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Properties of integral operators in complex variable boundary integral equation in plane elasticity

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Abstract. This paper investigates properties of integral operators in complex variable boundary integral equation in plane elasticity, which is derived from the Somigliana identity in the complex variable form. The generalized Sokhotski-Plemelj's formulae are used to obtain the BIE in complex variable. The properties of some integral operators in the interior problem are studied in detail. The Neumann and Dirichlet problems are analyzed. The prior condition for solution is studied. The solvability of the formulated problems is addressed. Similar analysis is carried out for the exterior problem. It is found that the properties of some integral operators in the exterior boundary value problem (BVP) are quite different from their counterparts in the interior BVP.

Keywords: properties of integral operators; complex variable boundary integral equation; Somigliana identity; interior boundary value problem; exterior boundary value problem

1. Introduction

The boundary integral equation (BIE) was initiated by some pioneer researchers (Rizzo 1967, Cruse 1969, Jaswon and Symm 1977, Brebbia *et al*. 1984). If one compares BIE with the finite element method (FEM), the number of unknowns in BIE can be reduced significantly. It has now recognized that the numerical procedures based on BIE become the third important technique in the numerical analysis of elasticity problem (Cheng and Cheng 2005).

In the BIE, there are two kinds of formulation. One is the direct BIE method, and other is the indirect BIE method (Chen and Hong, Cheng and Cheng 2005). In the direct BIE method, the unknown functions are the displacements and tractions. However, in the indirect BIE method the unknown function is an intermediate function. Since both methods reflect the nature of the governing equation, for example, the Laplace equation, both methods can be used to solve the boundary value problem (BVP). For the boundary value problem of Laplace equation, the direct and indirect BIE methods were summarized (Cheng and Cheng 2005).

The Somigliana identity is generally used in the direct method of BIE (Brebbia *et al*. 1984, Cheng and Cheng 2005). The Somigliana identity is actually a result of using the Betti's reciprocal theorem between the fundamental field and the physical field. Previously, the Somigliana

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identity in plane elasticity was formulated on the real variable (Brebbia *et al*. 1984, Cheng and Cheng 2005). The real variable BIE in plane elasticity has some unsuitable points. When the source point in the region approaches the boundary point, one needs to make some explanation for the process in the resulting BIE. In fact, this process is a result of limit value for the Cauchy type integral, or the Sokhotski-Plemelj formula in the complex variable. Secondly, the character of the involved integral operator is not easy to see. For example, it is not easy to extract the singular portion in the integral kernel.

Cruse and Suwito presented an analytical investigation of the properties of the integral terms in the Somigliana stress identity in elasticity (Cruse and Suwito 1993). A weakly-singular form of the Somigliana stress identity was found. A kind of the Somigliana identity using the complex variable was suggested (Mogilevskaya and Linkov 1998, Mogilevskaya 2000). In the formulation, the force component in the physical field was expressed by the resultant force function rather than the traction itself. Thus, this formulation is not easy to compare with its counterpart in the real variable. A kind of complex variable BIE was suggested (Linkov 2002). In the formulation, the generalized Sokhotski-Plemelj's formulae have not been used. A unified discussion of real and complex boundary integral equations (BIEs) for two-dimensional potential problems was presented (Kolte *et al*. 1996).

On the other hand, formulations of BIE may rely on the real or complex variables. In many cases, it is straightforward to formulate the BIE using the complex variable. The dual boundary element method in the real domain was extended to the complex variable dual boundary element domain (Chen and Chen 2000). A review was given to the complex variable based numerical solutions for Dirichlet potential problem in two and higher dimensions (Whitley and Hromadka 2006).

Recently, the regularity condition at infinity in the exterior boundary value problem of plane elasticity was examined (Chen and Lin 2008, Chen *et al*. 2009, 2010). It is proved that the usual kernel which is acted upon the tractions can not be used to the exterior boundary value problem when the applied tractions on contour are not in equilibrium.

Recently, a null-field approach for the multi-inclusion problem under antiplane shears was suggested (Chen and Wu 2007). In addition, the torsional rigidity of a circular bar with multiple circular inclusions was investigated, which is based on the null-field integral approach (Chen and Lee 2009). Free terms in hypersingular boundary integral equations were studied and evaluated (Davey and Farooq 2011). An exact solution was proposed for the hypersingular boundary integral equation of two-dimensional elastostaticcs (Zhang and Zhang 2008).

A dual integral formulation for the interior problem of the Laplace equation with a smooth boundary is extended to the exterior problem (Chen *et al*. 1995). Two regularized versions were proposed. The spectral properties for the influence matrices in the dual boundary integral equation (dual BEM) are investigated for the Laplace and Helmholtz equations of a circular domain (Chen and Chiu 2002a). Many results were obtained. For example, Eqs. (60) to (63) in the paper represent some relations between four influence matrices in the dual BEM formulation..

This paper investigates properties of integral operators in complex variable boundary integral equation in plane elasticity, which is derived on the Somigliana identity in the complex variable form. In the derivation, the displacement component at a point of an interior finite region is evaluated by using the Somigliana identity, or the Betti's reciprocal theorem in elasticity. Letting a moving point approach the boundary and using the generalized Sokhotski-Plemelj's formulae, the complex variable boundary integral equation in plane elasticity is obtained. It is found that the complex variable BIE is equivalent to its counterpart in real variable case. However, in the complex variable BIE, the singular portion and the regular portion in the integral kernel are easy to distinguish. The properties of some integral operators are studied in detail. The Neumann and Dirichlet problems are analyzed. The solvability of the formulated problems is addressed. Similar analysis is carried out for BIE in the exterior region. However, the properties of some integral operators in the exterior boundary value problem (BVP) are quite different from their counterparts in the interior BVP. Numerical examination for some properties are also carried out.

2. Formulation of complex variable BIE for the interior boundary value problem

2.1 Some preliminary knowledge in complex variable method of plane elasticity

The complex variable function method plays an important role in plane elasticity. Fundamental of this method is introduced. In the method, the stresses (σ_x, σ_y, σ_{xy}), the resultant forces (X, Y) and the displacements (u, v) are expressed in terms of complex potentials $\phi(z)$ and $\psi(z)$ such that (Muskhelishvili 1953)

$$\sigma_x + \sigma_y = 4\, Re\,\Phi(z)$$

$$\sigma_y - \sigma_x + 2i\sigma_{xy} = 2[\bar{z}\Phi'(\mathrm{z}) + \Psi(\mathrm{z})] \tag{1}$$

$$f = -Y + iX = \phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)} \tag{2}$$

$$2G(u+iv) = \kappa\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)} \tag{3}$$

where $\Phi(z) = \phi'(z)$, $\Psi(z) = \psi'(z)$, a bar over a function denotes the conjugated value for the function, G is the shear modulus of elasticity, $\kappa = (3-\nu)/(1+\nu)$ in the plane stress problem, $\kappa = 3 - 4\nu$ in the plane strain problem, and ν is the Poisson's ratio. Sometimes, the displacements u and v are denoted by u_1 and u_2, the stresses σ_x, σ_y and σ_{xy} by σ_1, σ_2 and σ_{12}, the coordinates x and y by x_1 and x_2.

Except for the physical quantities mentioned above, from Eqs. (2) and (3) two derivatives in specified direction (abbreviated as DISD) are introduced as follows (Savruk 1981, Chen *et al.* 2003)

$$J_1(z) = \frac{d}{dz}\{-Y + iX\} = \Phi(z) + \overline{\Phi(z)} + \frac{d\bar{z}}{dz}(z\overline{\Phi'(z)} + \overline{\Psi(z)}) = \sigma_N + i\sigma_{NT} \tag{4}$$

$$J_2(z) = 2G\frac{d}{dz}\{u + iv\} = \kappa\Phi(z) - \overline{\Phi(z)} - \frac{d\bar{z}}{dz}(z\overline{\Phi'(z)} + \overline{\Psi(z)}) = (\kappa + 1)\Phi\Phi(z) - J_1 \tag{5}$$

It is easy to verify that $J_1 = \sigma_N + i\sigma_{NT}$ denotes the normal and shear tractions along the segment $z, z+dz$. Secondly, the J_1 and J_2 values depend not only on the position of a point "z", but also on the direction of the segment "$d\bar{z}/dz$".

In plane elasticity, the following integrals are useful (Muskhelishvili 1953, Savruk 1981)

$$F(z) = \frac{1}{2\pi i}\int_L \frac{f(t)dt}{t - z} \tag{6}$$

$$G(z)=\frac{1}{2\pi i}\int_L \frac{g(t)d\bar{t}}{t-z} \tag{7}$$

$$H(z,\bar{z})=\frac{1}{2\pi i}\int_L \frac{\bar{t}-\bar{z}}{(t-z)^2}h(t)dt \tag{8}$$

where L is a smooth curve or a closed contour Γ in Fig. 1. Also, we assume that the function $f(t)$, $g(t)$ and $h(t)$ satisfy the Hölder condition (Muskhelishvili 1953). Sometimes, the functions $f(t)$, $g(t)$ and $h(t)$ are called the density functions hereafter. Clearly, the two integrals defined by Eqs. (6) and (7) are analytic functions, and one defined by Eq. (8) is not. The integral (6) is precisely the well-known Cauchy type integral.

Generally speaking, these integrals take different values when $z \to t_0^+$ and $z \to t_0^-$, ($t_0 \in L$) respectively. The limit values of these functions from the upper and lower sides of the curve L are found to be (Muskhelishvili 1953, Savruk 1981)

$$F^{\pm}(t_o)=\pm\frac{f(t_o)}{2}+\frac{1}{2\pi i}\int_L \frac{f(t)\,dt}{t-t_o} \tag{9}$$

$$G^{\pm}(t_o)=\pm\frac{g(t_o)}{2}\frac{d\bar{t}_o}{dt_o}+\frac{1}{2\pi i}\int_L \frac{g(t)d\bar{t}}{t-t_o} \tag{10}$$

$$H^{\pm}(t_o,\bar{t}_o)=\pm\frac{h(t_o)}{2}\frac{d\bar{t}_o}{dt_o}+\frac{1}{2\pi i}\int_L \frac{\bar{t}-\bar{t}_o}{(t-t_o)^2}h(t)dt \tag{11}$$

In Eqs. (9) to (11), all the integrals should be understood in the sense of principal value of the integral. Note that the notations of $f(t)$, $g(t)$, $h(t)$, $F(z)$, $G(z)$ and $H(z,\bar{z})$ used in Eqs. (6) to (11) have no relation with those mentioned in other places.

2.2 Formulation of BIE for the interior region

In the following analysis, the α-field shown by Fig. 1(a) is relating to the fundamental field caused by concentrated force at the point $z = \tau$. The relevant complex potentials are as follows (Muskhelishvili 1953)

$$\phi(z)=F\,ln(z-\tau)\,,\quad \phi'(z)=\Phi(z)=\frac{F}{z-\tau}\,,\quad \phi''(z)=-\frac{F}{(z-\tau)^2} \tag{12}$$

$$\psi(z)=-\kappa\bar{F}\,ln(z-\tau)-\frac{F\bar{\tau}}{z-\tau}\,,\quad \psi'(z)=\Psi(z)=-\frac{\kappa\bar{F}}{z-\tau}+\frac{F\bar{\tau}}{(z-\tau)^2} \tag{13}$$

where

$$F=-\frac{P_x+iP_y}{2\pi(\kappa+1)} \tag{14}$$

In Eq. (14), P_x+iP_y is the concentrated force applied at the point $z=\tau$ in Fig. 1(a). Note that the complex potentials shown by Eq. (12), (13) are expressed in a pure deformable form (see Appendix A or Chen *et al*. 2009).

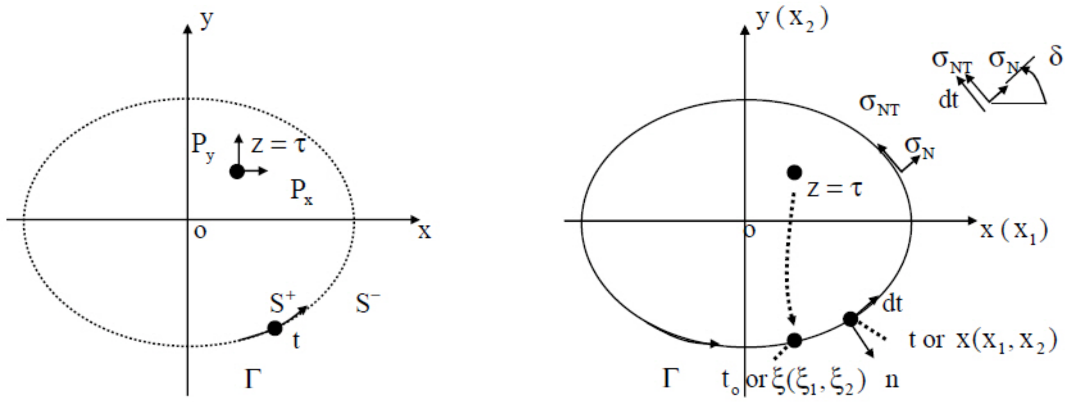
The complex potentials shown by Eqs. (12) and (13) are defined in full infinite plane. From Eqs. (3), (12) and (13), we can evaluate the relevant displacement at the point "t" as follows (Fig. 1)

$$2G(u+iv)_{*} = 2\kappa F\, ln|t-\tau| - \overline{F}\,\frac{t-\tau}{\bar{t}-\bar{\tau}} \tag{15}$$

Similarly, from Eqs. (4), (12) and (13) we can evaluate the relevant boundary traction at the point "t" as follows (Fig. 1)

$$(\sigma_N + i\sigma_{NT})_{*} = \frac{F}{t-\tau} + \frac{\overline{F}}{\bar{t}-\bar{\tau}} + \frac{d\bar{t}}{dt}\left(-\frac{\kappa F}{\bar{t}-\bar{\tau}} - \frac{\overline{F}(t-\tau)}{(\bar{t}-\bar{\tau})^2}\right) \tag{16}$$

In Eqs. (15) and (16), the subscript "*" denotes that the arguments are derived from the fundamental solution.



(a) The α-field (b) The β-field

Fig. 1 (a) The α-field with the concentrated forces applied at $z=\tau$, (b) The β-field, or the physical field defined on a finite region

After using the Betti's reciprocal theorem, or the Somigliana identity, between the fundamental field (or the α-field in Fig. 1(a)) and the physical field (or the β-field in Fig. 1(b)), we have

$$P_x u(\tau) + P_y v(\tau) + Re\left(\int_{\Gamma}(dX+idY)_{*}(u-iv)\right) = Re\left(\int_{\Gamma}(u-iv)_{*}(dX+idY)\right)$$

$$(\tau \in S^{+}) \tag{17}$$

where the left hand term represents the work done by traction in the fundamental field (the α-field) to the displacement of the physical field (the β-field). In addition, the right hand term represents the work done by traction in the physical field to the displacement of the fundamental field.

In Eq. (17), $dX+idY$ denotes the force applied on the segment "dt" (Fig. 1). From Eqs. (2), (4) and Fig. 1, we find

$$dX+idY = (\sigma_N + i\sigma_{NT})e^{i\delta}ds\,,\ \ dt = ie^{i\delta}ds\,,\ \ dX+idY = -i(\sigma_N + i\sigma_{NT})dt \tag{18}$$

Thus, Eq. (17) can be rewritten as

$$P_x u(\tau) + P_y v(\tau) + Re\int_{\Gamma} (-i)(\sigma_N + i\sigma_{NT})_*(u - iv)dt = Re\int_{\Gamma} (-i)(u - iv)_*(\sigma_N + i\sigma_{NT})dt$$

$$(\tau \in S^+) \tag{19}$$

In the following analysis, one can let

$$U(t) = u(t) + iv(t)\,,\ Q(t) = \sigma_N(t) + i\sigma_{NT}(t)\ \ ,(t \in \Gamma) \tag{20}$$

Substituting the explicit form for $(\sigma_N + i\sigma_{NT})_*$ and $(u - iv)_*$ and Eq. (20) into Eq. (19) yields

$$P_x u(\tau) + P_y v(\tau) + Re\int_{\Gamma} (-i)\left(\frac{F}{t-\tau} + \frac{\bar{F}}{\bar{t}-\bar{\tau}} + \frac{d\bar{t}}{dt}\left(-\frac{\kappa F}{\bar{t}-\bar{\tau}} - \frac{\bar{F}(t-\tau)}{(\bar{t}-\bar{\tau})^2}\right)\right)\overline{U(t)}dt$$

$$= \frac{1}{2G} Re\int_{\Gamma} (-i)\left(2\kappa\bar{F}\,ln|t-\tau| - F\frac{\bar{t}-\bar{\tau}}{t-\tau}\right)Q(t)dt\,,\ \ (\tau \in S^+) \tag{21}$$

In Eq. (21), if we let $P_x = 1$, $P_y = 0$ and $F = -1/2\pi(\kappa+1)$, we can find an equation for $u(\tau)$. Similarly, if we let $P_x = 0$, $P_y = 1$ and $F = -i/2\pi(\kappa+1)$, we can find an equation for $v(\tau)$. Thus, we will find the displacement at a domain point τ

$$U(\tau) = u(\tau) + iv(\tau) = B_1 i\int_B \left(-\frac{(\kappa-1)}{t-\tau}U(t)dt + L_1(t,\tau)U(t)dt - L_2(t,\tau)\overline{U(t)}dt\right)$$

$$+ B_2 i\int_B 2\kappa\, ln|t-\tau|Q(t)dt \ \ + B_2 i\int_{\Gamma} \frac{t-\tau}{\bar{t}-\bar{\tau}}\overline{Q(t)}d\bar{t}\,,\ \ (\tau \in S^+) \tag{22}$$

where

$$B_1 = \frac{1}{2\pi(\kappa+1)},\quad B_2 = \frac{1}{4\pi G(\kappa+1)} \tag{23}$$

$$L_1(t,\tau) = -\frac{d}{dt}\left\{ln\frac{t-\tau}{\bar{t}-\bar{\tau}}\right\} = -\frac{1}{t-\tau} + \frac{1}{\bar{t}-\bar{\tau}}\frac{d\bar{t}}{dt} \tag{24}$$

$$L_2(t,\tau) = \frac{d}{dt}\left\{\frac{t-\tau}{\bar{t}-\bar{\tau}}\right\} = \frac{1}{\bar{t}-\bar{\tau}} - \frac{t-\tau}{(\bar{t}-\bar{\tau})^2}\frac{d\bar{t}}{dt} \tag{25}$$

In Eq. (22), letting $\tau \to t_o$ ($\tau \in S^+$, $t_o \in \Gamma$) and using the generalized Sokhotski-Plemelj's formulae shown by Eqs. (9), (10) and (11) and Appendix B, yields

$$\frac{U(t_o)}{2} + B_1 i\int_{\Gamma}\left(\frac{(\kappa-1)}{t-t_o}U(t)dt - L_1(t,t_o)U(t)dt + L_2(t,t_o)\overline{U(t)}dt\right)$$

$$= B_2 i\int_{\Gamma}\left(2\kappa\, ln|t-t_o|Q(t)dt + \frac{t-t_o}{\bar{t}-\bar{t}_o}\overline{Q(t)}d\bar{t}\right),\ \ (t_o \in \Gamma) \tag{26}$$

where

$$L_1(t,t_o) = -\frac{d}{dt}\left\{ln\frac{t-t_o}{\bar{t}-\bar{t}_o}\right\} = -\frac{1}{t-t_o} + \frac{1}{\bar{t}-\bar{t}_o}\frac{d\bar{t}}{dt} \quad (27)$$

$$L_2(t,t_o) = \frac{d}{dt}\left\{\frac{t-t_o}{\bar{t}-\bar{t}_o}\right\} = \frac{1}{\bar{t}-\bar{t}_o} - \frac{t-t_o}{(\bar{t}-\bar{t}_o)^2}\frac{d\bar{t}}{dt} \quad (28)$$

Eq. (26) represents the complex variable boundary integral equation (CVBIE) for the interior problem in plane elasticity based on the Somigliana identity. An alternative expression of Eq. (26) was obtained previously (Linkov 2002), and the author did not use the generalized Sokhotski-Plemelj's formulae in the derivation.

In the following analysis, we prefer to write Eq. (26) in the following form

$$\Lambda_i(t_o) = B_2 iJ(t_o)\,,\ (\ \Lambda_i(t_o)\{U(t)\to\Lambda_i(t_o)\}\ \text{abbreviating as}\ \Lambda_i(t_o)\,)\quad (t_o\in\Gamma) \quad (29)$$

where

$$\Lambda_i(t_o) = \frac{U(t_o)}{2} + B_1 iI(t_o)\,,\quad (t_o\in\Gamma) \quad (30)$$

$$I(t_o) = \int_\Gamma \left(\frac{(\kappa-1)}{t-t_o}U(t)dt - L_1(t,t_o)U(t)dt + L_2(t,t_o)\overline{U(t)}dt\right),\quad (t_o\in\Gamma) \quad (31)$$

$$J(t_o) = \int_\Gamma \left(2\kappa\, ln\left|t-t_o\right|Q(t)dt + \frac{t-t_o}{\bar{t}-\bar{t}_o}\overline{Q(t)}d\bar{t}\right),\quad (t_o\in\Gamma) \quad (32)$$

In Eqs. (29) and (30), the subscript "*i*" is used to indicate the studied interior BVP.

Accordingly, the relevant homogenous equation takes the following form

$$\Lambda_i(t_o) = 0\,,\quad (t_o\in\Gamma) \quad (33)$$

In the meantime, from Eq. (26) and some manipulations, the real variable BIE for the plane strain case is as follows

$$\frac{1}{2}u_i(\xi) + \int_\Gamma P^*_{ij}(\xi,x)u_j(x)ds(x) = \int_\Gamma U^{*1}_{ij}(\xi,x)p_j(x)ds(x)\,,\quad (i=1,2,\ \xi\in\Gamma) \quad (34)$$

In Eq. (34), the point $x(x_1,x_2)$ (or t in Fig. 1) is the field point, and $\xi(\xi_1,\xi_2)$ (or t_o in Fig. 1) is the source point.

In Eq. (34), the kernel $P^*_{ij}(\xi,x)$ is defined by Brebbia *et al*. (1984)

$$P^*_{ij}(\xi,x) = -\frac{1}{4\pi(1-\nu)}\frac{1}{r}\left\{(r_{,1}n_1 + r_{,2}n_2)((1-2\nu)\delta_{ij} + 2r_{,i}r_{,j}) + (1-2\nu)(n_i r_{,j} - n_j r_{,i})\right\} \quad (35)$$

where Kronecker deltas δ_{ij} is defined as, $\delta_{ij}=1$ for $i=j$, $\delta_{ij}=0$ for $i\neq j$, and

$$r_{,1} = \frac{x_1-\xi_1}{r} = \cos\alpha\,,\quad r_{,2} = \frac{x_2-\xi_2}{r} = \sin\alpha \quad (36)$$

and $n(n_1, n_2)$ is a unit normal to the boundary at the filed point $x(x_1, x_2)$.

In Eq. (34), the kernel $U^{*1}_{ij}(\xi,x)$ is defined by (Chen *et al*. 2009)

$$U_{ij}^{*1}(\xi,x)=\frac{1}{8\pi(1-\nu)G}\left\{-(3-4\nu)\,ln(r)\delta_{ij}+r_{,i}r_{,j}-0.5\delta_{ij}\right\} \tag{37}$$

In the meantime, for the plane strain case, a real variable BIE was suggested as follows (Brebbia *et al.* 1984)

$$\frac{1}{2}u_i(\xi)+\int_\Gamma P_{ij}^{*}(\xi,x)u_j(x)ds(x)=\int_\Gamma U_{ij}^{*2}(\xi,x)p_j(x)ds(x)\,,\quad (i=1,2,\ \xi\in\Gamma\,) \tag{34a}$$

In Eq. (34a), the kernel $U_{ij}^{*2}(\xi,x)$ is defined by (Brebbia *et al.* 1984, Chen *et al.* 2009)

$$U_{ij}^{*2}(\xi,x)=\frac{1}{8\pi(1-\nu)G}\left\{-(3-4\nu)\,ln(r)\delta_{ij}+r_{,i}r_{,j}\right\} \tag{37a}$$

The difference between the kernels $U_{ij}^{*1}(\xi,x)$ and $U_{ij}^{*2}(\xi,x)$ was investigated (Chen *et al.* 2009).

2.3 Properties of integral operator $\Lambda_i(t_0)$

We see from Eq. (30) that the integral operator $\Lambda_i(t_0)$ is composed of the function $U(t_o)$ and the operator $I(t_0)$. In fact, the operator $I(t_0)$ has a detailed expression $I(t_o)\{U(t)\rightarrow I(t_o)\}$ shown by Eq. (31). Properties of the integral operators $I(t_0)$ and $\Lambda_i(t_0)$ are introduced below.

For finding the first property of $\Lambda_i(t_o)$, a property of the integral operator $I(t_0)$ is introduced below

$$I(t_o)\ \big|_{U(t)=1}=\pi(\kappa+1)i\,,\quad (t_o\in\Gamma\,) \tag{38}$$

In fact, from Eq. (31), the integral $I(t_o)\ \big|_{U(t)=1}$ can be decomposed as

$$I(t_o)\ \big|_{U(t)=1}=I_1(t_o)+I_2(t_o)+I_3(t_o)\,,\quad (t_o\in\Gamma\,) \tag{39}$$

In Eq. (39), three integrals can be integrated as follows

$$I_1(t_o)=(\kappa-1)\int_\Gamma\frac{dt}{t-t_o}=(\kappa-1)\pi i\ ,\quad (t_o\in\Gamma\,) \tag{40}$$

$$I_2(t_o)=-\int_\Gamma L_1(t,t_o)dt=\int_\Gamma\frac{dt}{t-t_o}-\int_\Gamma\frac{d\bar{t}}{\bar{t}-\bar{t}_o}=\int_\Gamma\frac{dt}{t-t_o}-\overline{\int_\Gamma\frac{dt}{t-t_o}}=2\pi i\,,\ (t_o\in\Gamma\,) \tag{41}$$

$$I_3(t_o)=\int_\Gamma L_2(t,t_o)dt=\int_\Gamma\left(\frac{dt}{\bar{t}-\bar{t}_o}-\frac{(t-t_o)d\bar{t}}{(\bar{t}-\bar{t}_o)^2}\right)=\int_\Gamma d\left(\frac{t-t_o}{\bar{t}-\bar{t}_o}\right)=0\,,\ (t_o\in\Gamma\,) \tag{42}$$

Therefore, the equality (38) is proved.

Letting $U(t)=1$ in the integral operator $\Lambda_i(t_o)$ and using Eq. (38), we will find

$$\Lambda_i(t_o)\ \big|_{U(t)=1}=\frac{1}{2}-\pi(\kappa+1)B_1=0\,,\quad (t_o\in\Gamma\,) \tag{43}$$

Eq. (43) reveals that the homogenous equation (33) has a non-trivial solution $U(t) = 1$.

Similarly, we can prove the second property of the integral operator $I(t_0)$, which is as follows

$$I(t_o)\ |_{U(t)=i} = -\pi(\kappa+1)\,, \quad (t_o \in \Gamma) \tag{44}$$

Letting $U(t) = i$ in the integral operator $\Lambda_i(t_o)$ and using Eq. (44), we will find

$$\Lambda_i(t_o)\ |_{U(t)=i} = \frac{i}{2} - \pi(\kappa+1)B_1 i = 0\,, \quad (t_o \in \Gamma) \tag{45}$$

Eq. (45) reveals that the homogenous equation (33) has a non-trivial solution $U(t) = i$.

Similarly, the third property of the integral operator $I(t_0)$ is introduced as follows

$$I(t_o)\ |_{U(t)=it} = -(\kappa+1)t_o\pi\,, \quad (t_o \in \Gamma) \tag{46}$$

In fact, the integral $I(t_o)\ |_{U(t)=it}$ can be rewritten as

$$I(t_o)\ |_{U(t)=it} = I_1(t_o) + I_2(t_o)\,, \quad (t_o \in \Gamma) \tag{47}$$

In Eq. (47), two integrals $I_1(t_0)$ and $I_2(t_0)$ can be integrated as follows

$$I_1(t_o) = (\kappa+1)i\int_\Gamma \frac{tdt}{t-t_o} = (\kappa+1)i\int_\Gamma \frac{t_o dt}{t-t_o} = -(\kappa+1)t_o\pi\,, \quad (t_o \in \Gamma) \tag{48}$$

$$\begin{aligned} I_2(t_o) &= i\int_\Gamma \left(-\frac{tdt}{t-t_o} - \frac{td\bar{t}}{\bar{t}-\bar{t}_o} - \frac{\bar{t}dt}{\bar{t}-\bar{t}_o} + \frac{(t-t_o)\bar{t}d\bar{t}}{(\bar{t}-\bar{t}_o)^2} \right) \\ &= i\int_\Gamma \left(-\frac{t_o dt}{t-t_o} - \frac{td\bar{t}}{\bar{t}-\bar{t}_o} \right) - i\int_\Gamma \bar{t}d\left(\frac{t-t_o}{\bar{t}-\bar{t}_o} \right) = i\int_\Gamma \left(-\frac{t_o dt}{t-t_o} - \frac{td\bar{t}}{\bar{t}-\bar{t}_o} \right) + i\int_\Gamma \left(\frac{t-t_o}{\bar{t}-\bar{t}_o} \right) d\bar{t} \\ &= -t_o i\int_\Gamma \left(\frac{dt}{t-t_o} + \frac{d\bar{t}}{\bar{t}-\bar{t}_o} \right) = 0\,, \quad (t_o \in \Gamma) \end{aligned} \tag{49}$$

In the derivation, the equality $\int_\Gamma dt = 0$ is used. Therefore, the equality (46) is proved.

Letting $U(t) = it$ in the integral operator $\Lambda_i(t_o)$ and using Eq. (46), we will find

$$\Lambda_i(t_o)\ |_{U(t)=it} = \frac{it_o}{2} - B_1(\kappa+1)t_o\pi i = 0\,, \quad (t_o \in \Gamma) \tag{50}$$

Eq. (50) reveals that the homogenous equation (33) has a non-trivial solution $U(t) = it$.

It is seen that any boundary values for $U(t)$ and $Q(t)$ from an elastic solution satisfy BIE shown by Eq. (33) accordingly. Particularly, the following three pairs of solution: (1) $U(t) = 1$, $Q(t) = 0$, (2) $U(t) = i$, $Q(t) = 0$ and (2) $U(t) = it$, $Q(t) = 0$ satisfy the BIE shown by Eq. (29).

2.4 Numerical examination for properties of the integral operator $\Lambda_i(t_0)$

For examining the equalities shown by Eqs. (43), (45) and (50), an elliptic contour with two half-axes a = 20 and b = 10 is adopted in the numerical examination.

For the first two cases, we assume $U(t)$ = 1 or $U(t)$ = i. Therefore, the maximum element takes the value of "1". It is seen that the function $\Lambda_i(t_0)$ is composed of the u- and v-component, which are defined on the contour ($t_o \in \Gamma$). After discretization, we can compute the value of $\Lambda_i(t_0)$ at many discrete points along the contour. For a point $x = a \cos\theta$ $y = b \sin\theta$ ($t_o = x + iy \in \Gamma$) along the contour, the relative error is defined by

$$f_1(\theta) = max|u_{numer} - u_{exact}|,$$

$$g_1(\theta) = max|v_{numer} - v_{exact}| = max|v_{numer}|, \qquad \text{(for } U(t) = 1 \text{ case)} \tag{51}$$

$$f_2(\theta) = max|u_{numer} - u_{exact}| = max|u_{numer}|,$$

$$g_2(\theta) = max|v_{numer} - v_{exact}|, \qquad \text{(for } U(t) = i \text{ case)} \tag{52}$$

Note that, the results v_{exact} = 0 and u_{exact} = 0 (see Eqs. (43) and (45)) have been used in Eqs. (51) and (52).

For the third case, we assume $U(t) = it$. Therefore, the maximum element takes the value of "a". In this case, the relative error is defined by

$$f_3(\theta) = max|u_{numer} - u_{exact}|/a,$$

$$g_3(\theta) = max|v_{numer} - v_{exact}|/a, \text{ (for } U(t) = it \text{ case)} \tag{53}$$

For three cases, the relative errors are tabulated in Table 1. From tabulated results we see that the errors for the first two cases (or $U(t)$ = 1 and $U(t)$ = i) are extremely small. In addition, the errors for the third cases (or $U(t) = it$) is also small. Thus, three equalities shown by Eqs. (43), (45) and (50) are examined numerically in the mentioned example.

Table 1 Relative errors for some integration operations: $f_1(\theta) = \mathrm{Re}\,\Lambda_i(t_o)\,|_{U(t)=1}$, $g_1(\theta) = \mathrm{Im}\,\Lambda_i(t_o)\,|_{U(t)=1}$, $f_2(\theta) = \mathrm{Re}\,\Lambda_i(t_o)\,|_{U(t)=i}$, $g_2(\theta) = \mathrm{Im}\,\Lambda_i(t_o)\,|_{U(t)=i}$, $f_3(\theta) = \mathrm{Re}\,\Lambda_i(t_o)\,|_{U(t)=it}$, $g_3(\theta) = \mathrm{Im}\,\Lambda_i(t_o)\,|_{U(t)=it}$ (see Eqs. (30), (31), (51), (52) and (53))

			$f_1(\theta)$			
θ (degree)	0	20	40	60	80	100
	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
θ	120	140	160	180	200	220
	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
θ	240	260	280	300	320	340
	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
			$g_1(\theta)$			
θ (degree)	0	20	40	60	80	100
	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
θ	120	140	160	180	200	220

Table 1 Continued

	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
θ	240	260	280	300	320	340
	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
			$f_2(\theta)$			
θ (degree)	0	20	40	60	80	100
	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
θ	120	140	160	180	200	220
	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
θ	240	260	280	300	320	340
	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
			$g_2(\theta)$			
θ (degree)	0	20	40	60	80	100
	-0.000002	0.000000	0.000000	0.000000	0.000000	0.000000
θ	120	140	160	180	200	220
	0.000000	0.000000	0.000000	-0.000002	0.000000	0.000000
θ	240	260	280	300	320	340
	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
			$f_3(\theta)$			
θ (degree)	0	20	40	60	80	100
	0.000000	-0.000825	-0.001546	-0.002082	-0.002367	-0.002367
θ	120	140	160	180	200	220
	-0.002082	-0.001546	-0.000825	0.000000	0.000825	0.001546
θ	240	260	280	300	320	340
	0.002082	0.002367	0.002367	0.002082	0.001546	0.000825
			$g_3(\theta)$			
θ (degree)	0	20	40	60	80	100
	0.001518	0.001422	0.001156	0.000754	0.000262	-0.000262
θ	120	140	160	180	200	220
	-0.000754	-0.001156	-0.001422	-0.001518	-0.001422	-0.001156
θ	240	260	280	300	320	340
	-0.000754	-0.000262	0.000262	0.000754	0.001156	0.001422

2.5 The Neumann problem for the interior region

It is assumed that the boundary traction $\widetilde{Q}(t)=\widetilde{\sigma}_N(t)+i\widetilde{\sigma}_{NT}(t)$ is known beforehand. Substituting the known traction $\widetilde{Q}(t)=\widetilde{\sigma}_N(t)+i\widetilde{\sigma}_{NT}(t)$ in the right hand side of Eq. (29) yields the following BIE

$$\Lambda_i(t_o)=B_2 i\widetilde{J}(t_o)\ , \ \ \text{(where } \Lambda_i(t_o)\{U(t)\rightarrow \Lambda_i(t_o)\}\text{)},\ \ (t_o\in\Gamma) \tag{54}$$

where the integral operator $\Lambda_i(t_o)$ is defined by

$$\Lambda_i(t_o)=\frac{U(t_o)}{2}+B_1 iI(t_o)\ , \ \ (t_o\in\Gamma) \tag{55}$$

In Eq. (54), the right hand term is defined by

$$\widetilde{J}(t_o) = \int_\Gamma \left(2\kappa \ln|t - t_o| \widetilde{Q}(t)dt + \frac{t - t_o}{\bar{t} - \bar{t}_o} \overline{\widetilde{Q}(t)} d\bar{t} \right), \quad (t_o \in \Gamma) \tag{56}$$

Obviously, from Eq. (54), we can propose relevant homogenous BIE

$$\Lambda_i(t_o) = 0 \ , \quad (t_o \in \Gamma) \tag{57}$$

From Eq. (55) we see that the integral operator $\Lambda_i(t_o)$ is composed of the function $U(t_o)$ and the operator $I(t_0)$. In fact, the operator $I(t_0)$ has a detailed form $I(t_0)\{U(t) \rightarrow I(t_0)\}$ shown by Eq. (31).

In the Neumann problem of the interior region, the first studied problem is about the right hand term in Eq. (54). It is known that for a finite region, the boundary traction $\widetilde{Q}(t)$ must be in equilibrium. Clearly, this assertion is easy to see. However, it has not been proved theoretically by BIE (54) itself.

The mentioned condition for the boundary traction $\widetilde{Q}(t)$ in equilibrium takes the following form (Muskhelishvili 1953)

$$\int_\Gamma \left(\widetilde{\sigma}_N(t) + i\widetilde{\sigma}_{NT}(t)\right)dt = \int_\Gamma \widetilde{Q}(t)dt = 0 \tag{58}$$

$$Re\int_\Gamma \left(\widetilde{\sigma}_N(t) + i\widetilde{\sigma}_{NT}(t)\right)\bar{t}dt = Re\int_\Gamma \widetilde{Q}(t)\bar{t}dt = 0 \tag{59}$$

Eq. (58) represents the equilibrium condition for forces, and Eq. (59) represents the equilibrium condition for moment.

If the conditions (58) and (59) are satisfied, from property of the integral operator $\Lambda_i(t_o)$, the non-homogenous equation (54) has the following solution for displacement $U(t)$

$$U(t) = U_p(t) + c_1 + c_2 i + c_3 it, \quad (t \in \Gamma) \tag{60}$$

or in the form

$$\begin{Bmatrix} u(t) \\ v(t) \end{Bmatrix} = \begin{Bmatrix} u_p(t) \\ v_p(t) \end{Bmatrix} + c_1 \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} + c_2 \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} + c_3 \begin{Bmatrix} -y_t \\ x_t \end{Bmatrix}, \ \text{(letting } t = x_t + iy_t, \ t \in \Gamma) \tag{61}$$

where $u_p(t) + iv_p(t)$ is a particular solution for the non-homogenous equation and c_1, c_2 and c_3 are constant. Eq. (60) or (61) reveals that Eq. (54) has many solutions, and they may differ from each other by a rigid motion.

Particularly, for the relevant homogenous equation (57), we have the following solution

$$U(t) = c_1 + c_2 i + c_3 it, \quad (t \in \Gamma) \tag{62}$$

On contrary, it is assumed that the boundary tractions $\widetilde{Q}(t) = \widetilde{\sigma}_N(t) + i\widetilde{\sigma}_{NT}(t)$ do not satisfy the equilibrium conditions and the right hand term $\widetilde{J}(t_o)$ is derived from boundary traction $\widetilde{Q}(t) = \widetilde{\sigma}_N(t) + i\widetilde{\sigma}_{NT}(t)$ not in equilibrium. In this case, the non-homogenous equation (54) has no solution for $U(t)$. Physically, this assertion is easy to see. However, it has not been proved theoretically from the BIE shown by Eq. (54) itself.

2.6 The Dirichlet problem for the interior region

It is assumed that the boundary displacement $\widetilde{U}(t) = \widetilde{u}(t) + i\widetilde{v}(t)$ is known beforehand.

Substituting the known displacement $\tilde{U}(t) = \tilde{u}(t) + i\tilde{v}(t)$ in the left hand side of Eq. (29) yields the following BIE

$$B_2 iJ(t_o) = \tilde{\Lambda}_i(t_o), \quad (t_o \in \Gamma) \tag{63}$$

or in an explicit form

$$B_2 iJ(t_o) = B_2 i\int_\Gamma \left(2\kappa\, ln|t - t_o| Q(t)dt + \frac{t - t_o}{\bar{t} - \bar{t}_o}\overline{Q(t)}d\bar{t} \right) = \tilde{\Lambda}_i(t_o), \quad (t_o \in \Gamma) \tag{64}$$

where

$$\tilde{\Lambda}_i(t_o) = \frac{\tilde{U}(t_o)}{2} + B_1 i\tilde{I}(t_o), \quad (t_o \in \Gamma) \tag{65}$$

$$\tilde{I}(t_o) = \int_\Gamma \left(\frac{(\kappa - 1)}{t - t_o}\tilde{U}(t)dt - L_1(t,t_o)\tilde{U}(t)dt + L_2(t,t_o)\overline{\tilde{U}(t)}dt \right), \quad (t_o \in \Gamma) \tag{66}$$

Obviously, from Eq. (63), we can get the following homogenous BIE

$$J(t_o) = \int_\Gamma \left(2\kappa\, ln|t - t_o| Q(t)dt + \frac{t - t_o}{\bar{t} - \bar{t}_o}\overline{Q(t)}d\bar{t} \right) = 0, \quad (t_o \in \Gamma) \tag{67}$$

It is assumed that the scale has not reached the degenerate scale. In this case, the integral operator $J(t_0)$ possesses the invertibility (Vodicka and Mantic 2008). The input function $\tilde{U}(t)$ can be an arbitrary function. In this case, the non-homogenous equation (63) has a unique solution for $Q(t)$. Physically, the obtained boundary tractions $Q(t)$ from the non-homogenous equation (63) must be in equilibrium. Otherwise, the equilibrium condition for finite body is violated. However, this assertion has not been proved theoretically from BIE (63) itself.

Particularly, if the input function $\tilde{U}(t)$ takes the following form

$$\tilde{U}(t) = c_1 + c_2 i + c_3 it, \quad (t \in \Gamma) \tag{68}$$

we will obtain a unique solution $Q(t)$ =0. That is to say $\tilde{U}(t) = c_1$, $\tilde{U}(t) = c_2 i$ and $\tilde{U}(t) = c_3 it$ correspond to the same solution $Q(t)$ =0.

Similarly, If the real scale has not reached the degenerate scale, the homogenous equation (67) only has a solution $Q(t)$ = 0. However, from the homogenous equation (67), we can propose the following degenerate scale problem, which can be defined as follows. One wants to find a particular scale such that the homogenous integral equation (67) has a non-trivial solution for the function $Q(t)$. Here, $Q(t)$ = 0 is a trivial solution. The degenerate scale problem has been studied by many researchers using a variety of methods (Chen *et al.* 2002b, Chen *et al.* 2005, Chen *et al.* 2007, Vodicka and Mantic 2004, 2008). Since many researchers studied the degenerate scale problem, we do not study it here anymore.

3. Formulation of complex variable BIE for the exterior boundary value problem

3.1 Formulation of BIE for the exterior region

In the exterior BVP, we may use the Somigliana identity on the region bounded by contours Σ_1 and Σ_2 (Fig. 2). The contour Σ_2 is placed at a remote place, which may be assumed to be a sufficient large circle. The complex variable BIE for the exterior boundary value problem for $t_o \in \Sigma_1$ is similar to the one in the interior boundary value problem, which can be expressed as follows

$$\frac{U(t_o)}{2} + B_1 i \int_{\Sigma_1} \left(\frac{(\kappa-1)}{t-t_o} U(t)dt - L_1(t,t_o)U(t)dt + L_2(t,t_o)\overline{U(t)}dt \right)$$

$$= B_2 i \int_{\Sigma_1} \left(2\kappa \ln|t-t_o| Q(t)dt + \frac{t-t_o}{\bar{t}-\bar{t}_o} \overline{Q(t)}d\bar{t} \right) + D(t_o), \quad (t_o \in \Sigma_1) \tag{69}$$

where

$$D(t_o) = -B_1 i \int_{\Sigma_2} \left(\frac{(\kappa-1)}{t-t_o} U(t)dt - L_1(t,t_o)U(t)dt + L_2(t,t_o)\overline{U(t)}dt \right)$$

$$+ B_2 i \int_{\Sigma_2} \left(2\kappa \ln|t-t_o| Q(t)dt + \frac{t-t_o}{\bar{t}-\bar{t}_o} \overline{Q(t)}d\bar{t} \right), \quad (t_o \in \Sigma_1) \tag{70}$$

The term $D(t_0)$ denotes the contribution to the identity from the contour Σ_2. In Eqs. (69) and (70), the functions $U(t)$, $Q(t)$ and the elastic constants B_1, B_2 have been defined by Eqs. (20) and (23) previously.

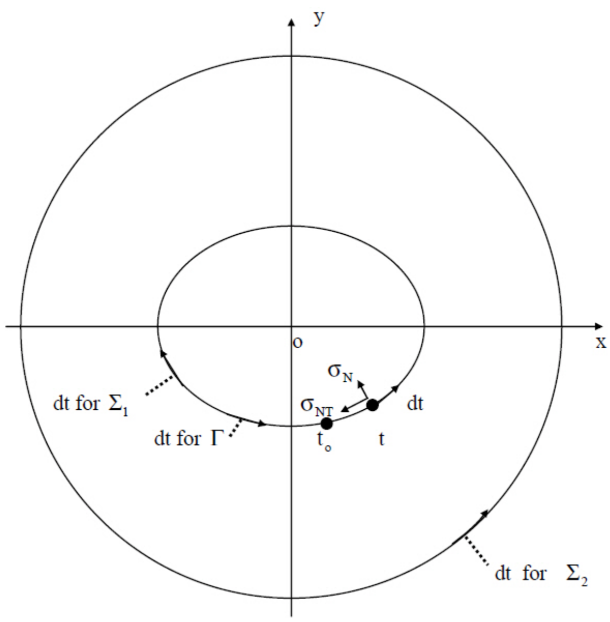


Fig. 2 Formulation of the Somigliana identity for exterior region

Clearly, the term $D(t_0)$ is not always equal to zero. It was proved that if and only if the function $U(t)$ ($U(t)$- the displacement along the boundary) is expressed in a pure deformable form (see Appendix A), we have $D(t_0) = 0$ (Chen *et al*. 2009). This is the regularity condition in the exterior BVP (Brebbia *et al*. 1984). In our formulation, the regularity condition means that one does not need to formulate any equation at remote place, for example, along a large circle. It is assumed that this condition $D(t_0) = 0$ is satisfied, Eq. (69) can be reduced to

$$\frac{U(t_o)}{2} + B_1 i \int_{\Sigma_1} \left(\frac{(\kappa-1)}{t-t_o} U(t)dt - L_1(t,t_o)U(t)dt + L_2(t,t_o)\overline{U(t)}dt \right)$$

$$= B_2 i \int_{\Sigma_1} \left(2\kappa\, ln|t-t_o| Q(t)dt + \frac{t-t_o}{\bar{t}-\bar{t}_o} \overline{Q(t)} d\bar{t} \right), \quad (t_o \in \Sigma_1) \tag{71}$$

One difference between Eq. (26) and (71) is that the increment "*dt*" in Eq. (71) is in the clockwise direction along the boundary Σ_1. One may change the integration path from the clockwise direction to the anti-clockwise direction. Therefore, Eq. (71) can be rewritten as (Fig. 2)

$$\frac{U(t_o)}{2} - B_1 i \int_{\Gamma} \left(\frac{(\kappa-1)}{t-t_o} U(t)dt - L_1(t,t_o)U(t)dt + L_2(t,t_o)\overline{U(t)}dt \right)$$

$$= -B_2 i \int_{\Gamma} \left(2\kappa\, ln|t-t_o| Q(t)dt + \frac{t-t_o}{\bar{t}-\bar{t}_o} \overline{Q(t)} d\bar{t} \right), \quad (t_o \in \Gamma) \tag{72}$$

Eq. (72) represents the complex variable boundary integral equation (CVBIE) for the exterior boundary value problem, which is based on the Somigliana identity. Note that the increment "*dt*" in Eq. (72) is in the anti-clockwise direction along the boundary Γ (Fig. 2).

In the following analysis, we prefer to write Eq. (72) in the following form

$$\Lambda_e(t_o) = -B_2 i J(t_o), \quad (t_o \in \Gamma) \tag{73}$$

where

$$\Lambda_e(t_o) = \frac{U(t_o)}{2} - B_1 i I(t_o), \quad (t_o \in \Gamma) \tag{74}$$

$$I(t_o) = \int_{\Gamma} \left(\frac{(\kappa-1)}{t-t_o} U(t)dt - L_1(t,t_o)U(t)dt + L_2(t,t_o)\overline{U(t)}dt \right), \quad (t_o \in \Gamma) \tag{75}$$

$$J(t_o) = \int_{\Gamma} \left(2\kappa\, ln|t-t_o| Q(t)dt + \frac{t-t_o}{\bar{t}-\bar{t}_o} \overline{Q(t)} d\bar{t} \right), \quad (t_o \in \Gamma) \tag{76}$$

In Eq. (75), the subscript "*e*" is used for the exterior BVP.

Accordingly, the relevant homogenous equation for Eq. (73) takes the following form

$$\Lambda_e(t_o) = 0, \quad (t_o \in \Gamma) \tag{77}$$

3.2 Properties of integral operator $\Lambda_e(t_o)$

Previously, the integral operator $\Lambda_e(t_o)$ is defined by

$$\Lambda_e(t_o) = \frac{U(t_o)}{2} - B_1 iI(t_o), \quad (t_o \in \Gamma) \tag{78}$$

where the integral operator $I(t_0)$ has been defined by Eq. (31) previously.

Note that the integral operator $\Lambda_i(t_o)$ takes the form $\Lambda_i(t_o) = U(t_0)/2 + B_1 iI(t_0)$ in the interior problem, and the integral operator $\Lambda_e(t_o)$ takes the form $\Lambda_e(t_o) = U(t_0)/2 - B_1 iI(t_0)$ in the exterior problem. Therefore, the properties of the two integral operators $\Lambda_i(t_o)$ and $\Lambda_e(t_o)$ must be different.

From properties for $I(t_0)$ shown by Eqs. (38), (44)and (46) and definition for $\Lambda_e(t_o)$ shown by Eq. (74), we will find

$$\Lambda_e(t_o)\{U(t) \to \Lambda_e(t_o)\} \, |_{U(t)=1} = 1, \quad (t_o \in \Gamma) \tag{79}$$

$$\Lambda_e(t_o)\{U(t) \to \Lambda_e(t_o)\} \, |_{U(t)=i} = i, \quad (t_o \in \Gamma) \tag{80}$$

$$\Lambda_e(t_o)\{U(t) \to \Lambda_e(t_o)\} \, |_{U(t)=it} = it_o, \quad (t_o \in \Gamma) \tag{81}$$

In fact, from Eqs. (38), we have

$$\Lambda_e(t_o) \, |_{U(t)=1} = \frac{1}{2} - B_1 iI(t_o) \, |_{U(t)=1} = \frac{1}{2} + B_1 \pi(\kappa + 1) = 1 \tag{82}$$

Therefore, the equality shown by Eq. (79) is proved. Clearly, Eqs. (80) and (81) can be proved in a similar manner. The result shown by Eqs. (79), (80) and (81) is called the identity property of the integral operator $\Lambda_e(t_o)$.

It is seen that only the boundary values for $U(t)$ derived from the pure deformable form (See appendix A) and $Q(t)$ from an elastic solution satisfy BIE shown by Eq. (73). In the meantime, the boundary values for $U(t)$ derived from the impure deformable form (See appendix A) and $Q(t)$ from an elastic solution do not satisfy BIE shown by Eq. (73). Since we have assumed that both the fundamental field and the physical field are expressed in the pure deformable form and the term $D(t_0)$ has been dropped in the derivation, it is natural to obtain this result. Particularly, we substitute $U(t) = 1$ in the left hand side of Eq. (73) and get $\Lambda_e(t_o) \, |_{U(t)=1} = 1$. However, we substitute $Q(t) = 0$ in the right hand side of Eq. (73) and get $-B_2 iJ(t_o) \, |_{Q(t)=0} = 0$. Alternative speaking, the following boundary values $U(t) = 1$ and $Q(t) = 0$ do not satisfy Eq. (73). Therefore, the three particular functions ($U(t) = 1$, $U(t) = i$ and $U(t) = it$) are not the solution of the homogenous equation $\Lambda_e(t_o) = 0$. Thus, It is expected that the homogenous equation $\Lambda_e(t_o) = 0$ only has a unique solution $U(t) = 0$.

3.3 The Neumann problem for the exterior region

It is assumed that the boundary traction $\widetilde{Q}(t) = \widetilde{\sigma}_N(t) + i\widetilde{\sigma}_{NT}(t)$ is known beforehand. Substituting the known traction $\widetilde{Q}(t) = \widetilde{\sigma}_N(t) + i\widetilde{\sigma}_{NT}(t)$ in the right hand side of Eq. (73) yields the following BIE

$$\Lambda_e(t_o) = -B_2 i\widetilde{J}(t_o)\,, \quad \text{where } \Lambda_e(t_o)\{U(t) \to \Lambda_e(t_o)\}\,)\,, \quad (t_o \in \Gamma\,) \tag{83}$$

where the integral operator $\Lambda_e(t_o)$ is defined by

$$\Lambda_e(t_o) = \frac{U(t_o)}{2} - B_1 iI(t_o)\,, \quad (t_o \in \Gamma\,) \tag{84}$$

In Eq. (83), the right hand term is defined by

$$\widetilde{J}(t_o) = \int_\Gamma \left(2\kappa\, ln|t - t_o| \widetilde{Q}(t)dt + \frac{t - t_o}{\bar{t} - \bar{t}_o} \overline{\widetilde{Q}(t)} d\bar{t} \right), \quad (t_o \in \Gamma\,) \tag{85}$$

If right hand term in Eq. (83) vanishes, we obtain the following homogenous equation

$$\Lambda_e(t_o) = 0\,, \quad (t_o \in \Gamma\,) \tag{86}$$

Contrary to the Neumann problem of the interior region (in & 2.4), the boundary tractions $\widetilde{Q}(t) = \widetilde{\sigma}_N(t) + i\widetilde{\sigma}_{NT}(t)$ in the exterior problem do not need to satisfy the equilibrium conditions. Particularly, for any assumed boundary traction $\widetilde{Q}(t) = \widetilde{\sigma}_N(t) + i\widetilde{\sigma}_{NT}(t)$, from Eq. (83) one has a unique solution for $U(t)$. In addition, the relevant homogenous equation has a unique solution $U(t) = 0$.

This situation is quite different from that in the interior problem. In fact, we know that the boundary tractions in the interior BVP must be in equilibrium, and solution for $U(t)$ takes the form of Eq. (60), which contains three constants c_1 ($i = 1, 2, 3$).

3.4 The Dirichlet problem for the exterior region

It is assumed that the boundary displacement $\widetilde{U}(t) = \widetilde{u}(t) + i\widetilde{v}(t)$ is known beforehand. Substituting the known displacement $\widetilde{U}(t) = \widetilde{u}(t) + i\widetilde{v}(t)$ in the left hand side of Eq. (73) yields the following BIE

$$-B_2 iJ(t_o) = \widetilde{\Lambda}_e(t_o)\,, \quad (t_o \in \Gamma\,) \tag{87}$$

or in an explicit form

$$-B_2 iJ(t_o) = -B_2 i \int_\Gamma \left(2\kappa\, ln|t - t_o| Q(t)dt + \frac{t - t_o}{\bar{t} - \bar{t}_o} \overline{Q(t)} d\bar{t} \right) = \widetilde{\Lambda}_e(t_o)\,, \quad (t_o \in \Gamma\,) \tag{88}$$

where

$$\widetilde{\Lambda}_e(t_o) = \frac{\widetilde{U}(t_o)}{2} - B_1 i\widetilde{I}(t_o)\,, (t_o \in \Gamma\,) \tag{89}$$

$$\widetilde{I}(t_o) = \int_\Gamma \left(\frac{(\kappa - 1)}{t - t_o} \widetilde{U}(t)dt - L_1(t,t_o)\widetilde{U}(t)dt + L_2(t,t_o)\overline{\widetilde{U}(t)dt} \right), \quad (t_o \in \Gamma\,) \tag{90}$$

It is assumed that the scale has not reached the degenerate scale. In this case, the integral operator $J(t_0)$ possesses the invertibility (Vodicka and Mantic 2008). The input function $\widetilde{U}(t)$ can be an arbitrary function. In this case, the non-homogenous equation (87) has a unique solution for $Q(t)$.

If right hand term in Eq. (87) vanishes, we obtain the following homogenous equation

$$J(t_o)=0\,,\quad (t_o\in\Gamma) \tag{91}$$

or in an explicit form

$$\int_\Gamma\left(2\kappa\, ln|t-t_o|Q(t)dt+\frac{t-t_o}{\bar{t}-\bar{t}_o}\overline{Q(t)}d\bar{t}\right)=0\,,\quad (t_o\in\Gamma) \tag{92}$$

Similarly, If the real scale has not reached the degenerate scale, Eq. (92) only has a solution $Q(t) = 0$. However, from the homogenous equation (92), we can propose the degenerate scale problem, which is same as in the interior problem.

4. Differences between formulations in the interior and the exterior problems

We may summarize the formulation of BIE in the interior and the exterior problems in Table 2. From Table 2 we see that only the $U(t)$ ($t\in\Gamma$) expressed in pure deformable form and the relevant $Q(t)$ ($t\in\Gamma$) from an elastic solution satisfy the BIE for the exterior region. However, in the interior problem any $U(t)$ and the relevant $Q(t)$ from an elastic solution satisfy governing equation.

Table 2 Comparison for BIE in the interior problem and the exterior problem

Form of governing equation	Result for substitution of an elastic solution
(1a) In interior problem $\Lambda_i(t_o)=\frac{U(t_o)}{2}+B_1 iI(t_o)=B_2 iJ(t_o)$	(1b) Any $U(t)$ and the relevant $Q(t)$ from an elastic solution satisfy governing equation
(2a) In exterior problem $\Lambda_e(t_o)=\frac{U(t_o)}{2}-B_1 iI(t_o)=-B_2 iJ(t_o)$	(2b) Only the $U(t)$ expressed in pure deformable form and the relevant $Q(t)$ from an elastic solution satisfy governing equation

where $\Lambda_i(t_o)\{U(t)\to\Lambda_i(t_o)\}$, $\Lambda_e(t_o)\{U(t)\to\Lambda_e(t_o)\}$, $J(t_o)\{Q(t)\to J(t_o)\}$

Table 3 Comparison for the Neumann problem in the interior region and the exterior region

Form of governing equation	Right hand term for boundary traction $\widetilde{Q}(t)$	Property of solution for $U(t)$
(1a) In interior problem $\Lambda_i(t_o)=\frac{U(t_o)}{2}+B_1 iI(t_o)=B_2 i\widetilde{J}(t_o)$	(1b) $\widetilde{Q}(t)$ must be in equilibrium	(1c) Many $U(t)$ solutions differing each other by a rigid motion
(2a) In exterior problem $\Lambda_e(t_o)=\frac{U(t_o)}{2}-B_1 iI(t_o)=-B_2 i\widetilde{J}(t_o)$	(2b) $\widetilde{Q}(t)$ may be arbitrary	(2c) Unique solution for $U(t)$ expressed in pure deformable form

where $\Lambda_i(t_o)\{U(t)\to\Lambda_i(t_o)\}$, $\Lambda_e(t_o)\{U(t)\to\Lambda_e(t_o)\}$, $\widetilde{J}(t_o)\{\widetilde{Q}(t)\to\widetilde{J}(t_o)\}$

Table 4 Comparison for the Dirichlet problem in the interior region and the exterior region

Forms of governing equations	Right hand term for boundary displacement $\widetilde{U}(t)$	Property of solution for $Q(t)$
(1a) In the interior problem with normal scale $B_2 iJ(t_o)=\widetilde{\Lambda}_i(t_o)$	(1b) $\widetilde{U}(t)$ may be arbitrary	(1c) Unique solution for $Q(t)$ satisfying the equilibrium condition
(2a) In the exterior problem with normal scale $-B_2 iJ(t_o)=\widetilde{\Lambda}_e(t_o)$	(2b) $\widetilde{U}(t)$ may be arbitrary	(2c) Unique solution for $Q(t)$
(3a) Degenerate scale problem $J(t_o)=0$	(3b) Vanishing right hand term	(3c) Find two non-trivial solutions for $Q(t)$ under two critical sizes

where $J(t_o)\{Q(t)\to J(t_o)\}$, $\widetilde{\Lambda}_i(t_o)\{\widetilde{U}(t)\to \widetilde{\Lambda}_i(t_o)\}$, $\widetilde{\Lambda}_e(t_o)\{\widetilde{U}(t)\to \widetilde{\Lambda}_e(t_o)\}$,

In addition, a comparison for the Neumann problem in the interior region and the exterior region is presented in Table 3. From Table 3 we see that in the interior problem the input traction $\widetilde{Q}(t)$ must be in equilibrium. However, in the exterior problem the input traction $\widetilde{Q}(t)$ may be arbitrary.

Moreover, a comparison for the Dirichlet problem in the interior region and the exterior region is presented in Table 4. From Table 4 we see that in the normal scale case, in the interior problem the input displacement $\widetilde{U}(t)$ may be arbitrary and obtained unique solution $Q(t)$ satisfies the equilibrium condition. From the interior BVP or the exterior BVP, the same degenerate scale problem can be defined.

5. Conclusions

For the interior problem, this paper provides a compact derivation for the displacement expression (22) at the domain point, which is directly from the Betti's reciprocal theorem. When $\tau \to t_o$ ($\tau \in S^+$, $t_o \in \Gamma$), the usage of the generalized Sokhotski-Plemelj's formulae shown by Eqs. (9), (10) and (11) and some results in Appendix B will yield the complex variable BIE (26). Therefore, the derivation is rigorous. Properties of the integral operators $\Lambda_i(t_o)$ and $I(t_0)$ are analyzed in detail. This will give a better understanding for BIE.

However, in the exterior problem, the identity (71) for the exterior region is obtained after dropping the term $D(t_0)$ in Eq. (69). Therefore, only the boundary displacements expressed in a pure deformable form and relevant tractions will satisfy the identity (71). Properties of the integral operators $\Lambda_e(t_o)$ are also analyzed in detail.

The differences between the BIEs for the interior problem and the exterior problem are clearly indicated. For example, the left hand term in the BIE for the interior problem is $\Lambda_i(t_0)=U(t_0)/2+B_1 iI(t_0)$, and for the exterior problem is $\Lambda_e(t_0)=U(t_0)/2-B_1 iI(t_0)$.

This situation will cause a significant difference in the solutions for the interior problem and the exterior problem.

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Appendix A

Concept of the pure deformable form of displacements

The concept of the pure deformable form of displacements plays an important role in the present study. The definition for the pure deformable form of displacements is introduced as follows

For a finite or an infinite region, the complex potentials are expressed in the form

$$\phi(z)=\sum_{k=1}^{N}\ \phi_k(z),\quad \psi(z)=\sum_{k=1}^{N}\ \psi_k(z) \tag{a1}$$

It is assumed that all pairs of complex potentials ($\phi_k(z),\psi_k(z)$) (k = 1, 2,…N) satisfy the single- valued condition of displacements. Now we can propose the following definitions.

If and only if all pairs of complex potentials ($\phi_k(z),\psi_k(z)$) (k = 1, 2,…N) cause non-vanishing stresses σ_{ij} ($\sigma_{ij} \neq 0$) anywhere in the defined region with exception of several singular points , the relevant displacements are said to be expressed in the pure deformable form.

If among all pairs of complex potentials ($\phi_k(z),\psi_k(z)$) (k = 1, 2,…N), there is at least one pair of the complex potentials (for example, $\phi_3(z),\psi_3(z)$) causes vanishing stresses σ_{ij} (σ_{ij} = 0) anywhere, and other pairs ($\phi_k(z),\psi_k(z)$) (k = 1, 2, 4,… N, no term for k = 3) cause non-vanishing stresses σ_{ij} ($\sigma_{ij} \neq 0$) anywhere in the defined region, the relevant displacements are said to be expressed in the impure deformable form.

Below, we introduce two cases to define the pure deformable form of displacements. In the first case, a finite plate is subjected to some loading along the contour. In this case, the complex potentials in finite region can be expressed in the form

$$\phi(z)=\sum_{k=0}^{\infty}\ (a_{k(r)}+ia_{k(i)})z^k$$

$$\psi(z)=\sum_{k=0}^{\infty}\ (b_{k(r)}+ib_{k(i)})z^k \tag{a2}$$

where all the coefficients $a_{k(r)}$, $a_{kir)}$, $b_{k(r)}$, $b_{k(i)}$ (k = 0, 1, 2…) take real value.

In Eq. (a2), the following five pairs of complex potentials are relating to vanishing stresses (or σ_{ij} = 0)

$$\phi(z)=ia_{1(i)}z\ ,\ \psi(z)=0\ ,\quad \text{(corresponding to a rigid rotation)} \tag{a3}$$

$$\phi(z)=a_{0(r)}\ ,\ \psi(z)=0\ ,\quad \text{(corresponding to a rigid translation)} \tag{a4}$$

$$\phi(z)=ia_{0(i)}\ ,\ \psi(z)=0\ ,\quad \text{(corresponding to a rigid translation)} \tag{a5}$$

$$\phi(z)=0\ ,\ \psi(z)=b_{0(r)}\ ,\quad \text{(corresponding to a rigid translation)} \tag{a6}$$

$$\phi(z)=0\ ,\ \psi(z)=ib_{0(i)}\ ,\quad \text{(corresponding to a rigid translation)} \tag{a7}$$

Therefore, the relevant displacements from complex potentials (a2) belong to the impure deformable form.

Alternatively, after deleting terms with the coefficients $A_{1(i)}$, $a_{0(r)}$, $a_{0(i)}$, $b_{0(r)}$, $b_{0(i)}$ in Eq. (a2), we have

$$\phi(z) = a_{1(r)}z + \sum_{k=2}^{\infty} (a_{k(r)} + ia_{k(i)})z^k$$

$$\psi(z) = (b_{1(r)} + ib_{1(i)})z + \sum_{k=2}^{\infty} (b_{k(r)} + ib_{k(i)})z^k \tag{a8}$$

Clearly, the relevant displacements from complex potentials (a8) belong to the pure deformable form.

In the second case, an infinite plate contains many voids and inclusions with the complicated loads. In this case, the complex potentials in remote place can be expressed in the form

$$\phi(z) = (A_{1(r)} + iA_{1(i)})z + (A_{2(r)} + iA_{2(i)})\ln z + a_{o(r)} + ia_{o(i)} + \sum_{k=1}^{\infty} \frac{a_{k(r)} + ia_{k(i)}}{z^k}$$

$$\psi(z) = (B_{1(r)} + iB_{1(i)})z + (B_{2(r)} + iB_{2(i)})\ln z + b_{o(r)} + ib_{o(i)} + \sum_{k=1}^{\infty} \frac{b_{k(r)} + ib_{k(i)}}{z^k} \tag{a9}$$

where all the coefficients from $A_{1(r)}$, … to $b_{k(i)}$ take real value. We assume that, the following single-valued condition of displacement, or $B_{2(r)} + iB_{2(i)} = -\kappa\,(A_{2(r)} - iA_{2(i)})$ is satisfied.

In Eq. (a9), the following five pairs of complex potentials are relating to vanishing stresses (or $\sigma_{ij} = 0$)

$$\phi(z) = iA_{1(i)}z\,,\ \psi(z) = 0\,,\ \text{(corresponding to a rigid rotation)} \tag{a10}$$

$$\phi(z) = a_{o(r)}\,,\ \psi(z) = 0\,,\ \text{(corresponding to a rigid translation)} \tag{a11}$$

$$\phi(z) = ia_{o(i)}\,,\ \psi(z) = 0\,,\ \text{(corresponding to a rigid translation)} \tag{a12}$$

$$\phi(z) = 0\,,\ \psi(z) = b_{o(r)}\,,\ \text{(corresponding to a rigid translation)} \tag{a13}$$

$$\phi(z) = 0\,,\ \psi(z) = ib_{o(i)}\,,\ \text{(corresponding to a rigid translation)} \tag{a14}$$

Therefore, the relevant displacements from complex potentials (a9) belong to the impure deformable form.

Alternatively, after deleting terms with the coefficients $A_{1(i)}$, $a_{0(r)}$, $a_{0(i)}$, $b_{0(r)}$, $b_{0(i)}$ in Eq. (a9), we have

$$\phi(z) = A_{1(r)}z + (A_{2(r)} + iA_{2(i)})\ln z + \sum_{k=1}^{\infty} \frac{a_{k(r)} + ia_{k(i)}}{z^k}$$

$$\psi(z) = (B_{1(r)} + iB_{1(i)})z + (B_{2(r)} + iB_{2(i)})\ln z + \sum_{k=1}^{\infty} \frac{b_{k(r)} + ib_{k(i)}}{z^k} \tag{a15}$$

Clearly, the relevant displacements from complex potentials (a15) belong to the pure deformable form.

In fact, the kernel $U_{ij}^{*1}(\xi,x)$ shown by Eq. (37) is the displacement in an elasticity solution caused by concentrated forces at certain point, and it has been expressed in the pure deformable form.

Appendix B

Properties for some integrals with kernel functions $L_1(t,z),L_2(t,z)$ defined by Eqs. (24) and (25)

Two integrals with the kernel functions $L_1(t,z),L_2(t,z)$, shown by Eqs. (24) and (25) are defined as follows

$$W_1(z)=\frac{1}{2\pi i}\int_\Gamma L_1(t,z)f(t)dt\,,\quad (\,z\in S^+ \text{ or } z\in S^-\,) \tag{b1}$$

$$W_2(z)=\frac{1}{2\pi i}\int_\Gamma L_2(t,z)f(t)dt\,,\quad (\,z\in S^+ \text{ or } z\in S^-\,) \tag{b2}$$

where

$$L_1(t,z)=-\frac{d}{dt}\left\{ln\frac{t-z}{\bar{t}-\bar{z}}\right\}=-\frac{1}{t-z}+\frac{1}{\bar{t}-\bar{z}}\frac{d\bar{t}}{dt} \tag{b3}$$

$$L_2(t,z)=\frac{d}{dt}\left\{\frac{t-z}{\bar{t}-\bar{z}}\right\}=\frac{1}{\bar{t}-\bar{z}}-\frac{t-z}{(\bar{t}-\bar{z})^2}\frac{d\bar{t}}{dt} \tag{b4}$$

In Eqs. (b1) and (b2), Γ denotes a closed contour and $f(t)$ is an arbitrary function. It is assumed that "dt" goes forward in an anti-clockwise direction, S^+ and S^- are the inside finite region and the outside infinite region, respectively.

In Eqs. (b1) and (b2), letting $z\to t_0$ ($z\in S^+$, $t_o\in\Gamma$) and letting $z\to t_0$ ($z\in S^-$, $t_o\in\Gamma$) , and using the generalized Sokhotski-Plemelj's formulae shown by Eqs. (9), (10) and (11), we will find

$$W_1^{\pm}(t_o)=\mp f(t_o)+\frac{1}{2\pi i}\int_\Gamma L_1(t,t_o)f(t)dt\ \ ,(\,t_o\in\Gamma\,) \tag{b5}$$

$$W_2^{\pm}(t_o)=\frac{1}{2\pi i}\int_\Gamma L_2(t,t_o)f(t)dt\ \ ,(\,t_o\in\Gamma\,) \tag{b6}$$

We can prove the assertion shown by Eq. (b5) as follows. In fact, we can rewrite $W_1(z)$ as

$$W_1(z)=I_1+I_2\ ,\quad (\,z\in S^+ \text{ or } z\in S^-\,) \tag{b7}$$

where

$$I_1(z)=\frac{1}{2\pi i}\int_\Gamma\left(-\frac{1}{t-z}\right)f(t)dt\ \ ,(\,z\in S^+ \text{ or } z\in S^-\,) \tag{b8}$$

$$I_2(z) = \frac{1}{2\pi i}\int_\Gamma \left(\frac{1}{\bar{t} - \bar{z}} \frac{d\bar{t}}{dt} \right) f(t)dt \ , (\, z \in S^+ \text{ or } z \in S^- \,) \tag{b9}$$

For convenience in derivation, we can define

$$I_3(z) = -\overline{I_2(z)} = \frac{1}{2\pi i}\int_\Gamma \frac{1}{t - z} \overline{f(t)}dt \ , (\, z \in S^+ \text{ or } z \in S^- \,) \tag{b10}$$

In Eqs. (b8) and (b10), letting $z \to t_0$ ($z \in S^+$, $t_o \in \Gamma$) and letting $z \to t_0$ ($z \in S^-$, $t_o \in \Gamma$), and using the generalized Sokhotski-Plemelj's formulae shown by Eqs. (9), (10) and (11), we will find

$$I_1^{\pm}(t_o) = \mp \frac{f(t_o)}{2} + \frac{1}{2\pi i}\int_\Gamma \left(-\frac{1}{t - z} \right) f(t)dt \ , (\, t_o \in \Gamma \,) \tag{b11}$$

$$I_3^{\pm}(t_o) = -\overline{I_2^{\pm}(t_o)} = \pm \frac{\overline{f(t_o)}}{2} + \frac{1}{2\pi i}\int_\Gamma \frac{1}{t - t_o} \overline{f(t)}dt \ , (\, t_o \in \Gamma \,) \tag{b12}$$

$$I_2^{\pm}(t_o) = -\overline{I_3^{\pm}(t_o)} = \mp \frac{f(t_o)}{2} + \frac{1}{2\pi i}\int_\Gamma \left(\frac{1}{\bar{t} - \bar{t}_o} \frac{d\bar{t}}{dt} \right) f(t)dt \ , (\, t_o \in \Gamma \,) \tag{b13}$$

From Eqs. (b7), (b11) and (b13), the validity of Eq. (b5) is proved. Similarly, we can prove the validity of Eq. (b6).