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Buckling and vibration of rectangular plates of variable thickness with different end conditions by finite difference technique

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Abstract. This paper is concerned with the determination of exact buckling loads and vibration frequencies of variable thickness isotropic plates using well known finite difference technique. The plates are subjected to uni, biaxial compression and shear loadings and various combinations of boundary conditions are considered. The buckling load is found out as the in plane load that makes the determinant of the stiffness matrix equal to zero and the natural frequencies are found out by carrying out eigenvalue analysis of stiffness and mass matrices. New and exact results are given for many cases and the results are in close agreement with the published results. In this paper, like finite element method, finite difference method is applied in a very simple manner and the application of boundary conditions is also automatic.

Keywords: finite difference; buckling load; natural frequency; stepped plate; stiffened plate; buckling coefficient; frequency parameter

1. Introduction

In the optimal design of structures, varying thickness plates are frequently used to economize on the plate materials or lighten the plates. Stiffened plates possess a number of attractive features, such as material saving, weight reduction, stiffness and strength improvement. Most research over the last 25 years has been based on theoretical approaches both approximate and exact and is computer based. Numerical methods characterize the behaviour of a structure at points or within regions of the structure that result in large order of system of equations whose coefficients are numerically evaluated and they are functions of materials, geometry and applied load parameters at these points or regions. Finite difference, finite element and finite strip are the foremost numerical methods. The finite element provides the most general frame work and many of the contributions of finite element and finite strip methods to plates are given by Azhari (1993). When dealing with non-uniform thickness plates, it is generally difficult to obtain exact solutions and hence many of the references used numerical methods for determining buckling and vibration solutions.

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Finite difference represents the traditional approach to numerical analysis based on well established principles. The method leads to solution of equations assuring convergence. The resulting algebraic equations are relatively simple. Even though finite element method is well established as a procedure for static and dynamic analysis for complex structures, in this paper, it is shown that finite difference method is applied in a simple manner and incorporating boundary conditions is also an easy task.

The buckling loads for uniform thickness plates have been summarised in the text by Timoshenko and Gere (1961), Allen and Bulson (1980), Szilard (2004). Many authors have investigated the problem of stability of variable thickness plates. Wittrick and Ellen (1962) solved the problem of plates tapering in one direction and uniformly compressed in that direction by using Galerkin's method. Navaneethakrishnan (1968) applied quintic – spline collocation technique by considering sinusoidal variation in the thickness in loaded direction and constant thickness in other direction to solve for buckling of plate with various support conditions. Chung and Cheung (1971) investigated the thin flat- walled structures with different boundary conditions in which they developed a finite strip local buckling approach for plates. The first use of the semi analytical finite strip method for local buckling appears to be in the work of Przemieniecki (1973). Perturbations technique was applied by Chehil and Dua (1973) to solve the stability problems of a plate. Hwang (1973) investigated the stability of plates with piecewise variation in thickness using energy method applying the principle of virtual displacement. Hancock (1978) used a sine curve for longitudinal variation of buckling displacement which is applicable for structures whose ends are simply supported. The implementation of energy method in stability of plates was explained by Chen and Lui (1987). Singh and Dey (1990) applied variational finite difference approach for bi-directionally stepped plates. Harik *et al.* (1991) used a semi numerical – semi analytical method for the analysis of plates with varying rigidities. They first reduced the governing partial differential equation to an ordinary differential equation using the method of separation of variables, and then finite difference technique was used to solve the ordinary differential equation. Subramanian *et al.* (1993) analyzed elastic stability of varying thickness plates using the finite element method with uniformly distributed compressive forces in one direction. Bradford and Ashari (1995) used finite strip method using two types of series functions to find the elastic local buckling of plates with different boundary conditions. Nerantzaki and Katsikadalils (1996) solved for the buckling load of simply supported plate with linear and exponential variation in loaded direction by using the analogue equation method. Bradford and Azhari (1997) used modified semi-analytical using bubble functions for the stability of plates with different boundary conditions. Yuan and Yin (1998) used Kantorovich method for computation of buckling loads. Xiang and Wang (2002) presented an analytical approach that combines Levy method and the state space technique for determining exact buckling and vibration solutions of uni – directional multi-stepped rectangular plates. The extended Kantorovich method in conjunction with the exact element method was used by Eisenberger and Alexandrov (2003). Exact solutions for buckling and vibration of stepped rectangular Mindlin plates were obtained by Xiang and Wei (2004) by using Levy's solution and a decomposition method. Xiang and Zhang (2005) presented results for the free vibration analysis of stepped circular Mindlin plates with multiple step-wise thickness variations using the domain decomposition technique. They have considered various plate boundary conditions and various ratios such as thickness location etc. Gupta *et al.* (2006) have analysed free axi-symmetric vibrations of non-homogeneous isotopic circular plates of nonlinear thickness variation by using differential quadrature method. In addition to variation in thickness, variation in Young's modulus and mass density of plate material is also considered. They have been presented and compared with thickness variation and boundary conditions .The thickness variations in one or two directions are taken as polynomial form. Mindlin's first order shear deformation theory and Reddy's higher order shear deformation theory has been applied to the plate analysis. Solution was obtained by using Extended Kantorovich method. Numerical results were compared with published results. Xiang (2007) applied Levy's solution method with domain decomposition method and the state space technique for free vibration of rectangular plates with abrupt changes in properties such as plate thickness, in shear force and in slopes. Matsunaga (2008) analysed for free vibration and buckling analysis of plates made of functionally graded (FG) material taking into account the effect of transverse shear and normal deformation and rotary inertia. Fundamental dynamic equation using 2D higher theory is derived using power series expansion of displacements and Hamilton's principle. Results show that 2D higher order theory predicts accurately the natural frequency of functionally graded simply supported plates. Nie and Zhong (2008) investigated the free and forced vibration of functionally graded annular sectorial plates with simply supported radial edges and arbitrary circular edges using a semi – analytical approach. Numerical results are compared with the existing exact solutions and the good agreement is obtained. Yalcin *et al.* (2009) carried out the free vibration analysis of thin circular plates for simply supported, clamped and free boundary conditions using differential transform method (DTM). They reduced the governing differential equation into recurrent relation using (DTM) finally reducing those to algebraic equations. Their results are compared with Bessel solutions. Hashami *et al.* (2011) presented exact closed –form solution of free vibration analysis of thick FG rectangular plates with different conditions of free, simply supported and clamped boundary conditions. Vibration frequencies were obtained by solving five coupled partial differential equations and using the potential functions and separation of variables. Felix *et al.* (2011) investigated the buckling and vibration of clamped orthotropic plate under linearly varying in-plane load using Ritz technique. John Wilson and Rajasekaran (2012) investigated elastic stability of all edges simply supported, stepped and stiffened rectangular plate under uni-axial loading. The objective of this paper is to present the results of buckling and vibration of plates subjected to in-plane loads using finite difference technique. Various numerical examples are solved and the results are compared with the earlier work wherever possible.

2. Formulation of the problem

Consider a rectangular plate of side 'a' and 'b' as shown in Fig. 1. Divide the plate into $M \times N$ rectangular meshes of length $l = a/M$; $m = b/N$.

2.1 Plate supported on all edges clamped (C) or simply supported (S)

To solve the problem by finite difference approach one has to assume imaginary points as shown in Fig. 2(a). The total number of unknowns will be the deflections at all the $nj = MN + 3M + 3N - 3$ joints. (Total no. of joints = internal points+ boundary points+ imaginary points).

Equations available are

1. Equilibrium equations to be provided at all the internal points $neq = (M-1)(N-1)$ as

$$\underset{neq \times nj}{\mathbf{M_0}} \ \underset{nj \times 1}{\underline{\mathbf{w}}} + \underset{neq \times nj}{\mathbf{h_0}} \ \underset{nj \times 1}{\underline{\mathbf{w}}} = \mathbf{0} \tag{1}$$

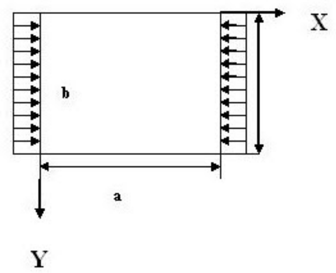


Fig. 1 Uniform rectangular plate

where $\mathbf{M}_0$ and $\mathbf{h}_0$ represent flexural and geometric stiffness matrix in case of buckling problems and flexural and mass matrices in case of free vibration problems.

2. Displacements are zero along all the four boundaries of the plate giving $2(M+N)$ equations
3. If the edges are simply supported, (moments are zero at the points on the support)

3a) If edges $x = 0$ and $x = a$ are simply supported

$$M_x = D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right) \tag{2a}$$

Since $\frac{\partial^2 w}{\partial y^2} = 0$ on those edges

$\left.\frac{\partial^2 w}{\partial x^2}\right|_{i,j} = \frac{w_{i-1,j} - 2\,w_{i,j} + w_{i+1,j}}{l^2} = 0$ where (i, j) is the point on the boundary and $w_{i,j} = 0$ and

hence
$$w_{i-1,j} = -\,w_{i+1,j} \tag{2b}$$

Similarly if on edges $y = 0$ and $y = b$ are simply supported $w_{i,j} = 0$ and

hence
$$w_{i,j-1} = -\,w_{i,j+1} \tag{2c}$$

3b) If the edges are clamped on $x = 0$ and $x = a$ the slopes are zero.

$$\theta_x = \frac{\partial w}{\partial x} = \frac{w_{i+1,j} - w_{i-1,j}}{2l} = 0 \ or \ w_{i+1,j} = w_{i-1,j} \tag{2d}$$

If edges $y = 0$ and $y = b$ are clamped

$$\theta_y = \frac{\partial w}{\partial y} = \frac{w_{i,j+1} - w_{i,j-1}}{2m} = 0 \ or \ w_{i,j+1} = w_{i,j-1} \tag{2e}$$

The above conditions a) or b) lead to $2(M + N - 2)$ equations.

4. Hence the total boundary condition equations are $4(M + N - 1)$ given by

$$\underset{4(M+N-1)\times nj}{\mathbf{M_1}} \quad \underset{nj\times 1}{\underline{\mathbf{w}}} = \mathbf{0} \tag{3a}$$

5. Combining the equilibrium equations and boundary condition equations we get total no of equations available are $nj = 4(M + N - 1) + (M - 1)(N - 1) = MN + 3M + 3N - 3$ which is the required no of unknowns given by

$$\underset{nj \times nj}{\begin{bmatrix} \mathbf{M_0} \\ \mathbf{M_1} \end{bmatrix}} \underset{nj \times 1}{\underline{\mathbf{w}}} + \underset{nj \times nj}{\begin{bmatrix} \mathbf{h_0} \\ \mathbf{0} \end{bmatrix}} \underset{nj \times 1}{\underline{\mathbf{w}}} = \mathbf{0} \tag{3b}$$

2.2 For plates supported on three edges (x=0; y=0; y=b) and free on edge x=a (see Fig. 2(b))

The total number of unknowns will be the deflections at all the $n_j = NM + 3N + 4M$ joints.
Equations available are

1. Equilibrium equations to be provided at all the internal points $neq = MN - N$ written as

$$\underset{neq \times nj}{\mathbf{M_0}} \underset{nj \times 1}{\underline{\mathbf{w}}} + \underset{neq \times nj}{\mathbf{h_0}} \underset{nj \times 1}{\underline{\mathbf{w}}} = \mathbf{0} \tag{4}$$

2. Displacements are zero along all the three boundaries of the plate giving $(2M + N + 1)$ constraint equations.
3. If the edges are simply supported, (moments are zero at the points on the support)

$$w_{i-1}, j = -w_{i+1,j}; \quad w_{i,j-1} = -w_{i,j+1} \tag{5a}$$

or if the edges are clamped (slopes are zero at the points on the support)(for three edges)

$$w_{i-1}, j = w_{i+1,j}; \quad w_{i,j-1} = w_{i,j+1} \tag{5b}$$

leads to $(2M + N - 1)$ equations.

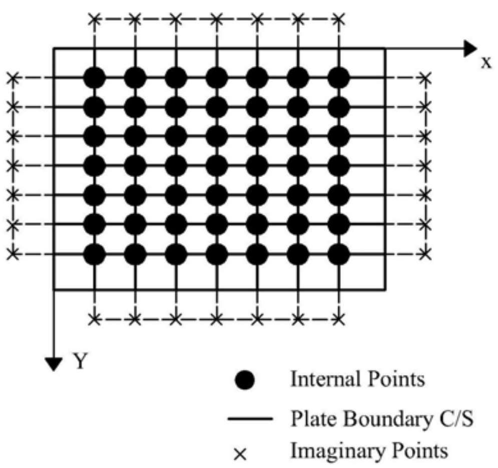


Fig. 2 (a) Internal and boundary points of a plate (edges S or C)

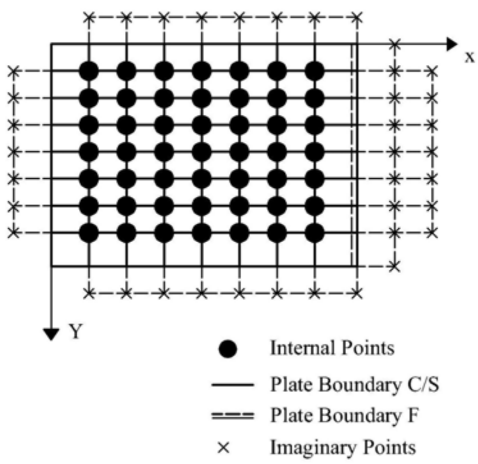


Fig. 2 (b) Internal and boundary points of a plate (edges $y = 0$ and $y = b$ C/S; $x = 0$ – S, $x = a$ – F

4. Moment is zero on the free edge leading to (N + 1) equations and shear is zero on the free edge leading to (N – 1) equations.

5. Hence the total constraint equations are 4(N + M) given by

$$\underset{4(M+N)\times nj}{\mathbf{M_1}} \quad \underset{nj\times 1}{\underline{\mathbf{w}}} = \mathbf{0} \tag{6a}$$

6. Combining the equilibrium equations and constraint equations we get total no of equations available are $nj = (MN - N) + (4M + 4N) = MN + 4M + 3N$ which is the required no of unknowns given by

$$\underset{nj\times nj}{\begin{bmatrix}\mathbf{M_0}\\ \mathbf{M_1}\end{bmatrix}} \quad \underset{nj\times 1}{\underline{\mathbf{w}}} + \underset{nj\times nj}{\begin{bmatrix}\mathbf{h_0}\\ \mathbf{0}\end{bmatrix}} \quad \underset{nj\times 1}{\underline{\mathbf{w}}} = \mathbf{0} \tag{6b}$$

which is equivalent to

$$\mathbf{K}\,\underline{\mathbf{w}} + \mathbf{K_G}\,\underline{\mathbf{w}} = \mathbf{0} \tag{7a}$$

or

$$\mathbf{K}\,\underline{\mathbf{w}} + P\,\overline{\mathbf{K}}_G\,\underline{\mathbf{w}} = \mathbf{0} \tag{7b}$$

and

$$\mathbf{K}\,\underline{\mathbf{w}} + \omega^2\,\mathbf{M}\,\underline{\mathbf{w}} = \mathbf{0} \tag{7c}$$

where $\mathbf{K}$, $\overline{\mathbf{K}}_G$ and $\mathbf{M}$ are the flexural stiffness, geometric stiffness and mass matrices respectively. Eq. 7(b) is the governing equation for buckling problems and Eq. 7(c) for free vibration problems where P and ω denote the buckling load and the natural frequency of the plate.

2.3 Derivation of matrix coefficients: (buckling problem)

In order to derive the coefficient matrices for buckling problems, let us consider the fundamental differential equation for the deflection 'w' of a thin plate under the action of forces in the middle plane and this is given by

$$\frac{\partial^2 M_x}{\partial x^2} - 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = N_x\frac{\partial^2 w}{\partial x^2} + N_y\frac{\partial^2 w}{\partial y^2} + 2N_{xy}\frac{\partial^2 w}{\partial y \partial x} = P\left(\beta\frac{\partial^2 w}{\partial x^2} + \gamma\frac{\partial^2 w}{\partial y^2} + 2\delta\frac{\partial^2 w}{\partial y \partial x}\right) \tag{8}$$

where β, γ, δ are the tracers that take the value of either 0 or 1 for different in-plane load combinations and

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right), M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right), M_{xy} = D(1-\nu)\left(\frac{\partial^2 w}{\partial x \partial y}\right) \; and \; D = \frac{Et^3}{12(1-\nu^2)} \tag{9}$$

D is the flexural rigidity of the plate and N_x, N_y and N_{xy} are the in- plane forces per unit length.

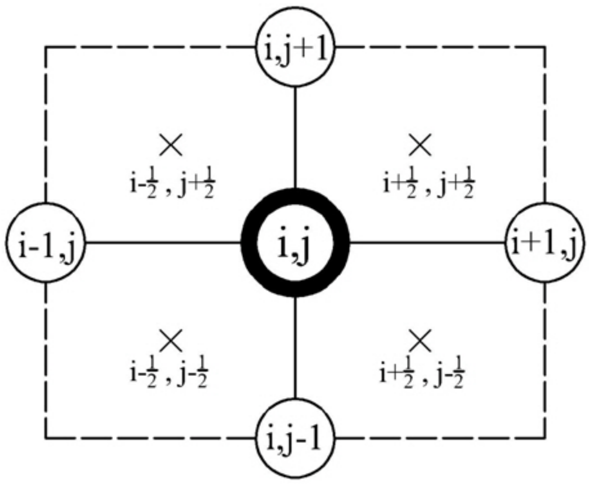


Fig. 3 Definition of subscripts of w

By considering the in-plane forces the governing differential equation becomes

$$\frac{\partial^2}{\partial x^2}\left\{t^3\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)\right\}+2\frac{\partial^2}{\partial x\partial y}\left\{t^3\left(1-\nu\right)\frac{\partial^2 w}{\partial x\partial y}\right\}+\frac{\partial^2}{\partial y^2}\left\{t^3\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right)\right\}$$
$$=-\frac{12\left(1-\nu^2\right)}{E}P\left\{\beta\frac{\partial^2 w}{\partial x^2}+\gamma\frac{\partial^2 w}{\partial y^2}+2\delta\frac{\partial^2 w}{\partial x\partial y}\right\} \tag{10}$$

At the node (i, j) for varying thickness $t_{i,j}$ the above equation can be transformed into system of equations using the following central difference formulae for the derivatives. The subscripts of w given in Eqs. (11) to (22) are shown in Fig. 3.

$$\left(\frac{\partial w}{\partial x}\right)_{i,j}=\frac{1}{l}\left(w_{i+\frac{1}{2},j}-w_{i-\frac{1}{2},j}\right)\quad,\left(\frac{\partial w}{\partial y}\right)_{i,j}=\frac{1}{m}\left(w_{i,j+\frac{1}{2}}-w_{i,j-\frac{1}{2}}\right) \tag{11}$$

$$\left(\frac{\partial^2 w}{\partial x^2}\right)_{i,j}=\frac{1}{l^2}\left(w_{i+1,j}-2w_{i,j}+w_{i-1,j}\right)\quad,\left(\frac{\partial^2 w}{\partial y^2}\right)_{i,j}=\frac{1}{m^2}\left(w_{i,j+1}-2w_{i,j}+w_{i,j-1}\right) \tag{12}$$

$$\left(\frac{\partial^2 w}{\partial x\partial y}\right)_{i,j}=\frac{1}{lm}\left(w_{i+\frac{1}{2},j+\frac{1}{2}}-w_{i-\frac{1}{2},j+\frac{1}{2}}-w_{i+\frac{1}{2},j-\frac{1}{2}}+w_{i-\frac{1}{2},j-\frac{1}{2}}\right) \tag{13}$$

$$
\begin{aligned}
\frac{\partial^2}{\partial x^2}\left\{t^3\left(\frac{\partial^2 w}{\partial x^2}=\nu\frac{\partial^2 w}{\partial y^2}\right)\right\}_{i,j} &= \frac{1}{l^2}\left\{\; t^3_{i+1,j}\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)_{i+1,j}\right. \\
&\quad \left. -\,2t^3_{i,j}\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)_{i,j} + t^3_{i-1,j}\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)_{i-1,j}\;\right\} \\
&= t^3_{i+1,j}\left\{\frac{1}{l^4}\left(w_{i+2,j}-2w_{i+1,j}+w_{i,j}\right)+\frac{\nu}{l^2m^2}\left(w_{i+1,j+1}-2w_{i+1,j}+w_{i+1,j-1}\right)\right\} \\
&\quad -2t^3_{i,j}\left\{\frac{1}{l^4}\left(w_{i+1,j}-2w_{i,j}+w_{i-1,j}\right)+\frac{\nu}{l^2m^2}\left(w_{i,j+1}-2w_{i,j}+w_{i,j-1}\right)\right\} \\
&\quad +t^3_{i-1,j}\left\{\frac{1}{l^4}\left(w_{i,j}-2w_{i-1,j}+w_{i-2,j}\right)+\frac{\nu}{l^2m^2}\left(w_{i-1,j+1}-2w_{i-1,j}+w_{i-1,j-1}\right)\right\}
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
2\frac{\partial^2}{\partial x\partial y}\left\{t^3\left(1-\nu\right)\frac{\partial^2 w}{\partial x\partial y}\right\}_{i,j} &= \frac{2\left(1-\nu\right)}{lm}\left\{\begin{matrix} t^3_{i+\frac{1}{2},j+\frac{1}{2}}\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)_{i+\frac{1}{2},j+\frac{1}{2}} - t^3_{i-\frac{1}{2},j+\frac{1}{2}}\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)_{i-\frac{1}{2},j+\frac{1}{2}} \\ -t^3_{i+\frac{1}{2},j-\frac{1}{2}}\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)_{i+\frac{1}{2},j-\frac{1}{2}} + t^3_{i-\frac{1}{2},j-\frac{1}{2}}\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)_{i-\frac{1}{2},j-\frac{1}{2}}\end{matrix}\right\} \\
&= \frac{2\left(1-\nu\right)}{l^2m^2}\left\{t^3_{i+\frac{1}{2},j+\frac{1}{2}}\left(w_{i+1,j+1}-w_{i+1,j}-w_{i,j+1}+w_{i,j}\right)-t^3_{i-\frac{1}{2},j+\frac{1}{2}}\left(w_{i,j+1}-w_{i-1,j+1}-w_{i,j}+w_{i-1,j}\right)\right. \\
&\quad \left. -t^3_{i+\frac{1}{2},j-\frac{1}{2}}\left(w_{i+1,j}-w_{i+1,j-1}-w_{i,j}+w_{i,j-1}\right)+t^3_{i-\frac{1}{2},j-\frac{1}{2}}\left(w_{i,j}-w_{i-1,j}-w_{i,j-1}+w_{i-1,j-1}\right)\;\right\}
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
\frac{\partial^2}{\partial y^2}\left\{t^3\left(\frac{\partial^2 w}{\partial y^2}=\nu\frac{\partial^2 w}{\partial x^2}\right)\right\}_{i,j} &= \frac{1}{m^2}\left\{\; t^3_{i,1+j}\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right)_{i,1+j}\right. \\
&\quad \left. -\,2t^3_{i,j}\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)_{i,j} + t^3_{i,1+j}\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right)_{i,1+j}\;\right\} \\
&= t^3_{i,j+1}\left\{\frac{1}{m^4}\left(w_{i,j+2}-2w_{i,j+1}+w_{i,j}\right)+\frac{\nu}{l^2m^2}\left(w_{i+1,j+1}-2w_{i,j+1}+w_{i-1,j+1}\right)\right\} \\
&\quad -2t^3_{i,j}\left\{\frac{1}{m^4}\left(w_{i,j+1}-2w_{i,j}+w_{i,j-1}\right)+\frac{\nu}{l^2m^2}\left(w_{i+1,j}-2w_{i,j}+w_{i-1,j}\right)\right\} \\
&\quad +t^3_{i,j-1}\left\{\frac{1}{m^4}\left(w_{i,j}-2w_{i,j-1}+w_{i,j-2}\right)+\frac{\nu}{l^2m^2}\left(w_{i+1,j-1}-2w_{i,j-1}+w_{i-1,j-1}\right)\right\}
\end{aligned}
\tag{16}
$$

Substitute Eqs. (14) – (16) in Eq. (10)
we get the following equilibrium equations for different values of i and j.
With $Z = 2(1 - \nu)\alpha^2$, $X = -2\alpha^4 - 2\nu\alpha^2$, $Y = -2 - 2\nu\alpha^2$, $Q = 4\alpha^4 + 8\nu\alpha^2 + 4$, $\alpha = m/l$

$$w_{i,j}\left[Qt^3_{i,j} + \alpha^4\left(t^3_{i+1,j} + t^3_{i-1,j}\right) + t^3_{i,j+1} + t^3_{i,j-1} + Z\left(t^3_{i-\frac{1}{2},j+\frac{1}{2}} + t^3_{i+\frac{1}{2},j-\frac{1}{2}} + t^3_{i+\frac{1}{2},j+\frac{1}{2}} + t^3_{i-\frac{1}{2},j-\frac{1}{2}}\right)\right]$$

$$+w_{i+1,j}\left\{\left(t^3_{i,j} + t^3_{i+1,j}\right)X - \left(t^3_{i+\frac{1}{2},j-\frac{1}{2}} + t^3_{i+\frac{1}{2},j+\frac{1}{2}}\right)Z\right\} + w_{i-1,j}\left\{\left(t^3_{i,j} + t^3_{i-1,j}\right)X - \left(t^3_{i-\frac{1}{2},j+\frac{1}{2}} + t^3_{i-\frac{1}{2},j-\frac{1}{2}}\right)Z\right\}$$

$$+w_{i,j+1}\left\{\left(t^3_{i,j} + t^3_{i,j+1}\right)Y - \left(t^3_{i+\frac{1}{2},j+\frac{1}{2}} + t^3_{i-\frac{1}{2},j+\frac{1}{2}}\right)Z\right\} + w_{i,j-1}\left\{\left(t^3_{i,j} + t^3_{i,j-1}\right)Y - \left(t^3_{i+\frac{1}{2},j-\frac{1}{2}} + t^3_{i-\frac{1}{2},j-\frac{1}{2}}\right)Z\right\}$$

$$+w_{i+2,j}\alpha^4 t^3_{i+1,j} + w_{i-2,j}\alpha^4 t^3_{i-1,j} + w_{i,j+2}t^3_{i,j+1} + w_{i,j-2}t^3_{i,j-1}$$

$$+w_{i+1,j+1}\left\{\left(t^3_{i+1,j} + t^3_{i,j+1}\right)\nu\alpha^2 + Zt^3_{i+\frac{1}{2},j+\frac{1}{2}}\right\} + w_{i+1,j-1}\left\{\left(t^3_{i+1,j} + t^3_{i,j-1}\right)\nu\alpha^2 + Zt^3_{i+\frac{1}{2},j-\frac{1}{2}}\right\}$$

$$+w_{i-1,j+1}\left\{\left(t^3_{i-1,j} + t^3_{i,j+1}\right)\nu\alpha^2 + Zt^3_{i-\frac{1}{2},j+\frac{1}{2}}\right\} + w_{i-1,j-1}\left\{\left(t^3_{i-1,j} + t^3_{i,j-1}\right)\nu\alpha^2 + Zt^3_{i-\frac{1}{2},j-\frac{1}{2}}\right\}$$

$$= \left(\frac{-12\left(1-\nu^2\right)P}{E}\right)\begin{Bmatrix} \frac{\beta m^4}{l^2}\left(w_{i+1,j} - 2w_{i,j} + w_{i-1,j}\right) + \gamma m^2\left(w_{i,j+1} - 2w_{i,j} + w_{i,j-1}\right) + \\ \frac{\delta m^3}{2l}\left(w_{i-1,j-1} - w_{i+1,j-1} - w_{i-1,j+1} + w_{i+1,j+1}\right) \end{Bmatrix} \quad (17)$$

2.4 Moment and shear on the free edge

Since ($x = a$) is considered as free edge, the moment

$$M_x = -t^3\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) = 0 \quad (18)$$

or

$$w_{i-1,j}\left(\frac{t^3{}_{i,j}}{l^2}\right) + w_{i+1,j}\left(\frac{t^3{}_{i,j}}{l^2}\right) + w_{i,j+1}\left(\frac{\nu t^3{}_{i,j}}{m^2}\right) + w_{i,j-1}\left(\frac{\nu t^3{}_{i,j}}{m^2}\right) - w_{i,j}2t^3{}_{i,j}\left(\frac{1}{l^2} + \frac{\nu}{m^2}\right) = 0 \quad (19)$$

Again on the free edge shear $V_x = 0$ and it is given by Szilard (2004)

$$V_x = -2\frac{\partial M_{xy}}{\partial y} + \frac{\partial M_x}{\partial x} - \beta P\frac{\partial w}{\partial x} = 0 \tag{20}$$

Writing the finite difference equation at the point (i, j) on the free edge we get

$$w_{i-2,j}A_1 + w_{i-1,j}A_2 + w_{i,j}A_3 + w_{i+1,j}A_4 + w_{i+2,j}A_5 + w_{i-1,j+1}A_6 + w_{i-1,j-1}A_9 + w_{i,j+1}A_7$$
$$+ w_{i,j-1}A_{10} + w_{i+1,j+1}A_8 + w_{i+1,j-1}A_{11} = -\beta P(w_{i+1,j} - w_{i-1,j}) \tag{21}$$

where

$$A_1 = -\frac{t^3_{i,j}}{2l^3};\ A_2 = \frac{t^3_{i,j}}{l^3} + (2-\nu)\frac{t^3_{i,j}}{lm^2} + \frac{t^3_x}{2l^3};\ A_3 = -\frac{t^3_x}{l^3} - \frac{\nu t^3_x}{lm^2}$$
$$A_4 = -\frac{t^3_{i,j}}{l^3} - (2-\nu)\frac{t^3_{i,j}}{lm^2} + \frac{t^3_x}{2l^3};\ \ A_5 = \frac{t^3_{i,j}}{2l^3};\ \ A_6 = -(2-\nu)\frac{t^3_{i,j}}{2lm^2} + (1-\nu)\frac{t^3_y}{4lm^2};$$
$$A_7 = \frac{\nu t^3_x}{2lm^2};\ A_8 = (2-\nu)\frac{t^3_{i,j}}{2lm^2} + (1-\nu)\frac{t^3_y}{4lm^2};\ A_9 = -(2-\nu)\frac{t^3_{i,j}}{2lm^2} + (1-\nu)\frac{t^3_y}{4lm^2};$$
$$A_{10} = \frac{\nu t^3_x}{2lm^2};\ \ A_{11} = (2-\nu)\frac{t^3_{i,j}}{2lm^2} + (1-\nu)\frac{t^3_y}{4lm^2};$$
$$with\ \ \ t^3_x = (t^3_{i+1,j} - t^3_{i-1,j});\ t^3_y = (t^3_{i,j+1} - t^3_{i,j-1}) \tag{22}$$

The coefficients of Eq. 22 will contribute to flexural stiffness and the coefficients of the right hand side of Eq. 21 will contribute to the geometric stiffness.

2.5 Derivation of matrix coefficients: (free vibration problem)

Based on the classical thin plate theory, the governing differential equation in harmonic vibration is given by Leissa (1973)

$$\frac{\partial^2 M_x}{\partial x^2} - 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} - \rho t\ \omega^2\ w = 0 \tag{23}$$

The coefficients of the flexural stiffness matrix will be the same while the mass matrix can be derived as follows. The right side of the Eq. 17 is given as $\omega^2 \rho t_{i,j}\ w_{i,j}$ which are coefficients of the of $\mathbf{h}_0$ matrix.

3. Convergence study

The analysis for the buckling coefficient of a simply supported plate with uniform thickness is carried out using the approach explained in section 2. Table 1 and Fig. 4 show the monotonic convergence when the grid size is increased from 2 × 2 to 32 × 32 and the percentage error reduced for 32 × 32 grid size. Hence in this paper different grid sizes are taken from 20 to 32 depending on the problem in which case the error is less than 0.205 %.

4. Types of problems considered (buckling of plates)

4.1 Uniform rectangular plate:

The rectangular plate is compressed by forces uniformly distributed along the sides $x = 0$ and $x = a$ as shown in Fig. 1. Using finite difference method 841 equilibrium equations at the internal nodes and equations for 236 boundary conditions are written to solve the resulting matrix of 1077 × 1077 as an eigenvalue problem to find the buckling load. Tables 2,3,4,5,6 and 7 give the buckling loads for a) all edges are simply supported b) all edges are clamped c) loaded edges supported and longitudinal edges clamped d) loaded edges clamped and longitudinal edges are simply supported e) loaded edges are simply supported and longitudinal edges are clamped – simply supported f) shear buckling for different boundary conditions and the values are compared with earlier results (Anonymous Hand book of Structural Stability 1971, Gambir 2004).The values

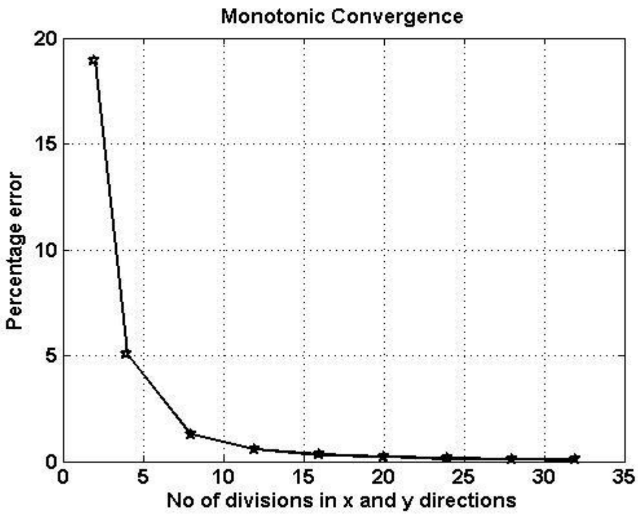


Fig. 4 Monotonic convergence of buckling coefficient

Table 1 Convergence of buckling coefficient

Grid size	Buckling coefficient(k)	% of error
2 × 2	3.2423	18.9400
4 × 4	3.7986	5.0350
8 × 8	3.9489	1.2775
12 × 12	3.9772	0.5700
16 × 16	3.9872	0.3200
20 × 20	3.9918	0.2050
24 × 24	3.9944	0.1400
28 × 28	3.9959	0.1025
32 × 32	3.9979	0.0803
Theoretical value 4.0000 Timoshenko and Gere (1961)		

Table 2 Local buckling coefficient *k* of plate with all edges simply supported Number in bracket ( ) shows the no of sine waves (β = 1, γ = 1, δ = 0, SSS)

a/b	This analysis	Gambir(2004)
0.4	8.4023(1)	8.4100
0.6	5.1331(1)	5.1378
0.8	4.1987(1)	4.2025
1.0	3.9963(1)	4.0000
1.2	4.1307(1)	4.1344
1.4	4.4661(1)	4.4702
1.6	4.1961(2)	4.2025
1.8	4.0397(2)	4.0446
2.0	3.9964(2)	4.0000
2.4	4.1327(2)	4.1344
2.8	4.0134(3)	4.2191
3.0	3.9964(3)	4.0000
4.0	3.9965(4)	4.0000

Table 3 Local buckling coefficient *k* of plate with all edges clamped (β = 1, γ = 0, δ = 0, CCCC)

a/b	This analysis	Bradford and Azhari (1997)	Anonymous Hand Book(1971)
0.50	19.2510(1)	19.20	19.20
0.75	11.6039(1)	11.70	11.40
1.00	10.0094(1)	10.31	10.08
1.25	9.1923(2)	9.28	9.94
1.50	8.2913(2)	8.40	8.32
1.75	8.0502(2)	8.28	8.08
2.00	7.7996(3)	7.89	7.88
3.00	7.3013(4)	-	-
4.00	7.1391(4)	-	-

Table 4 Local buckling coefficient *k* of a plate loaded edges simply supported longitudinal edges clamped (β = 1, γ = 0, δ = 0, SSCC)

a/b	This analysis	Bradford and Azhari (1997)	Anonymous Hand Book(1971)
0.4	9.4119(1)	9.46	9.45
0.5	7.6553(1)	7.71	7.69
0.6	7.0168(1)	7.07	7.06
0.7	6.9584(1)	7.02	7.00
0.8	7.2559(1)	7.32	7.30
0.9	7.8027(1)	7.87	7.83
1.0	7.6480(2)	7.71	7.69
1.2	7.0144(2)	7.07	7.06
2.0	6.9313(3)	6.99	6.99
3.0	6.9985(5)	7.40	7.40
4.0	6.9351(6)	-	-

obtained using finite difference method agrees with the published results. SSCS - means loaded edges simply supported and longitudinal edges clamped and simply supported. Fig. 5 shows the buckled shape for a fixed plate *a/b* = 4 subjected to shear on all edges.

Table 5 Local buckling coefficient k of a plate loaded edges clamped and longitudinal edges simply supported (CCSS)

a/b	This analysis	Bradford and Azhari (1997)	Anonymous Hand Book(1971)
0.4	27.0271(1)	27.12	27.12
0.6	13.3389(1)	13.38	13.38
0.8	8.7044(1)	8.73	8.73
1.0	6.7278(1)	6.75	6.74
1.2	5.8258(1)	5.85	5.84
1.4	5.4441(1)	5.50	5.45
1.6	5.3298(1)	5.48	5.34
1.8	5.1584(2)	5.17	5.18
2.0	4.8316(2)	4.85	4.85
3.0	4.3902(3)	4.41	4.42
4.0	4.2200(4)	-	-

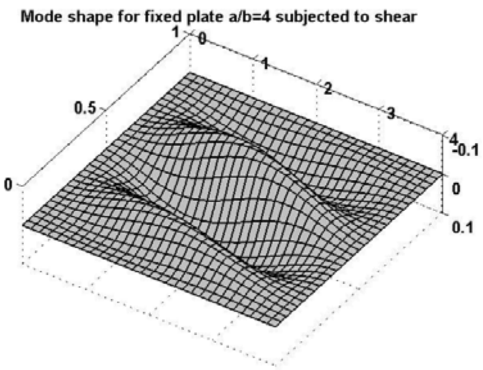


Fig. 5 Buckled shape of a fixed plate a/b = 4 subjected to shear

Table 6 Local buckling coefficient k of a plate loaded edges simply supported and longitudinal edges simply supported and clamped ($\beta = 1$, $\gamma = 0$, $\delta = 0$, SSSC)

a/b	This analysis	Bradford and Azhari (1997)
0.728	5.4426	5.47
0.790	5.3935	5.41
0.889	5.4692	5.50

4.2 Buckling of stepped thin plates

A one step rectangular Levy plate subjected to in-plane load as shown in Fig. 6 is considered. For the buckling analysis of thin plates, we consider three in-plane loading cases, namely (1) uni-axial in-plane compressive load in the x direction ($\beta = 1$; $\gamma = 0$); (2) uni-axial in-plane compressive load in the y direction ($\beta = 0$; $\gamma = 1$); and (3) equi-biaxial in-plane compressive loads ($\beta = 1$; $v = 1$).

Table 8 presents the buckling coefficient k generated by the present finite difference approach and compared with Xiang and Wei (2004) for the case of (a = 2, b = 1, a_1 = 1, v = 0.25) and varying the thickness ratio t_1/t_0 and the results are in close agreement with Xiang and Wei (2004).

Table 7 Local buckling coefficient k of a plate buckling of rectangular plates due to uniform shearing stress on all edges $\beta = 0$, $\gamma = 0$, $\delta = 1$(first two letters correspond to short edges and last two letters correspond to long edges)

a/b	CCCC		SSCC		SSSS	
	This analysis	Gambir (2004)	This analysis	Gambir (2004)	This analysis	Gambir (2004)
1.0	14.6517	14.71	12.5870	12.280	9.3763	9.338
1.5	11.4776	11.50	10.8273	11.120	7.1104	7.070
2.0	10.3039	10.34	10.0660	10.210	6.5886	6.590
2.5	9.9501	9.82	9.7507	9.810	6.0908	6.066
3.0	9.6797	9.62	9.6262	9.610	5.9047	5.890
4.0	9.5656	-	9.5489	-	5.7249	-

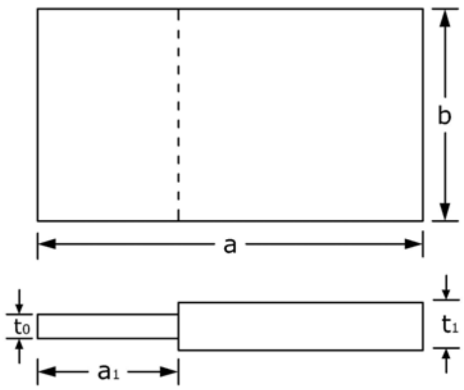


Fig. 6 One-step Levy rectangular plate

Table 8 Comparison of buckling coefficient for a one step SSSS rectangular plate subjected to uniaxial load (($\beta = 1$, $\gamma = 0$, $\delta = 0$) $a = 2.0$, $a_1 = 1.0$, $\nu = 0.25$, thin plate) as given in Fig. 6

t_1/t_0	This analysis	Xiang and Wei (2004)
0.4	0.3113	0.3082
0.6	1.0246	1.0244
0.8	2.3339	2.3439
1.0	3.9918	3.9995
1.2	4.5125	4.5315
1.4	4.6506	4.6651
1.6	4.7252	4.7280
1.8	4.7744	4.7639
2.0	4.8097	4.7865
2.2	4.8357	4.8014

Table 9 presents the buckling factors for the four symmetric Levy square plates (SS, CC, SF and CS) are the boundary conditions for the loaded edges and SS conditions for the longitudinal edges with one- step (see Fig. 6). The step length parameter a_1 varies from 0.3, 0.5 to 0.7. The step

Table 9 Comparison of buckling coefficients for a thin square plate with one step subjected to uniaxial and bi-axial load - δ = 0 (longer edges are simply supported) (values in brackets from Xiang and Wei (2004)

(β, γ)	t_1/t_0	a_1	SS	CC	SF	CS
(1,0)	1.2	0.3	5.7462 (5.7389)	10.1042 (10.1929)	4.0030 (3.9616)	7.7157 (7.7310)
		0.5	4.9635 (4.9616)	8.3380 (8.3862)	3.8047 (3.7668)	6.8194 (6.8186)
		0.7	4.5090 (4.5093)	7.5152 (7.5966)	3.4948 (3.4598)	5.8168 (5.8171)
(1,0)	2.0	0.3	10.4230 (10.3908)	19.7319 (19.6097)	10.3688 (10.3339)	19.2377 (18.7209)
		0.5	7.7450 (7.7348)	13.2327 (13.7249)	7.7069 (7.7133)	12.2088 (12.2933)
		0.7	5.9160 (5.8800)	9.7683 (9.8258)	5.9079 (5.8587)	9.0300 (9.0179)
(0,1)	1.2	0.3	5.9812 (5.9772)	11.5977 (11.7662)	2.2593 (2.2603)	8.5837 (8.6478)
		0.5	5.1995 (5.1961)	9.7893 (9.9132)	2.0874 (2.0894)	7.6112 (7.6544)
		0.7	4.6025 (4.6011)	8.4425 (8.5567)	1.8626 (1.8654)	6.6480 (6.6832)
(0,1)	2.0	0.3	16.6111 (16.4087)	30.1867 (33.8744)	8.5741 (8.5261)	25.4753 (25.1480)
		0.5	10.8251 (10.8319)	18.2350 (18.6525)	6.2770 (6.3249)	15.8639 (15.9189)
		0.7	7.5661 (7.5468)	11.2123 (11.4514)	4.3250 (4.3931)	10.9108 (10.9912)
(1,1)	1.2	0.3	2.9549 (2.9524)	5.6183 (5.7385)	1.7343 (1.7328)	4.0987 (4.1147)
		0.5	2.5599 (2.5584)	4.7838 (4.8650)	1.6245 (1.6246)	3.6197 (3.6270)
		0.7	2.2852 (2.2848)	4.2988 (4.3902)	1.4638 (1.4652)	3.1300 (3.1367)
(1,1)	2.0	0.3	6.7211 (6.6697)	12.6067 (13.0149)	5.8980 (5.8610)	11.5045 (11.2214)
		0.5	4.6999 (4.7072)	8.0630 (8.7214)	4.1439 (4.1886)	7.0590 (7.0850)
		0.7	3.4426 (3.4286)	6.0514 (6.4258)	2.9694 (3.0002)	5.0707 (5.0902)

thickness ratio of the plates are set to be t_1/t_0 = 1.2 and 2.0 for the thin plates. It is observed that the buckling factors decrease as the step length parameter a_1 increases for all cases. The rate of decrease is more pronounced for plates subjected to the uni-axial in-plane load in the y-direction (β = 0; γ = 1). The buckling coefficients increase as the step thickness ratio changes from 1.2 to 2.0. Even in this case, it is observed that the rate of increase is more significant for plates subjected to the uni-axial in-plane load in the y-direction (β = 0; γ = 1).

4.3 Buckling of one, two and three even step plates

Table 10 compares the buckling coefficients for two, three and four even step plates in which the step thickness variation for plates is moderate. i.e., $t_i/t_0 = 1 + i \times 0.1$ where i(= 1, 2 and 3) referring to the plate of i^{th} step. It is observed that the increase in the number of steps has insignificant effect on the buckling factors for SS, CC and CS plates. It is due to the fact that the

Table10 Comparison of buckling coefficients for thin rectangular plates having one, two and three even steps (Longer edges are simply supported $\delta = 0$) ($t_1/t_0 = 1.1$, $t_2/t_0 = 1.2$, $t_3/t_0 = 1.3$)) are the thicknesses ratio of first, second and third steps respectively) (values in brackets from Xiang and Wei (2004) for thick Mindlin plates)

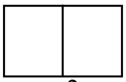
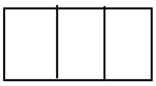
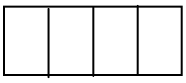
Cases	(β, γ)	SS	CC	SF	CS
 $a = 2$	(1,0)	4.3694 (4.1143)	5.3284 (4.9620)	3.0597 (2.6686)	5.0288 (4.7129)
	(0,1)	1.7886 (1.7246)	2.1634 (2.0864)	1.3503 (1.3125)	1.9872 (1.914)
	(1,1)	1.4173 (1.3669)	1.6670 (1.6056)	1.2485 (1.2069)	1.5632 (1.5075)
 $a = 3$	(1,0)	4.3743 (4.1435)	5.1749 (4.8472)	3.9700 (3.4063)	5.1656 (4.8382)
	(0,1)	1.5686 (1.5232)	1.6942 (1.6425)	1.4892 (1.4475)	1.6682 (1.6150)
	(1,1)	1.3645 (1.3256)	1.4813 (1.4347)	1.3570 (1.3190)	1.4709 (1.4244)
 $a = 4$	(1,0)	4.3743 (4.1437)	5.1655 (4.8403)	4.3742 (4.1436)	5.1655 (4.8402)
	(0,1)	1.5415 (1.4976)	1.6277 (1.5778)	1.5313 (1.4919)	1.6247 (1.5744)
	(1,1)	1.3623 (1.3236)	1.4654 (1.4198)	1.3616 (1.3235)	1.4649 (1.4193)

Table 11 Comparison of buckling coefficient value for uniformly compressed SSSS plate with bilinear thickness variation $e_x = 0.5$ and e_y varied ($\beta = 1$, $\gamma = 0$, $\delta = 0$) (values in brackets are from Eisenberger and Alexandrov (2003))

a/b	e_y values				
	0.125	0.25	0.5	0.75	1.0
0.5	13.8962 (13.9322)(1,1)	16.3048 (16.3557)(1,1)	21.5645 (21.6681)(1,1)	27.4250 (27.0622)(1,1)	33.84444 (34.1667)(1,1)
1.0	8.4910 (8.5212)(1,1)	10.0173 (10.0634)(1,1)	13.4742 (13.5826)(1,1)	17.4858 (17.7076)(1,1)	22.0681 (22.4688)(1,1)
1.5	7.7779 (7.8130)(2,1)	9.1700 (9.2266)(2,1)	12.3097 (12.4502)(2,1)	15.9352 (16.2238)(2,1)	20.0594 (20.5704)(2,1)
2.0	7.0958 (7.1297)(2,1)	8.3688 (8.4195)(2,1)	11.2481 (11.3611)(2,1)	14.5839 (14.8056)(2,1)	18.3903 (18.7760)(2,1)
3.0	6.4588 (6.4838)(3,1)	7.6198 (7.6595)(3,1)	10.2501 (10.3356)(3,1)	13.3048 (13.4752)(3,1)	16.7988 (17.0947)(3,1)
4.0	6.1177 (6.1462)(3,1)	7.2186 (7.2590)(3,1)	9.7153 (9.7989)(3,1)	12.6190 (12.7763)(4,1)	15.9445 (16.2122)(4,1)

buckling behaviour of the plates is dominated by the first two steps of the plate. For SF plates, however, the buckling factors increase significantly as the number of steps increase, especially when the plates are subjected to uni-axial in-plane load in the *x*-direction ($\beta = 1$; $\gamma = 0$).

4.4 Buckling of a plate with bi-linear thickness variation loaded in x – direction

The values of the dimensionless buckling load (buckling coefficients) for SSSS plate with bilinear thickness variation in the x and y directions as $t = t_0(1 + e_x x/a)(1 + e_y y/b)$ are given in Table 11 for values of $e_x = 0.5$ and $e_y = 0.125$, 0.25, 0.25, 0.5, 0.75, 1.0. The values computed using finite difference approach agrees with the values obtained by Eisenberger and Alexandrov (2003) who used Kantorovich method. The buckled shape for $a/b = 3$ and $e_y = 0.75$ for a simply supported plate uniformly compressed in x direction is shown in Fig. 7.

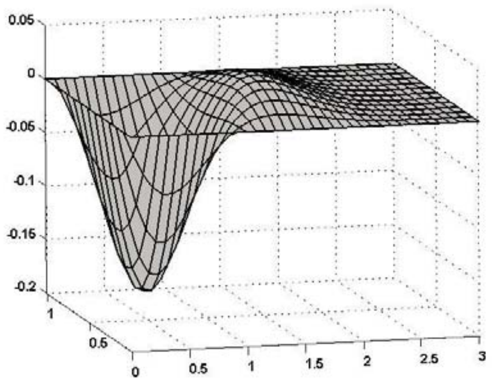


Fig. 7 Buckled shape (a/b = 3 thickness vary in both direction) SSS condition

Table 12 Buckling coefficient *k* for horizontally and vertically stiffened in the centre of the plate as shown in Fig. 8

a/b	t_1/t_0			
	2	1.75	1.5	1.25
0.25	42.9882	37.9774	32.3491	25.6451
0.50	17.7850	14.2768	11.2548	8.6047
0.75	11.6463	9.3406	7.4086	5.7657
0.80	11.1662	8.9654	7.1236	5.5593
0.90	10.5927	8.5221	6.7908	5.5211
1.00	10.3904	8.3735	6.6858	5.2509
1.10	10.4423	8.4288	6.7404	5.2999
1.20	10.6656	8.6262	6.9101	5.4379
1.30	10.9912	8.9164	7.1616	5.6442
1.40	11.3478	9.2494	7.4633	5.9021
1.50	11.3688	9.1433	7.2701	5.6908
1.75	10.1344	8.2653	6.6763	5.3027
2.00	9.3621	7.7664	6.3919	5.1663

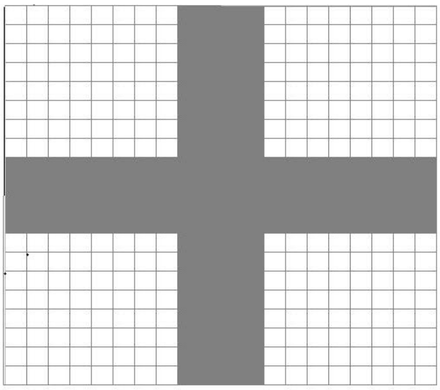


Fig. 8 Horizontally and vertically stiffened at centre of the plate

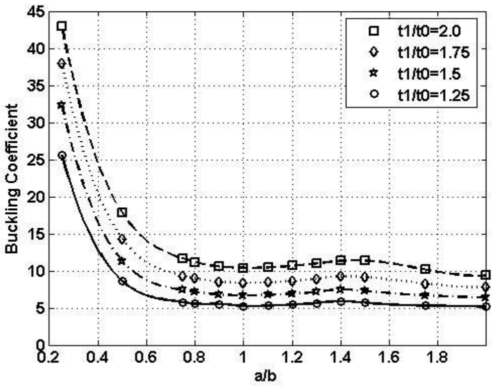


Fig. 9 Buckling coefficient vs a/b ration for horizontally and vertically stiffened at the centre of the plate

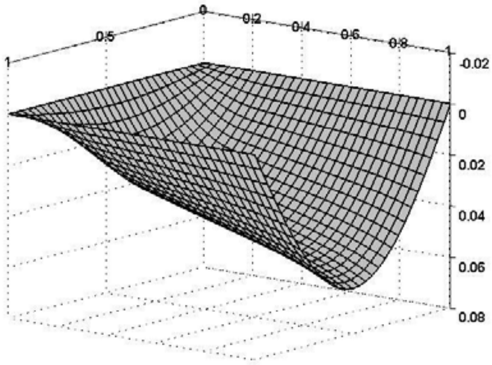


Fig. 10 Mode shape for a square plate ($x = 0 - S$; $x = a - F$, $y = 0 - S$, $y = b - S$)

4.5 Buckling of stiffened horizontally and vertically at the centre of the plate

Buckling is carried out for SSSS plate with 361 internal nodal points subject to uni-axial in-plane compressive load in the x direction ($\beta = 1$; $\gamma = 0$) ,the width of the horizontal strip is equal to four times the width of a mesh ($4l$) with thickness t_1 covering the centre of the plate and the width of the vertical strip equal to four times the length of the mesh ($4m$) with thickness t_1 as shown in Fig. 8 and the critical values are calculated and tabulated in Table 12 and plotted in Fig. 9.

Vibration of plates

4.6 Vibration of a rectangular plate of uniform thickness with different boundary conditions

Table 13 shows the comparison of frequency parameter values of a rectangular plate with different boundary conditions compared with Leissa (1973) who employed the Ritz method with

36 terms containing the product of beam functions. The results obtained by using finite difference method discussed in this paper more or less agree with the values obtained by Leissa (1973).The mode shape for shorter edges (simply supported and free) with longitudinal edges simply supported for $\alpha = 0.3$ and $t_1/t_0 = 2$ is shown in Fig. 10

4.7 Free vibration of stepped thin plates

The free vibration of thin square and rectangular plate of multi steps is studied. Table 14 presents frequency parameters Λ obtained using present finite difference method and from Xiang *et al.* (2004), Yuan *et al.* (1992) for one step (SSSS) square plate. The vibration solutions by finite difference solution are in close agreement with the results of Xiang and Wei (2004) and Yuan and Dickinson (1992). Results are given for one step with step length parameter a_1 varying as 0.25 and 0.75 and $t_1/t_0 = 0.5$ and 0.8. Table 15 presents first four frequency parameters for the Levy square plate with one step. The step length parameter a_1 varies for 0.3, 0.5 and 0.7 and the step thickness ratios t_1/t_0 is set to be 1.2 and 2.0. In all the cases we obtain that the frequency parameters decrease with increase in step length parameter a_1. It is due to the overall stiffness of the plate as the step length parameter a_1 increases. On the other hand, the frequency parameter Λ increases as the step thickness t_1/t_0 changes from 1.2 to 2.0. The values given in Table 15 are the same order of magnitude by values given by Xiang and Wei (2004) who analyzed thick plates using Levy type solution method with domain decomposition. Table 16 presents first four frequency parameters for thin rectangular plates with one, two and three even steps. The thickness ratio of the steps are 1.1, 1.2, 1.3 the plate aspect ratios a/b for the two, three and four even step plates are taken as 2, 3 and 4 respectively. The frequency parameters change significantly as the number of steps increase.

Table 13 Comparison of fundamental six frequency parameters with Leissa (1973) for a rectangular plate of uniform thickness

Mode	Analysis	**a/b**			**SSSS**	
		0.4	2/3	1	1.5	2.5
1	FD	7.2397	3.2431	1.9971	1.4444	1.1600
	Leissa(1973)	7.2499	3.2468	1.9999	1.4424	1.1583
2	FD	10.2183	6.2217	4.9758	2.7758	1.6399
	Leissa(1973)	10.2651	6.2425	4.9999	2.7662	1.6349
3	FD	15.1260	9.9383	7.9544	4.4413	2.4398
	Leissa(1973)	15.2510	9.9880	7.9999	4.4210	2.4202
4	FD	21.8789	11.1295	9.8835	4.9965	3.5597
	Leissa(1973)	22.2515	11.2365	9.9999	4.9474	3.5006
5	FD	25.8561	12.9170	9.8835	5.7737	4.1596
	Leissa(1973)	26.0017	12.9844	9.9331	5.7449	4.1371
6	FD	28.8347	17.8247	12.8621	7.9944	4.6396
	Leissa(1973)	29.0100	17.9715	13.0000	7.9261	4.6136
Mode	Analysis	**a/b**			**SCSC**	
		0.4	2/3	1	1.5	2.5
1	FD	7.6563	3.9377	2.9333	2.5202	2.3420
	Leissa(1973)	7.6813	3.9566	2.9149	2.5357	2.3583
2	FD	11.5137	7.9508	5.5455	3.5304	3.6510
	Leissa(1973)	11.6318	8.0479	5.5067	3.5543	2.6704
3	FD	17.3609	10.2774	7.0228	5.4803	4.1634
	Leissa(1973)	17.7107	10.3443	6.9236	5.5427	3.2784

Table 13 Continued

Mode	Analysis	0.4	2/3	1	1.5	2.5
4	FD	24.9707	13.8112	9.7517	6.4880	5.3969
	Leissa 1973)	25.7941	14.1291	9.4583	6.5820	4.2325
5	FD	27.0372	14.0383	10.3545	7.5507	6.2716
	Leissa(1973)	26.2048	14.1887	10.2238	7.6550	5.5462
6	FD	29.5029	19.8437	13.0774	8.3408	6.7383
	Leissa (973)	29.7708	20.2208	12.7524	8.5101	6.3685

Mode	Analysis	a/b			SCSS	
		0.4	2/3	1	1.5	2.5
1	FD	7.4410	3.5412	2.3679	1.9081	1.6782
	Leissa(1973)	7.4238	3.5511	2.3954	1.9138	1.6846
2	FD	10.8842	7.0262	5.2058	3.0908	2.0709
	Leissa 1973)	10.7186	7.0845	5.2346	3.1051	2.0804
3	FD	16.4133	10.0997	5.8878	5.1782	2.7707
	Leissa(1973)	16.1911	10.1607	5.9409	5.2320	2.7943
4	FD	23.9588	12.4178	8.6496	5.3957	3.7836
	Leissa(1973)	23.3800	12.6295	8.7254	5.4441	3.8458
5	FD	26.0964	13.4463	10.0365	6.5850	5.0911
	Leissa(1973)	25.9412	13.5750	10.1573	6.3436	5.1989
6	FD	29.3611	18.7949	11.2661	8.1229	5.1484
	Leissa(1973)	29.1511	19.0622	11.4700	8.2845	5.2352

Mode	Analysis	a/b			SSFS	
		0.4	2/3	1	1.5	2.5
1	FD	1.0242	1.0795	1.1821	1.3873	1.9025
	Leissa(1973)	1.0260	1.0900	1.1836	1.3889	1.9045
2	FD	1.3194	1.8483	2.7982	4.3908	5.0960
	Leissa(1973)	1.3227	1.8537	2.8120	4.4139	5.1198
3	FD	1.8964	3.3386	4.1395	4.8309	10.0378
	Leissa(1973)	1.9084	3.4135	4.1740	4.8479	10.1535
4	FD	2.7535	4.0416	5.9350	8.2103	11.1252
	Leissa(1973)	2.7916	4.0652	5.9840	8.2538	11.1659
5	FD	3.8881	4.8753	6.1850	9.2742	14.8834
	Leissa(1973)	3.9849	4.9038	6.2670	9.3848	14.9551
6	FD	3.9878	5.7376	8.9785	12.4982	16.7682
	Leissa(1973)	4.3254	5.8342	9.1480	12.6183	17.1301

Mode	Analysis	a/b			CCCC	
		0.4	2/3	1	1.5	2.5
1	FD	14.8587	6.1070	3.6160	2.7142	2.3774
	Leissa(1973)	14.9763	6.1501	3.646	2.7347	2.3958
2	FD	17.4250	9.3795	7.3244	4.1687	2.7880
	Leissa(1973)	17.6155	9.4987	7.4367	4.2237	2.8181
3	FD	22.0348	14.8110	10.7708	6.5827	3.5256
	Leissa(1973)	22.4480	15.0608	10.9678	6.6969	3.5912
4	FD	28.6960	17.8772	12.9901	6.5953	4.5914
	Leissa(1973)	29.5764	18.1800	13.3351	6.7383	4.7315
5	FD	37.2239	22.1453	13.0566	7.9454	5.9558
	Leissa(1973)	38.9821	22.9635	13.3959	8.0847	6.2361
6	FD	39.3218	22.9697	16.2942	9.8423	6.2915
	Leissa(1973)	39.9612	-	-	10.2114	6.3927

Table 14 Comparison of frequency parameters Λ for one-step SSSS square plate

a_1	t_1/t_0	Source	1	2	3	4
0.25	0.5	Present	1.2882	2.8753	2.8798	4.8542
		Xiang and Wei(2004)	1.2924	2.8708	2.8986	4.9192
		Yuan and Dickinson(1992)	1.2933	2.8718	2.8998	4.9225
	0.8	Present	1.7007	4.1572	4.1733	6.7158
		Xiang and Wei(2004)	1.7037	4.1863	4.1962	6.7643
		Yuan and Dickinson(1992)	1.7039	4.1872	4.1969	6.7661
0.75	0.5	Present	1.6463	4.0524	4.3001	6.7990
		Xiang and Wei(2004)	1.6289	4.0472	4.3404	6.8642
		Yuan and Dickinson(1992)	1.6290	4.0489	4.3414	6.8692
	0.8	Present	1.8869	4.6688	4.7487	7.5059
		Xiang and Wei(2004)	1.8892	4.6884	4.7823	7.5577
		Yuan and Dickinson(1992)	1.8894	4.6898	4.7833	7.5602

Table 15 Frequency parameter   for one step square plate (longitudinal edges simply supported)

Shorter edges	a_1	$t_1/t_0 = 1.2$ (modes)				$t_1/t_0 = 2$ (modes)			
		1	2	3	4	1	2	3	4
SS	0.3	2.2650	5.6119	9.0277	11.1747	3.1688	8.0397	13.3579	16.0208
	0.5	2.1849	5.4351	8.7123	10.6742	2.8915	7.0764	7.1739	11.2606
	0.7	2.1167	5.2001	5.2371	8.3769	2.6574	5.8818	9.9530	10.9108
CC	0.3	3.1969	6.2384	7.5730	10.5282	4.0715	8.9757	9.7633	14.7852
	0.5	3.1085	5.9675	7.3802	10.1611	3.7118	7.7805	9.1033	12.6790
	0.7	3.0496	5.7115	7.0913	9.7571	3.5792	6.5442	7.4741	11.0630
CS	0.3	2.6913	5.9918	6.0281	9.8224	3.7639	8.8821	9.1861	14.4076
	0.5	2.6132	5.7414	6.4454	9.4980	3.3755	7.7065	8.6613	12.4415
	0.7	2.5320	5.4816	6.2068	9.1107	3.1620	6.4738	6.9586	10.7626
SF	0.3	1.3756	3.1847	4.8881	6,7678	2.1256	4.4794	7.6870	9.9965
	0.5	1.3381	3.0752	4.7605	6.5452	1.9517	4.2179	6.7799	8.6474
	0.7	1.2914	2.9625	4.9678	6.3319	1.7664	3.6765	5.7610	7.5183

Table 16 Frequency parameter Λ for thin plate having one, two and three even step (longitudinal edges simply supported) b =1 having thickness ratios (t_1/t_0 = 1.1, t_1/t_0 = 1.2, t_1/t_0 = 1.3) of first, second and third steps

Λ	steps	mode	Shorter edges			
			SS	CC	CS	SF
2	1	1	1.3080	1.4393	1.3740	1.1212
		2	2.0883	2.4530	2.2715	1.5696
		3	3.3616	3.9446	3.6627	2.5074
		4	4.3952	4.4596	4.4445	3.9242
3	2	1	1.2098	1.2520	1.2395	1.1584
		2	1.5838	1.7109	1.6546	1.3517
		3	2.1839	2.4146	2.3113	1.8001
		4	3.0154	3.3444	3.1973	2.4711
4	3	1	1.1915	1.2157	1.2135	1.2173
		2	1.4344	1.4909	1.4721	1.4211
		3	1.7877	1.8993	1.8540	1.8819
		4	2.2778	2.4463	2.3780	2.5846

4.8 Simply supported square and rectangular plate with variable thickness

Numerical solutions for the lowest six frequency parameters defined by $\Lambda = \sqrt[4]{\frac{\rho_0 t_0 \omega^2 b^4}{D_0\left(1-\nu^2\right)}}$ of a simply supported square and rectangular plate of aspect ratio; *a/b* = 1, *a/b* =2 with thickness variation in *y* direction given by $t(x, t) = t_0(1 + b_y{\cdot}y/b)$ as shown in Table 17 for two cases $b_y = 0.1$ and 0.8 and these values agree well with the results of Sakiyama and Huang (1998).

4.9 Vibration of Fixed square plate with variable thickness

Numerical solutions for the lowest six natural frequency parameters Λ for a fixed square plate with a sinusoidal thickness variation in the *x*, *y* direction given by $t(x, y) = t_0(1- b_x \sin(\pi x/a))(1-b_y$

Table 17 Frequency parameter $\Lambda = \sqrt[4]{\rho_0 t_0 \omega^2 b^4 / D_0(1-\nu^2)}$ for simply supported square and rectangular plate with variable thickness

a/b	*M*	$b_y = 0.1$		$b_y = 0.8$	
		This analysis	Sakiyama and Huang (1998)	This analysis	Sakiyama and Huang (1998)
1	1	4.6417	4.660	5.227	5.354
	2	7.1786	7.342	8.1980	8.404
	3		7.363	8.2303	8.437
	4	9.0777	9.311	10.4194	10.980
	5		10.389	11.4417	11.742
	6	10.1118	10.393	11.5828	11.886
2	1	3.5957	3.684	4.1196	4.220
	2	4.5434	4.659	5.2215	5.352
	3	5.7753	5.930	6.6218	6.797
	4	6.6170	6.789	7.5816	7.775
	5	7.1296	7.322	8.1432	8.359
	6	7.1763	7.362	8.2267	8.436

Table 18 Frequency parameter $\Lambda = \sqrt[4]{\rho_0 t_0 \omega^2 b^4 / D_0(1-\nu^2)}$ for fixed square plate with thickness varying in both directions

	$b_x = b_y = 0.3$		$b_x = b_y = 0.5$	
	This analysis	Sakiyama and Huanag (1998)	This analysis	Sakiyama and Huanag (1998)
1	5.130	5.038	4.339	4.262
2	7.125	7.003	6.010	5.758
3	8.690	8.570	7.380	7.093
4	9.350	9.185	7.751	7.415
5	9.360	9.220	7.760	8.631
6	10.630	10.55	9.160	9.108

sin($\pi y/b$)) are shown in Table 18 for the two cases of $b_x = b_y = 0.5$ and 0.5 and the results do not agree well with Sakiyama and Huang (1998) .

4.10 Vibration of a plate stiffened horizontally and vertically at the centres of the plate

Free vibration analysis is carried out for SSSS plate with 361 internal nodal points subject to uni-axial in-plane compressive load in the x direction ($\beta = 1$; $\gamma = 0$) ,the width of the horizontal strip is equal to four times the width of a mesh ($4l$) with thickness t_1 covering the centre of the plate and the width of the vertical strip equal to four times the length of the mesh ($4m$) with thickness t_1 as shown in Fig. 8 The frequency parameter Λ is calculated and tabulated in Table 19 and plotted in a graph shown in Fig. 11. The mode shape for the case $t_1/t_0 = 1.5$ and $a/b = 0.5$ is given in Fig. 12.

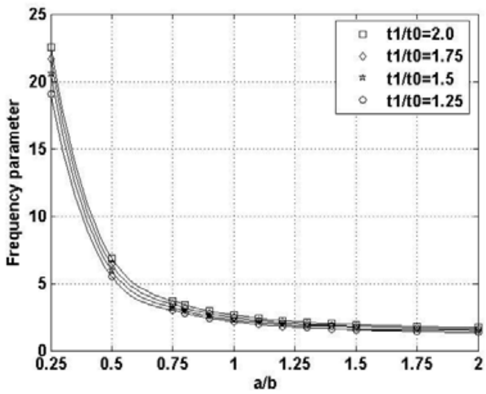


Fig. 11 Frequency parameter vs a/b ration for horizontally and vertically stiffened at the centre of the plate

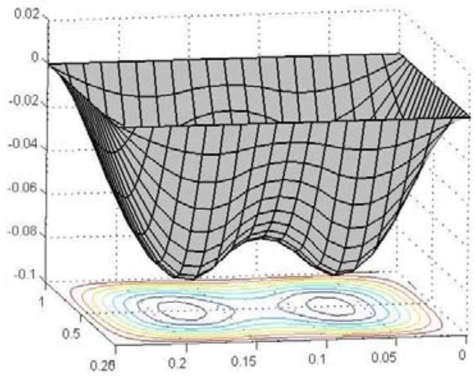


Fig. 12 Mode shape ($t_1/t_0 = 1.5$; $a/b = 0.5$) for SSSS plate horizontally and vertically stiffened at the centre

Table 19 Frequency parameter Λ for horizontally and vertically stiffened at the centre of the plate as shown in Fig. 8

a/b	t_1/t_0			
	2	1.75	1.5	1.25
0.25	22.5675	21.7081	20.5780	19.0322
0.50	6.8885	6.4181	5.9578	5.4901
0.75	3.6986	3.4537	3.2224	2.9984
0.80	3.3966	3.1734	2.9633	2.7608
0.90	2.9450	2.7535	2.5739	2.4018
1.00	2.6313	2.4607	2.3010	2.1481
1.10	2.4064	2.2500	2.1034	1.9629
1.20	2.2406	2.0940	1.9562	1.8273
1.30	2.1154	1.9758	1.8441	1.7168
1.40	2.0187	1.8842	1.7568	1.6329
1.50	1.9424	1.8118	1.6875	1.5660
1.75	1.8086	1.6853	1.5663	1.4478
2.00	1.7221	1.6045	1.4895	1.3725

5. Conclusions

This paper presents finite difference approach for studying the buckling and vibration behaviour of thin rectangular plates with variation in thickness and with multiple steps. Finite difference method is an extrapolation method whereas finite element is an interpolation method. Finite difference solves differential equations and finite element solves integral equations. In finite difference method boundary conditions are applied in discretized form whereas in finite element method we can use them as they are. Both methods can handle complex geometry. In finite element method, displacement functions are assumed a priori whereas such an assumption is unnecessary in finite difference method. For a given discretization scheme, finite difference underestimates buckling coefficient and frequency parameter whereas they are overestimated in finite element displacement approach. The results obtained in all the above numerical cases are tabulated in Tables 1-19 for ready use by the designer and they are compared with Bradford and Azhari (1997), Eisenberger and Alexandrov (2003), Xiang and Wei (2004), Leissa (1973), Gambir (2004), Yuan and Dickinson (1992), Sakiyama and Huang (1998). They are in close agreement except Sakiyama and Huang (1998). In this paper it is also shown how finite difference technique is made automatic like finite element procedure and the incorporation of boundary condition is also an easy task. The authors believe that the presented solutions for buckling and vibration of the stepped plates are very valuable as they may serve as bench mark results for future research in this area.

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Nomenclature

a - length of the plate
a_1 - step length
b - width of the plate
b_x and b_y - parameters for the definition of thickness variation
a/b - panel aspect ratio
D - flexural rigidity of the plate
E - Young's modulus
$\mathbf{h}_0$ - geometric /mass stiffness matrix
$\mathbf{K}$ - flexural stiffness matrix
$\overline{\mathbf{K}}_G$ - geometric stiffness matrix
$k = P\dfrac{b^2}{\pi^2 D_0}$ - buckling coefficient
$D_0 = \dfrac{Et_0^3}{12(1-\nu^2)}$
l - mesh length
m - mesh breadth
M - number of divisions in x direction
$\mathbf{M}$ - mass matrix
$\mathbf{M}_0$ - flexural matrix
M_x, M_y, M_{xy} - moments
N - number of divisions in y direction
nj - number of joints
N_x, N_y, N_{xy} - in-plane forces
t - thickness of the plate
t_i -step thickness
w - lateral deflection
$\underline{\mathbf{w}}$ - displacement vector
(x, y) - cartesian coordinates
α - ratio between the breadth and length of a mesh
β, γ, δ –tracers
$\Lambda = \dfrac{\omega b^2}{\pi^2}\sqrt{\dfrac{\rho_0 t_0}{D_0}}$ - frequency parameter
ν - Poisson's ratio
ω - natural frequency
θ_x, θ_y -slopes