\right]x^2}{12}\left(l^3-l^2x+\frac{l}{2}x^2-\frac{1}{10}x^3\right);$$
$$\sigma_{G_1G_3}(x)=\frac{\lambda E\left[F^2\right]x^5}{30}=\sigma_{G_3G_1}(x);$$
$$C_{G_1G_2}^{(2)}(x,l)=\lambda E\left[F^2\right]\int_0^x R_1(x-\rho)R_2(l-\rho)\mathrm{d}\rho=\frac{\lambda E\left[F^2\right]x^2}{2}\left(\frac{l^2}{2}-\frac{l}{3}x+\frac{1}{12}x^2\right);$$
$$\sigma_{G_1G_2}(x)=\frac{\lambda E\left[F^2\right]x^4}{8}=\sigma_{G_2G_1}(x);$$
$$C_{G_0G_2}^{(2)}(x,l)=\lambda E\left[F^2\right]\int_0^x R_0(x-\rho)R_2(l-\rho)\mathrm{d}\rho=\frac{\lambda E\left[F^2\right]x}{2}\left(l^2-lx+\frac{1}{3}x^2\right);$$

$$\sigma_{G_0G_2}(x)=\frac{\lambda E\left[F^2\right]x^3}{6}=\sigma_{G_2G_0}(x);$$

$$C^{(2)}_{G_0G_3}(x,l)=\lambda E\left[F^2\right]\int_0^x R_0(x-\rho)R_3(l-\rho)\mathrm{d}\rho=\frac{\lambda E\left[F^2\right]x}{6}\left(l^3-\frac{3}{2}xl^2+x^2l-\frac{l}{4}x^3\right);$$

$$\sigma_{G_0G_3}(x)=\frac{\lambda E\left[F^2\right]x^4}{24}=\sigma_{G_3G_0}(x) \tag{35a-g}$$

At last, the use of Eqs. (23) and (35), after some algebra, gives

$$\sigma_u^2(x)=\frac{\lambda E[F^2]}{(EI)^2}\frac{x^4}{1260}\left(-\frac{x^6}{l^3}+5\frac{x^5}{l^2}-7\frac{x^4}{l}-2x^3+13x^2l-11xl^2+3l^3\right);$$

$$\sigma_M^2(x)=\lambda E[F^2]\left(-\frac{1}{5}\frac{x^6}{l^3}+\frac{3}{5}\frac{x^5}{l^2}-\frac{1}{3}\frac{x^4}{l}-\frac{1}{3}x^3+\frac{13}{35}x^2l-\frac{11}{105}xl^2+\frac{1}{105}l^3\right);$$

$$\sigma_T^2(x)=\lambda E[F^2]\left(-\frac{x^4}{l^3}+2\frac{x^3}{l^2}-x+\frac{13}{35}l\right) \tag{36a-c}$$

In Fig. 2 the graphic representations of these laws are reported in a dimensionless form.

4. Continuous beams

In this section the case of the generic continuous beam of the type represented in Fig. 3 is taken into account.

The fourth order differential equation governing the behavior of the transverse deflection can be written as

$$u''''(x)=\frac{1}{EI}\sum_{i=1}^{N(l)}F_iR_{-1}(x-x_i)+\frac{1}{EI}\sum_{j=1}^{n-1}V_jR_{-1}(x-l_j) \tag{37}$$

where V_j is the reaction of the j-th intermediate constrain and $l_j=\sum_{k=1}^{j}\overline{l}_k$, $\overline{l}_k$ being the length of the k-th beam piece among a constrain and the successive one. The solution of Eq. (37) requires the following four integrations, some of which have to be considered in generalized sense

$$u'''(x)=\frac{1}{EI}G_0(x)+\frac{1}{EI}\sum_{j=1}^{n-1}V_jR_0(x-l_j)+D_1;$$

$$u''(x)=\frac{1}{EI}G_1(x)+\frac{1}{EI}\sum_{j=1}^{n-1}V_jR_1(x-l_j)+D_1x+D_2;$$

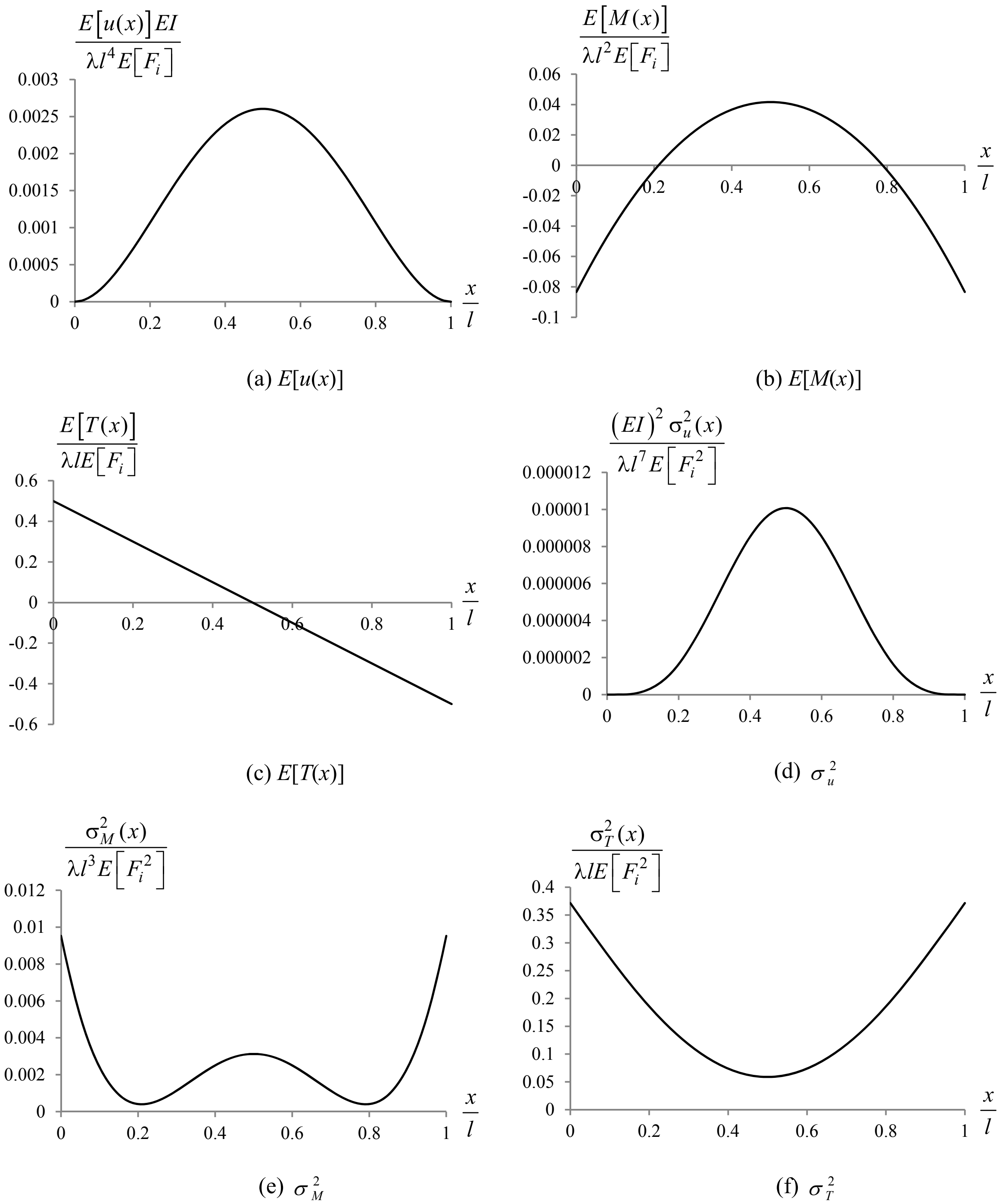


Fig. 2 Dimensionless means and variances for the response quantities of the clamped-clamped beam

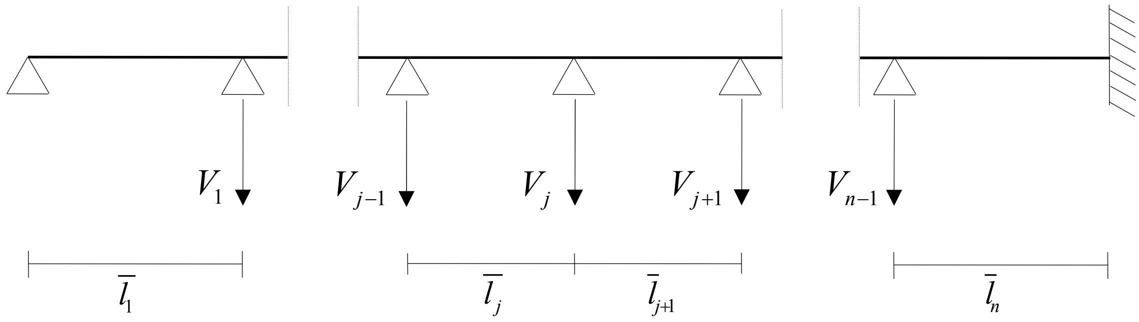


Fig. 3 Continuous multi-span beam.

$$u'(x) = \frac{1}{EI} G_2(x) + \frac{1}{EI} \sum_{j=1}^{n-1} V_j R_2\left(x - l_j\right) + \frac{1}{2} D_1 x^2 + D_2 x + D_3;$$

$$u(x) = \frac{1}{EI} G_3(x) + \frac{1}{EI} \sum_{j=1}^{n-1} V_j R_3\left(x - l_j\right) + \frac{1}{6} D_1 x^3 + \frac{1}{2} D_2 x^2 + D_3 x + D_4 \qquad \text{(38a-d)}$$

In these expressions ($n + 3$) random unknowns appear. They are the four integration constants D_i and the ($n - 3$) reactions V_j. In order to find their expressions, it is necessary to impose the four conditions at the ends of the beam and the ($n - 1$) conditions at the intermediate constrains, that are

$$u(x)|_{x=0} = 0 \quad \Rightarrow \quad D_4 = 0; \qquad M(x)|_{x=0} = 0 \quad \Rightarrow \quad u''(x)|_{x=0} = 0 \quad \Rightarrow \quad D_2 = 0;$$

$$u(x)|_{x=l_n} = 0 \quad \Rightarrow \quad \frac{1}{EI} G_3(l_n) + \frac{1}{EI} \sum_{j=1}^{n-1} V_j R_3\left(l_n - l_j\right) + \frac{1}{6} D_1 l_n^3 + D_3 l_n = 0;$$

$$M(x)|_{x=l_n} = 0 \quad \Rightarrow \quad = u''(x)|_{x=l} = 0 \quad \Rightarrow \quad \frac{1}{EI} G_1(l_n) + \frac{1}{EI} \sum_{j=1}^{n-1} V_j R_1\left(l_n - l_j\right) + D_1 l_n = 0;$$

$$u(x)|_{x=l_k} = 0 \quad \Rightarrow \quad \frac{1}{EI} G_3(l_k) + \frac{1}{EI} \sum_{j=1}^{n-1} V_j R_3\left(l_k - l_j\right) + \frac{1}{6} D_1 l_k^3 + D_3 l_k = 0; \quad k = 1, 2, \cdots, (n-1) \qquad \text{(39a-e)}$$

Eqs. (39a, b) evidence that D_2 and D_4 are deterministically zero, while the expressions of D_1, D_3 and of the reactions V_j must be obtained by solving Eqs. (39c-e) that are conveniently rewritten in matrix form as follows

$$\frac{1}{EI} \mathbf{r}_3^T \mathbf{v} + \mathbf{h}_3^T \mathbf{d} = -\frac{1}{EI} G_3\left(l_n\right); \qquad \frac{1}{EI} \mathbf{r}_1^T \mathbf{v} + \mathbf{h}_1^T \mathbf{d} = -\frac{1}{EI} G_1\left(l_n\right); \qquad \frac{1}{EI} \mathbf{R}_3 \mathbf{v} + \mathbf{H}_3 \mathbf{d} = -\frac{1}{EI} \mathbf{g}_3 \qquad \text{(40a-c)}$$

where

$$\mathbf{r}_i^T = \left(R_i\left(l_n - l_1\right) \quad R_i\left(l_n - l_2\right) \quad \cdots \quad R_i\left(l_n - l_{n-1}\right)\right); \qquad \mathbf{h}_1^T = \left(l_n \quad 0\right); \qquad \mathbf{h}_3^T = \begin{pmatrix} \frac{l_n^3}{6} & l_n \end{pmatrix};$$

$$\mathbf{d}^T = \left(D_1 \quad D_2\right); \qquad \mathbf{g}_3^T = \left(G_3\left(l_1\right) \quad G_3\left(l_2\right) \quad \cdots \quad G_3\left(l_{n-1}\right)\right);$$

$$\mathbf{R}_3 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ R_3(l_2 - l_1) & 0 & \cdots & 0 \\ R_3(l_3 - l_1) & R_3(l_3 - l_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ R_3(l_{n-1} - l_1) & R_3(l_{n-1} - l_2) & \cdots & 0 \end{pmatrix}; \quad \mathbf{H}_3 = \begin{pmatrix} \frac{l_1^3}{6} & l_1 \\ \frac{l_2^3}{6} & l_2 \\ \vdots & \vdots \\ \frac{l_{n-1}^3}{6} & l_{n-1} \end{pmatrix} \tag{41a-g}$$

Eq. (40) can be again compacted in the following form

$$\begin{pmatrix} \frac{1}{EI}\mathbf{R}_3 & \mathbf{H}_3 \\ \frac{1}{EI}\mathbf{R} & \mathbf{H} \end{pmatrix} \begin{pmatrix} \mathbf{v} \\ \mathbf{d} \end{pmatrix} = -\frac{1}{EI}\begin{pmatrix} \mathbf{g}_3 \\ \mathbf{g} \end{pmatrix}; \quad \mathbf{R} = \begin{pmatrix} \mathbf{r}_3^T \\ \mathbf{r}_1^T \end{pmatrix}; \quad \mathbf{H} = \begin{pmatrix} \mathbf{h}_3^T \\ \mathbf{h}_1^T \end{pmatrix}; \quad \mathbf{g} = \begin{pmatrix} G_3(l_n) \\ G_1(l_n) \end{pmatrix} \tag{42a-d}$$

that is able to give the expressions of the random variables included into $\mathbf{v}$ and $\mathbf{d}$ as follows

$$\mathbf{v} = \left(\mathbf{R}_3 - \mathbf{H}_3\mathbf{H}^{-1}\mathbf{R}\right)^{-1}\left(\mathbf{H}_3\mathbf{H}^{-1}\mathbf{g} - \mathbf{g}_3\right); \quad \mathbf{d} = -\frac{1}{EI}\mathbf{H}^{-1}\left(\mathbf{g} + \mathbf{R}\mathbf{v}\right) \tag{43a,b}$$

In order to find the response statistics of the continuous beam, it is convenient to rewrite the laws of the deflection, of the bending moment and of the shear force as follows

$$u(x) = \frac{1}{EI}G_3(x) + \frac{1}{EI}\mathbf{r}_3^T(x)\mathbf{v} + \mathbf{h}_3^T(x)\mathbf{d}; \quad M(x) = -G_1(x) - \mathbf{r}_1^T(x)\mathbf{v} - EID_1x;$$

$$T(x) = -G_0(x) - \mathbf{r}_0^T(x)\mathbf{v} - EID_1 \tag{44a-c}$$

where

$$\mathbf{r}_i^T(x) = \begin{pmatrix} R_i(x - l_1) & R_i(x - l_2) & \cdots & R_i(x - l_{n-1}) \end{pmatrix}; \quad i = 0,1,3; \quad \mathbf{h}_3^T(x) = \begin{pmatrix} \frac{x^3}{6} & x \end{pmatrix} \tag{45a,b}$$

Applying the mean operator to both the members of Eq. (44) the expressions of the beam mean responses are obtained as follows

$$E[u(x)] = \frac{1}{EI}E[G_3(x)] + \frac{1}{EI}\mathbf{r}_3^T(x)E[\mathbf{v}] + \mathbf{h}_3^T(x)E[\mathbf{d}];$$

$$E[M(x)] = -E[G_1(x)] - \mathbf{r}_1^T(x)E[\mathbf{v}] - EIxE[D_1];$$

$$E[T(x)] = -E[G_0(x)] - \mathbf{r}_0^T(x)E[\mathbf{v}] - EIE[D_1] \tag{46a-c}$$

in which the expressions of the mean values $E[G_i(x)]$ have been already given in the previous section, while the other mean values appearing before are obtained by applying the mean operator to both the members of Eq. (43), that are

$$E[\mathbf{v}] = \left(\mathbf{R}_3 - \mathbf{H}_3\mathbf{H}^{-1}\mathbf{R}\right)^{-1}\left(\mathbf{H}_3\mathbf{H}^{-1}E[\mathbf{g}] - E[\mathbf{g}_3]\right); \quad E[\mathbf{d}] = -\frac{1}{EI}\mathbf{H}^{-1}\left(E[\mathbf{g}] + \mathbf{R}E[\mathbf{v}]\right) \tag{47a,b}$$

At last $E[D_1]$ into Eq. (45) is the first element of the vector $E[\mathbf{d}]$. Thanks to the expressions given in the previous section for $E[G_i(x)]$ and to those given into Eqs. (41e) and (42d), the evaluation of the vectors $E[\mathbf{g}]$ and $E[\mathbf{g}_3]$ appearing in the previous relationships becomes very simple.

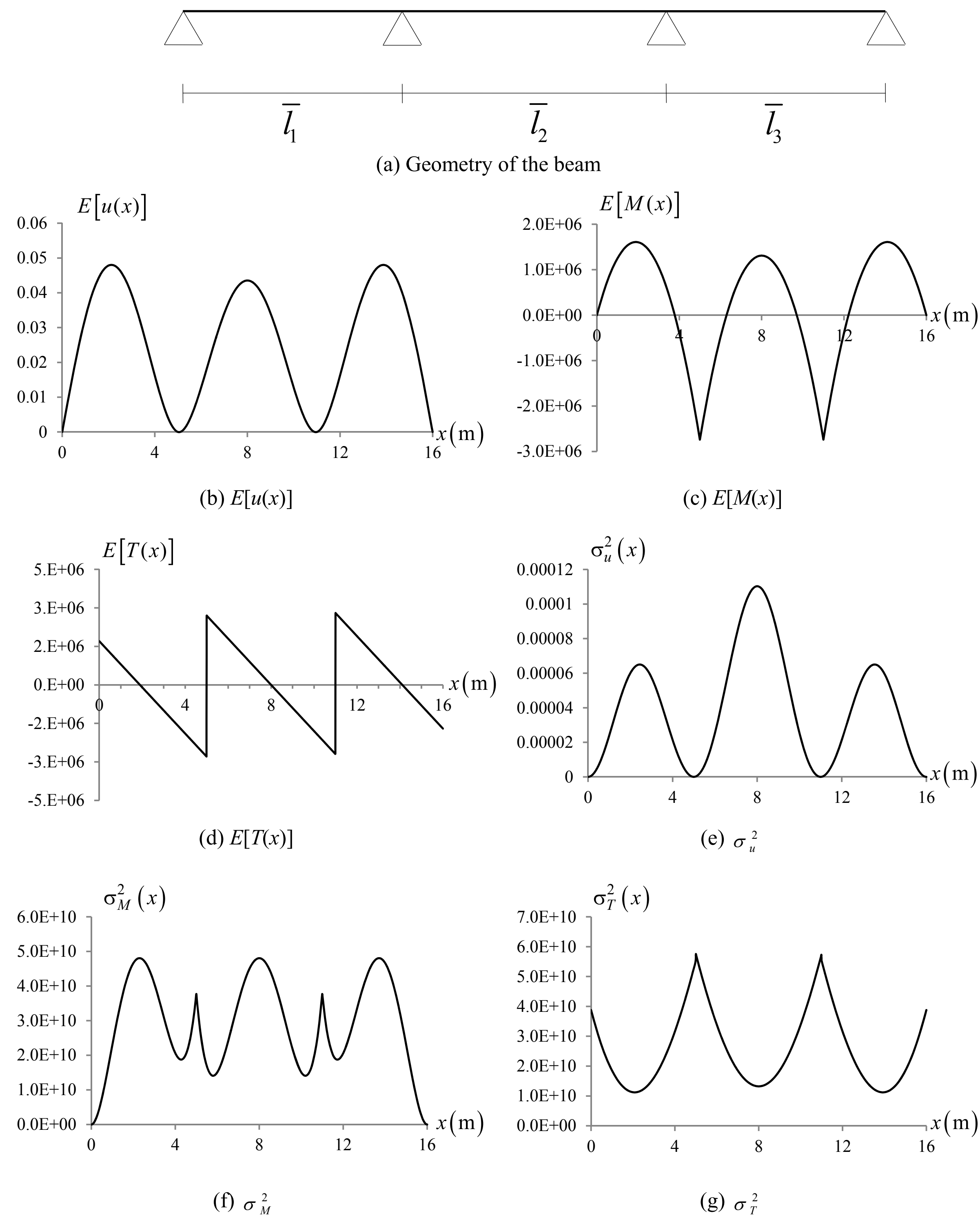
The expressions of the corresponding second order correlation functions can be obtained starting again from Eq. (44) and they are

$$\begin{aligned}
C_u^{(2)}(x_1,x_2) &= \frac{1}{(EI)^2}\left[C_{G_3}^{(2)}(x_1,x_2)+\mathbf{r}_3^T(x_1)\boldsymbol{\Sigma}_{\mathbf{vv}}\mathbf{r}_3(x_2)\right]+\mathbf{h}_3^T(x_1)\boldsymbol{\Sigma}_{\mathbf{dd}}\mathbf{h}_3(x_2)\\
&+\frac{1}{(EI)^2}\left[\mathbf{r}_3^T(x_2)\boldsymbol{\sigma}_{G_3\mathbf{v}}(x_1)+\mathbf{r}_3^T(x_1)\boldsymbol{\sigma}_{G_3\mathbf{v}}(x_2)\right]\\
&+\frac{1}{EI}\left[\mathbf{h}_3^T(x_2)\boldsymbol{\sigma}_{G_3\mathbf{d}}(x_1)+\mathbf{h}_3^T(x_1)\boldsymbol{\sigma}_{G_3\mathbf{d}}(x_2)\right]\\
&+\frac{1}{EI}\left[\mathbf{r}_3^T(x_1)\boldsymbol{\Sigma}_{\mathbf{vd}}\mathbf{h}_3(x_2)+\mathbf{r}_3^T(x_2)\boldsymbol{\Sigma}_{\mathbf{vd}}\mathbf{h}_3(x_1)\right];\\
C_M^{(2)}(x_1,x_2) &= C_{G_1}^{(2)}(x_1,x_2)+\mathbf{r}_1^T(x_1)\boldsymbol{\Sigma}_{\mathbf{vv}}\mathbf{r}_1(x_2)\\
&+(EI)^2\sigma_{D_1}^2 x_1x_2+\mathbf{r}_1^T(x_2)\boldsymbol{\sigma}_{G_1\mathbf{v}}(x_1)+\mathbf{r}_1^T(x_1)\boldsymbol{\sigma}_{G_1\mathbf{v}}(x_2)\\
&+EI\left(\sigma_{G_1D_1}(x_1)x_2+\sigma_{G_1D_1}(x_2)x_1+\mathbf{r}_1^T(x_2)\boldsymbol{\sigma}_{D_1\mathbf{v}}x_1+\mathbf{r}_1^T(x_1)\boldsymbol{\sigma}_{D_1\mathbf{v}}x_2\right);\\
C_T^{(2)}(x_1,x_2) &= C_{G_0}^{(2)}(x_1,x_2)+\mathbf{r}_0^T(x_1)\boldsymbol{\Sigma}_{\mathbf{vv}}\mathbf{r}_0(x_2)\\
&+(EI)^2\sigma_{D_1}^2+\mathbf{r}_0^T(x_2)\boldsymbol{\sigma}_{G_0\mathbf{v}}(x_1)+\mathbf{r}_0^T(x_1)\boldsymbol{\sigma}_{G_0\mathbf{v}}(x_2)\\
&+EI\left(\sigma_{G_0D_1}(x_1)+\sigma_{G_0D_1}(x_2)+\mathbf{r}_0^T(x_2)\boldsymbol{\sigma}_{D_1\mathbf{v}}+\mathbf{r}_0^T(x_1)\boldsymbol{\sigma}_{D_1\mathbf{v}}\right)
\end{aligned}\tag{48a-c}$$

where $\boldsymbol{\Sigma}_{\mathbf{vv}}$ and $\boldsymbol{\Sigma}_{\mathbf{dd}}$ are the covariance matrices of the random vectors $\mathbf{v}$ and $\mathbf{d}$, respectively, while $\boldsymbol{\Sigma}_{\mathbf{vd}}$ is their cross-covariance matrix; moreover, the vectors of the type $\boldsymbol{\sigma}_{A\mathbf{a}}(x)$ collect the covariances $\sigma_{Aa_i}(x)=E\left[A(x)a_i\right]-E\left[A(x)\right]E\left[a_i\right]$, with $i=1,2,\cdots n-1$; it must be noted that if A is independent by x also $\sigma_{Aa_i}(x)$ is.

It is not difficult to realize that the beam response variances, that are obtained by setting $x_1=x_2=x$ into Eq. (48), have the following form

$$\begin{aligned}
\sigma_u^2(x) &= \frac{1}{(EI)^2}\left[\sigma_{G_3}^2(x)+\mathbf{r}_3^T(x)\boldsymbol{\Sigma}_{\mathbf{vv}}\mathbf{r}_3(x)+2\mathbf{r}_3^T(x)\boldsymbol{\sigma}_{G_3\mathbf{v}}(x)\right]+\mathbf{h}_3^T(x)\boldsymbol{\Sigma}_{\mathbf{dd}}\mathbf{h}_3(x)\\
&+\frac{1}{(EI)^2}\left[2\mathbf{h}_3^T(x)\boldsymbol{\sigma}_{G_3\mathbf{d}}(x)+2\mathbf{r}_3^T(x)\boldsymbol{\Sigma}_{\mathbf{vd}}\mathbf{h}_3(x)\right];\\
\sigma_M^2(x) &= \sigma_{G_1}^2(x)+\mathbf{r}_1^T(x)\boldsymbol{\Sigma}_{\mathbf{vv}}\mathbf{r}_1(x)+(EI)^2\sigma_{D_1}^2x^2+2\mathbf{r}_1^T(x)\boldsymbol{\sigma}_{G_1\mathbf{v}}(x)\\
&+EI\left[2\sigma_{G_1D_1}(x)x+2\mathbf{r}_1^T(x)\boldsymbol{\sigma}_{D_1\mathbf{v}}x\right];\\
\sigma_T^2(x) &= \sigma_{G_0}^2(x)+\mathbf{r}_0^T(x)\boldsymbol{\Sigma}_{\mathbf{vv}}\mathbf{r}_0(x)+(EI)^2\sigma_{D_1}^2+2\mathbf{r}_0^T(x)\boldsymbol{\sigma}_{G_0\mathbf{v}}(x)+EI\left[2\sigma_{G_0D_1}(x)+2\mathbf{r}_0^T(x)\boldsymbol{\sigma}_{D_1\mathbf{v}}\right]
\end{aligned}\tag{49a-c}$$



(a) Geometry of the beam

(b) $E[u(x)]$ (c) $E[M(x)]$

(d) $E[T(x)]$ (e) σ_u^2

(f) σ_M^2 (g) σ_T^2

Fig. 4 Means and variances of a three-span beam

In Figs. 4(b)-(g) the means and the variances of the beam response quantities are represented for the continuous beam represented in Fig.4a, characterized by: $\overline{l_1} = \overline{l_3} = 5\text{m}$; $\overline{l_2} = 6\text{m}$ and bending stiffness $EI = 6.5625 \times 10^7$ N/mm^2. The first and second moments of the random variable F_i assumed in the analysis are $\mathrm{E}\left[F_i\right] = 3\times10^4\,\mathrm{N}$ and $\mathrm{E}\left[F_i^2\right] = 9\times10^8\,\mathrm{N}^2$, while $\lambda = 30$.

The knowledge of the mean values and of the variances of the internal forces $M(x)$ and $T(x)$ allows to apply the β -method for the reliability of the beam, as will be shown in the next section.

5. Application of the reliability β -method

The knowledge of the mean values and of the variances of the internal force $S(x)$, where $S(x)$ may be the absolute value of the bending moment $M(x)$ or the absolute value of the shear force $T(x)$, allows to apply the β -method approach for studying the beam reliability. It is obvious that it is also necessary the knowledge of the corresponding quantities referred to the resistance force $R(x)$, that indicates the resistance bending moment if $S(x) \equiv |M(x)|$ and the resistance shear force if $S(x) \equiv |T(x)|$. In the following, the case of deterministic resistance will be treated; but this does not invalidate the proposed approach that can be simply rearranged for taking into account the eventual randomness of $R(x)$.

It is known that in the cases treated in this work the β -method gives approximate results because of the non-Gaussianity of $S(x)$. As a matter of the fact, the beam responses would be Gaussian processes only if their input is Gaussian and this happens only if $\lambda \rightarrow \infty$ and $E[F^2] \rightarrow 0$ simultaneously. However, even if the beam response quantities are not Gaussian processes, for relatively high values of λ, the β -method gives sufficiently accurate results.

Once that the significance of the internal action $S(x)$ and of the resistance $R(x)$ are defined, it is possible to introduce the so-called success variable defined as follows

$$K(x) = R(x) - S(x) \tag{50}$$

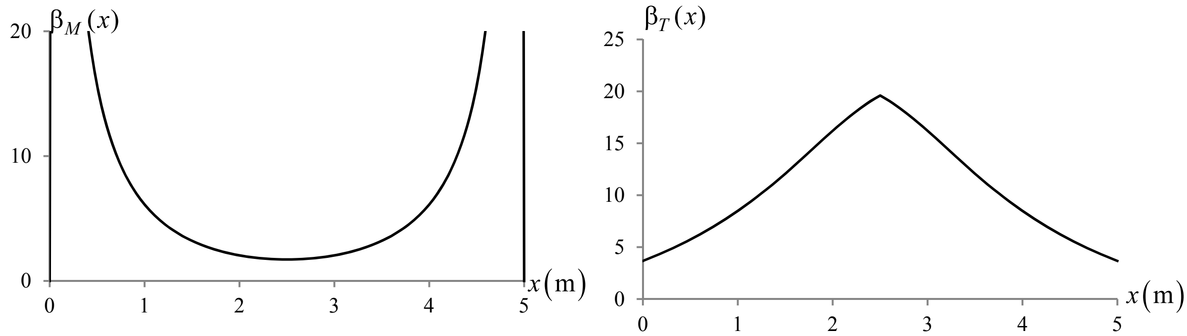
The corresponding β -coefficient is defined as follows (Papoulis and Pillai 2002)

$$\beta(x) = \frac{E\left[K(x)\right]}{\sigma_K(x)} = \frac{R - E\left[S(x)\right]}{\sigma_S(x)} \tag{51}$$

where the hypothesis that $R(x)$ is deterministic and independent by x has been taken into account. The knowledge of $\beta(x)$ allows to evaluate the beam failure probability against the given load conditions in the form

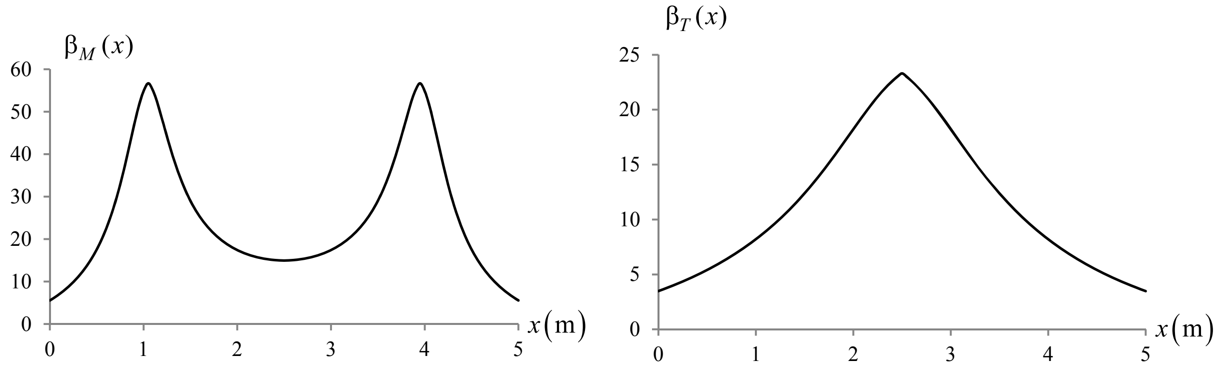
$$P_f(x) = \frac{1}{2} - \mathrm{erf}\left[\beta(x)\right] \tag{52}$$

It is interesting to identify the critical abscissa x_f where the failure probability above defined is maximum. Due to the properties of the error function $\mathrm{erf}\left[\square\right]$, the failure probability is maximum where $\beta(x)$ is minimum. For example, for the hinged-hinged beam treated in the section 3.1 it is possible to define the following two β -coefficients for the bending moment and the shear force, respectively



(a) $\beta_M(x)$ (b) $\beta_T(x)$

Fig. 5 β -coefficients for the bending moment and shear force of the hinged-hinged beam



(a) $\beta_M(x)$ (b) $\beta_T(x)$

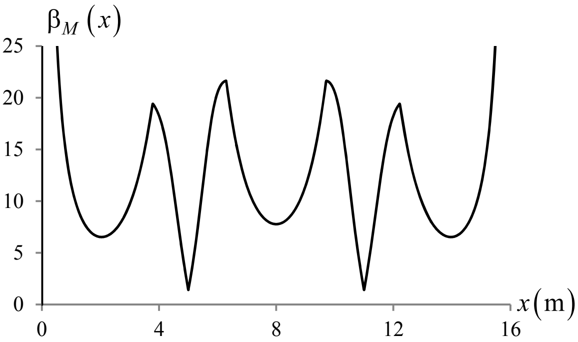
Fig. 6 β -coefficients for the bending moment and shear force of the clamped-clamped beam

$$\beta_M(x) = \frac{R_M - E\left[\left|M(x)\right|\right]}{\sigma_M(x)} = \frac{R_M - \frac{1}{2}\lambda E[F]x(l-x)}{\sqrt{\frac{\lambda E\left[F^2\right]}{3}x^2\left(\frac{1}{l}x^2 - 2x + l\right)}};$$

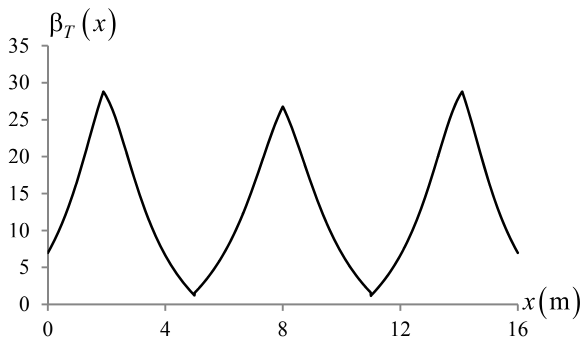
$$\beta_T(x) = \frac{R_T - E\left[\left|T(x)\right|\right]}{\sigma_T(x)} = \frac{R_T - \lambda E[F]\left(\left|\frac{l}{2} - x\right|\right)}{\sqrt{\lambda E\left[F^2\right]\left(\frac{1}{l}x^2 - x + \frac{l}{3}\right)}} \qquad (53a, b)$$

R_M and R_T being the values of the bending moment resistance and of the shear force resistance.

In Fig. 5(a), (b) the corresponding graphics of $\beta(x)$ for the bending moment and the shear force are reported, assuming $l = 5$m, $E[F] = 10^4$ N, $E[F^2] = 10^8$ N^2, $\lambda = 10$, $R_M = 4 \times 10^5$ Nm, $R_T = 4 \times 10^5$ N. In Fig. 6(a), (b) the same graphics referred to the clamped-clamped beam are given. The



(a) $\beta_M(x)$



(b) $\beta_T(x)$

Fig. 7 β -coefficients for the bending moment and shear force of the three-span beam

β -coefficients for the bending moment and the shear force of the continuous beam studied in the previous section are shown in Fig. 7(a), (b), where it has been assumed $R_M = 5 \times 10^6$ Nm and $R_T = 3 \times 10^6$ N. It is important to note that, for any example taken into account in this work, the most critical sections evidenced by the study of the β -coefficients are coincident with the most critical sections deriving by the application of an uniformly distributed deterministic load on the same beam.

6. Conclusions

The bending problem of Euler-Bernoulli discontinuous beams, where the discontinuity is due to the loads and eventually to essential constrains, has been dealt with. Studying this problem could be important in the static analysis of the bridge beams if the load is modeled as a delta-correlated process along the beam axis. In this case, it has been shown that the use of the generalized function has reduced the computational effort of the problem. In particular, it has made easily applicable the rules of the structural stochastic analysis, allowing to find the beam exact response in terms of means and variances of the deflections, the bending moments and the shear forces. The evaluation of these quantities has implied the possibility of applying the reliability β -method. In all the examples taken into account, it has been shown that the critical sections for the bending moments and the shear forces found with this approach coincide with those related to an uniformly distributed deterministic load.

References

Biondi, G. and Caddemi, S. (2007), “Euler-Bernoulli beams with multiple singularities in the flexural stiffness”, *European Journal of Mechanics, A/Solids*, **26**, 789-809.

Brungraber, R.J. (1965), “Singularity functions in the solution of beam-deflection problems”, *Journal of Engineering Education (mechanics Division Bulletin)*, **1.55**, 278-280.

Colajanni, P., Falsone, G. and Recupero, A. (2009), “Simplified formulation of solution for beams on Winkler foundation allowing discontinuities due to loads and constrains”, *International Journal of Engineering Education*, **25**, 75-83.

Dirac, P.A.M. (1947), *The principle of quantum mechanics*, Oxford University Press, Oxford.

Falsone, G. (2002), "The use of generalized functions in the discontinuous beam bending differential equations", *International Journal of Engineering Education*, **18**, 337-343.

Failla, G. (2011), "Closed-form solutions for Euler-Bernoulli arbitrary discontinuos beams", *Archives of Applied Mechanics*, **81**, 605-628.

Failla, G. and Impollonia, N. (2012), "General finite element description for non-uniform and discontinuous beam elements", *Archives of Applied Mechanics*, **82**, 43-67.

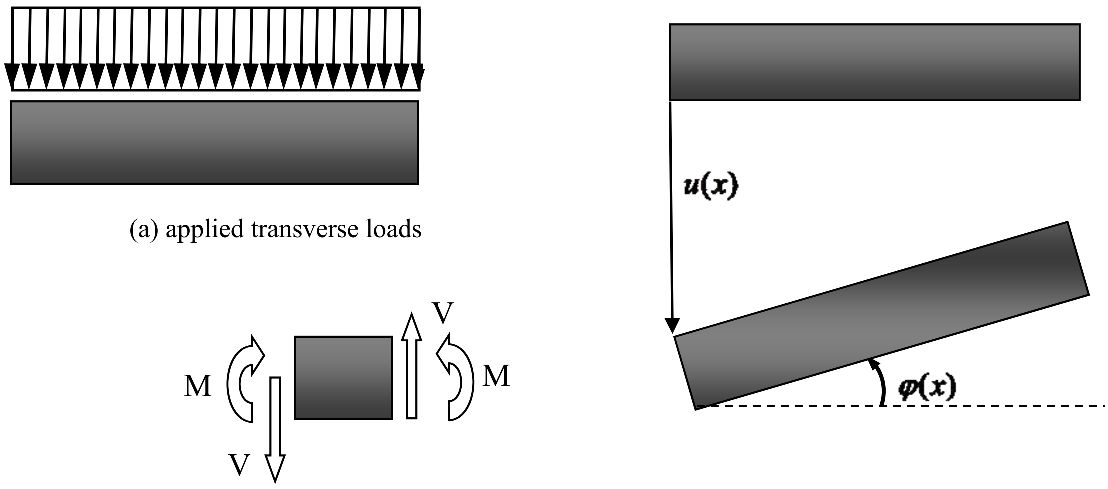
Failla, G. and Santini, A. (2007), "On Euler-Bernoulli discontinuous beam solutions via uniform-beam Green's functions", *International Journal of Solids and Structures*, **44**, 7666-7687.

Lighthill, M.J. (1959), *An Introduction to Fourier Analysis and Generalized Functions*, Cambridge University Press, Cambridge.

Macaulay, W.H. (1919), "Note on the deflection of the beams", *Messenger of Mathematics*, **48**, 129-130.

Papoulis, A. and Pillai, S.U. (2002), *Probability, Random Variables and Stochastic Processes*, 4th Edition, McGraw-Hill, Boston.

Appendix. Sign convention



(a) applied transverse loads

(b) generalized internal forces

(c) generalized displacements

Fig. A1 Sign convention