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Joint parameter identification of a cantilever beam using sub-structure synthesis and multi-linear regression

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Abstract. Complex structures are usually assembled from several substructures with joints connecting them together. These joints have significant effects on the dynamic behavior of the assembled structure and must be accurately modeled. In structural analysis, these joints are often simplified by assuming ideal boundary conditions. However, the dynamic behavior predicted on the basis of the simplified model may have significant errors. This has prompted the researchers to include the effect of joint stiffness in the structural model and to estimate the stiffness parameters using inverse dynamics. In the present work, structural joints have been modeled as a pair of translational and rotational springs and frequency equation of the overall system has been developed using sub-structure synthesis. It is shown that using first few natural frequencies of the system, one can obtain a set of over-determined system of equations involving the unknown stiffness parameters. Method of multi-linear regression is then applied to obtain the best estimate of the unknown stiffness parameters. The estimation procedure has been developed for a two parameter joint stiffness matrix.

Keywords: vibration; inverse dynamics; sub-structure synthesis; joint stiffness identification; linear parameters; multi-linear regression

1. Introduction

Modeling and dynamic response analysis of a complex structure has become a challenging task due to uncertainty in system parameters, particularly those associated with structural joints. Complex structures are usually composed of several substructures with joints to connect them together. These joints have significant effects on the behavior of the assembled structure and must be accurately modeled. In earlier analysis, joints were simplified by assuming ideal boundary conditions such as a fixed joint or a simply supported joint etc.

However, the analytical or FEM models with the simplified boundary conditions fail to predict the modal parameters accurately and often the deviation is significant enough asking for the need of proper joint modeling in the analysis. This has prompted the researchers to model the structural joint as an elastic support and to suggest suitable procedures for the joint parameter estimation.

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Among the early research works, authors (Wang and Liou 1991) have discussed a method for identification of joint properties using measured Frequency Response Functions (FRFs). The FRF sub-matrices connected with the joint co-ordinates were synthesized from two sets of measurement data for the structural system; one with the joint and another without the joint. Joint stiffness and damping parameters were then estimated from the differences of the FRF matrices. An iterative method have been suggested by some authors (Wang and Sas 1990) where in the multi degree of freedom model of a structure was converted in to several single degree of freedom systems by imposing selected eigenvector which was sensitive to a joint parameter. Joint parameters were solved by computing modal parameters of each single degree of freedom system and comparing them with measured modal parameters.

Use of model updating techniques (Friswell *et al*. 1995) in identification of unknown joint parameters initiated a series of research activities since past two decades. These research works can be broadly divided in to two areas. One is to investigate the sensitivity of modal parameters due to the joint parameters. The frequency equation of the structural systems with joint characteristics is formulated either analytically or through FEM. Second approach is to measure FRFs of the structure with the joint and comparing with the computed FRFs of the system without the joint. (Pabst and Hagedorn 1995) derived an analytical frequency equation of a cantilever beam with fixed end boundary conditions modeled through a translational and a rotational spring. The resulting nonlinear frequency equation was used to identify the stiffness values of the translational and the rotational springs from a pair of natural frequencies. They suggested measurement of more than two frequencies and then computing the joint parameters for each pair of frequencies and finally taking the average of all such computed values as the estimate of the stiffness parameters. (Mottershed *et al*. 1996) discussed a method for model updating of mechanical joints by using eigen value sensitivities to geometric parameters. The method was demonstrated for a beam with welded flange and a cantilever plate. The authors concluded that while measurements may sometime tend to be insensitive to stiffness parameters, geometric parameters have considerable potential in updating of the joint model. (Nobari *et al*. 1993) discussed the problem of stiff joints in updating process and presented a method for updating the joint model, based on accurate substructure models. (Ahmadian *et al*. 2001) simulated the effect of an elastic support using a set of nodal reaction forces on the boundary of a free structure. The nodal forces were related to support stiffness through a force-displacement relationship. The sub-system equations were combined to obtain a nonlinear frequency equation and it was demonstrated that by substituting measured natural frequency values in the equation, one can identify the joint parameters. The main advantage of the method is that no ill-conditioning occurs during the identification procedure. However, it requires the complexity of solving a nonlinear equation. (Li 2002) presented a model updating method on joint stiffness identification based on reduced order characteristics polynomial (ROCP) using natural frequency measurement only. It was observed that the accuracy in identification of rotational stiffness was less than that of the translational stiffness.

The second approach of joint identification is based on measurement of FRFs of the whole system and comparing it with joint excluded structural analytical model. (Yang and Park 1993) proposed a method based on subset frequency response function measurement. Experimentally an incomplete set of FRFs were measured and it was shown that if the number of measured FRFs are greater than or equal to number of joint related degree of freedom, then the unmeasured FRFs can be estimated from the measured FRFs and the analytical model of the structure. (Ren and Beard 1995) discussed on how to reduce the effect of measurement errors in the FRF measurement. Joints between substructures were modeled through stiffness, damping as well as mass matrices.

The matrices were identified from the FRF data by using a best solution criterion. Another identification method based on free-interface component mode synthesis method was reported by (Nobari *et al*. 1995). Further developments using sub-structure synthesis of joint matrix with structural matrices can be found in the research works (Yang *et al*. 2003, Ahmadian *et al*. 2007, Celic *et al*. 2008). Although these methods based on FRF measurements look attractive, one of the major practical constraints is the large number of FRF measurement and particularly the difficulty in measurement at the joint interface degree of freedoms.

Present work suggests a simple identification procedure based on only first few natural frequencies of the structure. The method employs the concept of Sub-Structure Synthesis (SSS) (Bishop 1960) to obtain the frequency equation of a structure connected with a joint. The frequency equation is formulated in terms of the unknown joint parameters. Only natural frequencies are measured and a set of over determined system of linear equations are constructed in terms of the unknown stiffness parameters, which are then estimated through regression analysis. The method is demonstrated for a cantilever beam with two parameter joint model, in which only the translational and the rotational spring stiffness are considered. The numerical simulation has been carried out with non-dimensional stiffness parameters so that the results are applicable for any size or dimension of the actual test specimen. The accuracy of parameter estimation has also been investigated for different range of joint parameters and a stiff region has been identified through natural frequency sensitivity study. The procedure has been also tested with frequency values perturbed with measurement error associated with typical frequency measurement instruments.

2. Sub-structure synthesis of systems joined by two interface co-ordinates

The concept of sub structure synthesis (Allen *et al.* 2010) provides a formulation for the derivation of Frequency Response Functions (FRFs) of a composite system from the knowledge of sub-system FRFs. Test data of the composite system can be used for identification of its subsystems (Sjovall Per *et al.* 2008). Figs. 1(a), (b) shows a composite system A consisting of two sub systems B and C connected by two joint co-ordinates X and θ at the interface. The forcing functions corresponding to these two co-ordinates are F and M respectively. Fig. 1(b) shows the sub-systems separated with corresponding displacements and forcing functions at the interface.

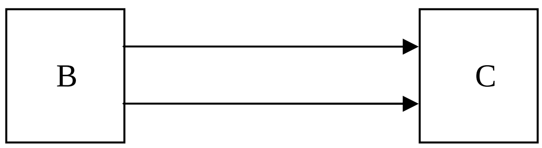


Fig. 1(a) A composite system with sub-systems B and C

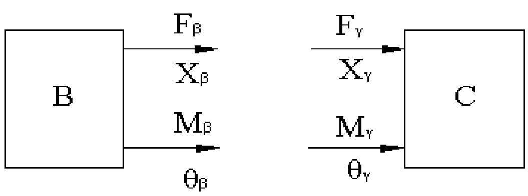


Fig. 1(b) Force and displacements acting on the sub-systems

Considering the dynamic equilibrium of the sub-systems B and C separately, one can obtain

$$X_\beta = \beta_{11}F_\beta + \beta_{12}M_\beta\,; \qquad X_\gamma = \gamma_{11}F_\gamma + \gamma_{12}M_\gamma \tag{1a}$$

$$\theta_\beta = \beta_{21}F_\beta + \beta_{22}M_\beta\,; \qquad \theta_\gamma = \gamma_{21}F_\gamma + \gamma_{22}M_\gamma \tag{1b}$$

where, β_{11}, β_{22}, γ_{11} and γ_{22} are the direct receptance functions and β_{12}, γ_{12}, β_{21}, γ_{21} are the cross receptance functions of sub-systems B & C, defined as

$$\beta_{11} = \frac{X_\beta}{F_\beta}\,;\quad \beta_{22} = \frac{\theta_\beta}{M_\beta}\,;\quad \beta_{12} = \frac{X_\beta}{M_\beta}\,;\quad \beta_{21} = \frac{\theta_\beta}{F_\beta} \tag{2a}$$

$$\gamma_{11} = \frac{X_\gamma}{F_\gamma}\,;\quad \gamma_{22} = \frac{\theta_\gamma}{M_\gamma}\,;\quad \gamma_{12} = \frac{X_\gamma}{M_\gamma}\,;\quad \gamma_{21} = \frac{\theta_\gamma}{F_\gamma} \tag{2b}$$

Eqs. (1a), (b) can be written in a matrix form as

$$\begin{Bmatrix} X_\beta \\ \theta_\beta \end{Bmatrix} = \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix} \begin{Bmatrix} F_\beta \\ M_\beta \end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix} X \\ \theta \end{Bmatrix}_\beta = [\beta] \begin{Bmatrix} F \\ M \end{Bmatrix}_\beta \tag{3}$$

and

$$\begin{Bmatrix} X_\gamma \\ \theta_\gamma \end{Bmatrix} = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} \begin{Bmatrix} F_\gamma \\ M_\gamma \end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix} X \\ \theta \end{Bmatrix}_\gamma = [\gamma] \begin{Bmatrix} F \\ M \end{Bmatrix}_\gamma \tag{4}$$

When the sub-systems are joined at the interface, the compatibility requirement of forces and displacements at the joint co-ordinate gives

$$X = X_\beta = X_\gamma \quad \text{and} \quad \theta = \theta_\beta = \theta_\gamma \tag{5a}$$

also

$$F = F_\beta + F_\gamma \quad \text{and} \quad M = M_\beta + M_\gamma \tag{5b}$$

Defining the direct and cross FRFs of composite system A as

$$\alpha_{11} = \frac{X}{F}\,;\quad \alpha_{22} = \frac{\theta}{M}\,;\quad \alpha_{12} = \frac{X}{M}\,;\quad \alpha_{22} = \frac{\theta}{F} \tag{6}$$

One obtains

$$\begin{Bmatrix} X \\ \theta \end{Bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \begin{Bmatrix} F \\ M \end{Bmatrix} = [\alpha] \begin{Bmatrix} F \\ M \end{Bmatrix} \tag{7}$$

Now, from Eqs. (3), (4) and (5)

$$\begin{Bmatrix} F \\ M \end{Bmatrix} = \begin{Bmatrix} F \\ M \end{Bmatrix}_\beta + \begin{Bmatrix} F \\ M \end{Bmatrix}_\gamma = \left\{ [\beta]^{-1} + [\gamma]^{-1} \right\} \begin{Bmatrix} X \\ \theta \end{Bmatrix} \tag{8}$$

which means

$$[\alpha] = \left\{ [\beta]^{-1} + [\gamma]^{-1} \right\}^{-1} \tag{9}$$

Simplifying Eq. (9) with Eqs. (3) and (4), receptance matrix $[\alpha]$ is obtained as

$$\alpha_{11} = \frac{\beta_{11}(\gamma_{11}.\gamma_{22} - \gamma_{12}{}^2) + \gamma_{11}(\beta_{11}.\beta_{22} - \beta_{12}{}^2)}{\Delta}$$

$$\alpha_{22} = \frac{\beta_{22}(\gamma_{11}.\gamma_{22} - \gamma_{12}{}^2) + \gamma_{22}(\beta_{11}.\beta_{22} - \beta_{12}{}^2)}{\Delta}$$

$$\alpha_{12} = \frac{\beta_{12}(\gamma_{11}.\gamma_{22} - \beta_{12}\gamma_{12}) + \gamma_{12}(\beta_{11}.\beta_{22} - \beta_{12}\gamma_{12})}{\Delta} = \alpha_{21}$$

Where

$$\Delta = (\beta_{11} + .\gamma_{11})(\beta_{22} + \gamma_{22}) - (\beta_{12} + \gamma_{12})^2 \tag{10}$$

The frequency equation for the composite system is obtained by setting the denominator of the receptance functions as zero, which gives

$$\Delta = (\beta_{11} + .\gamma_{11})(\beta_{22} + \gamma_{22}) - (\beta_{12} + \gamma_{12})^2 = 0 \tag{11}$$

Thus if the FRFs at the interface co-ordinates are known for the individual sub-systems, then the natural frequencies of the composite system can be obtained by solving the frequency equation, Eq. (11). If sub-system C is a structural component and sub-system B is a joint characterized by the joint parameters $\beta_{i,j}$, then Eq. (11) provides a basis to study the effect of joint parameters on the system natural frequencies. However, in the present work, Eq. (11) will be exploited for an inverse analysis, where one can estimate the joint parameters from the over determined set of measured frequency data, using multi-linear regression technique. In the following sections, a method is developed for a cantilever beam for characterization of its fixed end joint stiffness parameters. The joint stiffness matrix $[\beta]$is considered to be diagonal with two unknown joint parameters.

3. Joint parameter estimation of a cantilever beam

Many engineering applications such as a turbine blade, a tall chimney or a robot arm are often modeled as cantilever beams for the study of dynamic response characteristics. Their dynamic response under external excitation depends on the system natural frequencies and mode shapes, which are obtained through a computational method such as FEM, where the fixed end boundary conditions are generally taken as ideal joint. Although with this idealization of boundary condition, the computation becomes simple, measured modal parameters are often found to differ considerably from the computed one based on ideal model. A method is presented here, where the fixed end is modeled as an elastic support consisting of translational and a rotational spring. The concept of substructure synthesis, discussed in the previous section, is used to derive the frequency equation of the composite system consisting of a free-free beam interfaced with a joint at one end. An over determined system of linear equations involving the unknown joint parameters are formulated from the frequency equation using a set of measured natural frequency data. The equations are solved for best estimation of the support parameters using multi-linear regression procedure. In the model, joint sub-system is represented by the diagonal FRF matrix, in which the diagonal elements are the reciprocal of translational and rotational spring stiffness.

3.1 Estimation of two parameter joint model

In a two parameter joint system, the cantilever beam is modeled with elastic constraint at the fixed end represented by a translational spring (stiffness K_1) and a rotational spring (stiffness K_2) as shown in Fig. 2(a). The beam is considered as a composite system consisting of a free-free beam as sub-system C and the elastic springs as sub-system B, as shown in Fig. 2(b).

The FRF matrix for B is then represented by a diagonal matrix $[\beta]=\begin{bmatrix} 1/K_1 & 0 \\ 0 & 1/K_2 \end{bmatrix}$. The FRF matrix $[\gamma]$ for C can be obtained as (Bishop 1960)

$$\gamma_{11}=\left(\frac{-L^3}{EI}\right)\left(\frac{1}{\lambda^3}\right)\left(\frac{F_5}{F_3}\right);\ \gamma_{22}=\left(\frac{L}{EI}\right)\left(\frac{1}{\lambda}\right)\left(\frac{F_6}{F_3}\right)\text{ and }\gamma_{12}=\gamma_{21}=\left(\frac{L^2}{EI}\right)\left(\frac{1}{\lambda^2}\right)\left(\frac{F_1}{F_3}\right) \tag{12}$$

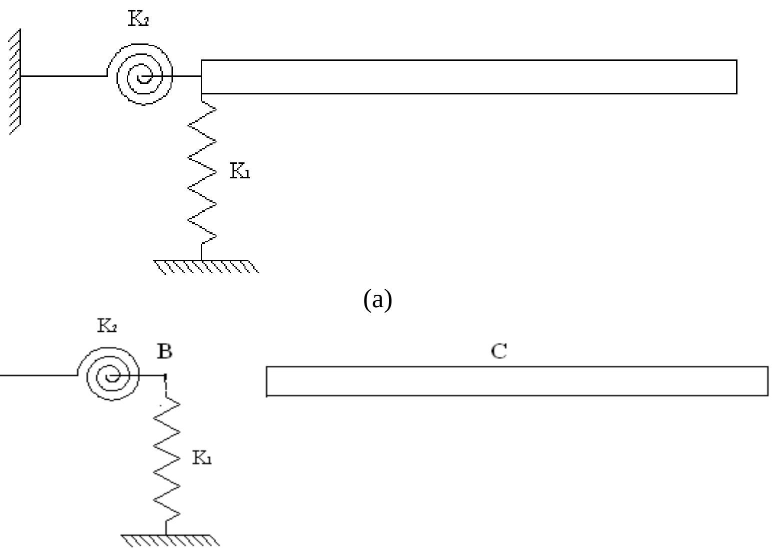
where, (EI) is bending stiffness of the beam, L is length of the beam and $\lambda=\left(\frac{\omega^2\rho A L^4}{EI}\right)^{1/4}$ in which (ρA) is the linear mass density function and ω is the natural frequency of flexural vibration of the cantilever beam. The functions F_1, F_3, F_5, F_6 are defined as (Bishop 1960)

$F_1 = sin(\lambda).sinh(\lambda)$; $F_3 = cos(\lambda).cosh(\lambda) - 1$;

$F_5 = cos(\lambda).sinh(\lambda) - sin(\lambda).cosh(\lambda)$; $F_6 = cos(\lambda).sinh(\lambda) + sin(\lambda).cosh(\lambda)$

Now, simplifying frequency equation by substituting direct and cross receptances of substructure B and C in Eq. (11), gives

$$-\left(\frac{L^2}{EI}\right)^2\left(\frac{1}{\lambda^4}\right)\left(\frac{F_4}{F_3}\right)K_1.K_2-\left(\frac{L^3}{EI}\right)\left(\frac{1}{\lambda}\right)\left(\frac{F_5}{F_3}\right)K_1+\left(\frac{L}{EI}\right)\left(\frac{1}{\lambda}\right)\left(\frac{F_6}{F_3}\right)K_2+1=0 \tag{13}$$



(b)

Fig. 2(a) Cantilever beam modeled with elastic support at fixed end, (b) cantilever beam, separated with sub system B and C where $F_4 = 1 + cos(\lambda)cosh(\lambda)$

The joint stiffness parameters can be represented as non-dimensional parameters

$K_x = \frac{K_1}{(EI/L^3)}$ and $K_y = \frac{K_2}{(EI/L\ )}$

Equation (13) then becomes

$$K_x.K_y + K_x.\lambda.\frac{F_5}{F_4} - K_y.\lambda^3.\frac{F_6}{F_4} - \lambda^4\frac{F_3}{F_4} = 0 \tag{14}$$

If we measure the first 'n' natural frequencies ω_i (i = 1, 2…..n), and λ_j, λ_k be the values corresponding to ω_j and ω_k, then

$$K_x.K_y + K_x.\lambda_j\left(\frac{F_5}{F_4}\right)_{\lambda=\lambda_j} - K_y.\lambda_j^3\left(\frac{F_6}{F_4}\right)_{\lambda=\lambda_j} - \lambda_j^4\left(\frac{F_3}{F_4}\right)_{\lambda=\lambda_j} = 0 \tag{15}$$

$$K_x.K_y + K_x.\lambda_k\left(\frac{F_5}{F_4}\right)_{\lambda=\lambda_k} - K_y.\lambda_k^3\left(\frac{F_6}{F_4}\right)_{\lambda=\lambda_k} - \lambda_k^4\left(\frac{F_3}{F_4}\right)_{\lambda=\lambda_k} = 0 \tag{16}$$

Subtracting Eq. (16) from Eq. (15), one obtains

$$\begin{aligned} K_x.\left\{\lambda_j\left(\frac{F_5}{F_4}\right)_{\lambda=\lambda_j} - \lambda_k\left(\frac{F_5}{F_4}\right)_{\lambda=\lambda_k}\right\} + K_y\left\{\lambda_k^3\left(\frac{F_6}{F_4}\right)_{\lambda=\lambda_k} - \lambda_j^3\left(\frac{F_6}{F_4}\right)_{\lambda=\lambda_j}\right\} \\ = \lambda_j^4\left(\frac{F_3}{F_4}\right)_{\lambda=\lambda_j} - \lambda_k^4\left(\frac{F_3}{F_4}\right)_{\lambda=\lambda_k} \end{aligned} \tag{17}$$

Or

$$A_i * K_x + B_i * K_y = C_i \tag{18}$$

Where A_i, B_i and C_i are the coefficients in Eq. (17) which can be computed from the pair λ_j, λ_k. For each pair of j and k, we get one equation from Eq. (18). Thus if we measure n natural frequencies, we get ${}^nC_2$ number of equations in two unknowns K_x and K_y. For n = 3, the number will be 3 and for n = 4, it will be 6. Since number of equations will be generally more than the number of unknowns, which are two here, one can employ the method of multi linear regression (Draper 1998) to get a least square error estimate of the unknown joint stiffness parameters. Taking $r = {}^nC_2$, Eq. (18) can be written as

$$\begin{bmatrix} A_1 & B_1 \\ A_2 & B_2 \\ \text{-------} & \\ A_r & B_r \end{bmatrix}\begin{Bmatrix} K_x \\ K_y \end{Bmatrix} = \begin{bmatrix} C_1 \\ C_2 \\ - \\ C_r \end{bmatrix} \quad \text{or} \quad [P]\{K\} = [Q] \tag{19}$$

The least square error criterion gives the best estimate of the joint parameter vector as

$$\{\hat{K}\} = pinv(P) * Q \tag{20}$$

Where *pinv* is the generalized inverse of a matrix

3.2 Numerical simulation

For numerical simulation, Eq. (14) is considered here again, in terms of non-dimensional stiffness parameters K_x and K_y and non-dimensional natural frequency λ given as

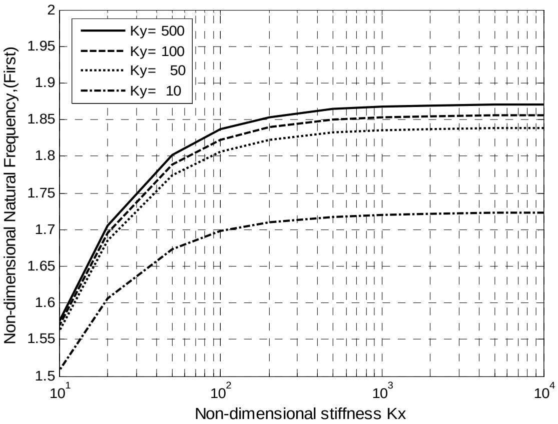
$$K_x.K_y + K_x.\lambda.\frac{F_5}{F_4} - K_y.\lambda^3.\frac{F_6}{F_4} - \lambda^4\frac{F_3}{F_4} = 0 \quad (21)$$

When K_x and K_y values are very high, the equation reduces to $F_4 = 1 + \cos(\lambda)\cosh(\lambda) = 0$, which is the frequency equation of an ideal cantilever beam, for which λ values are 1.875, 4.692, 7.855 and so on. However, in practical cases, K_x and K_y will have some finite values depending on type of the joint and λ values will be less than those of an ideal cantilever beam. For a given set of support stiffness, non-dimensional natural frequencies λ can be solved from Eq. (14). Figs. 3(a-e) show the variation of first five natural frequencies for a wide range of translational stiffness, K_x and rotational stiffness K_y.

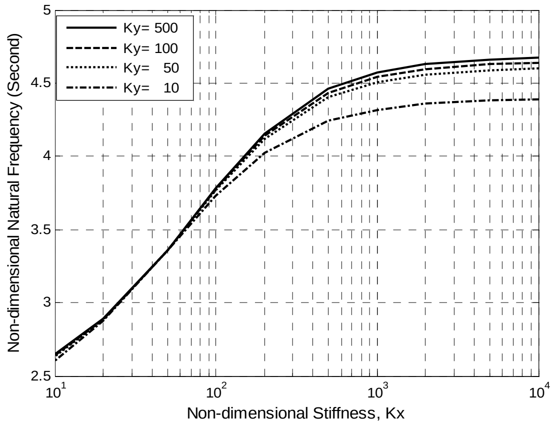
The support parameters are most important in lower modes and are less significant in higher modes (Ahmadian *et al.* 2001). From the Figs. 3(a-e) one can observe that, first three modes are significantly affected only for non-dimensional stiffness values K_x and K_y below 1000 and 500 respectively. Thus a joint having both these stiffness values in the higher range can be considered as stiff joint.

For identification of the support stiffness through Eqs. (19, 20); first five natural frequencies are computed from FE model (referred to as measured) with specified joint parameters. FE model is discretized into 40 finite elements. The dimensions of the beam are taken as Length = 1m, cross section = 2×10^{-4}m^2; density = 7800 Kg/m^3 and Young modulus $E = 2.07\times10^{11}$N/m^2. The values of support stiffness are considered corresponding to K_x = 1000, 500 and 100 and K_y = 500 and 100. The computed non-dimensional natural frequencies λ for various combinations of K_x and K_y values are given in Table 1a. Substitution of these λ values in Eq. (17) gives ten equations in two unknowns K_x and K_y, which are then estimated through multi-linear regression. Table 1b shows the estimated values and percentage error in estimation. It is observed that estimates are very much accurate, although the error is slightly more in the stiff region K_x = 1000 and K_y = 500 and above; the reason being lower frequencies are insensitive to joint parameters in stiff region (Lee *et al.* 2007).

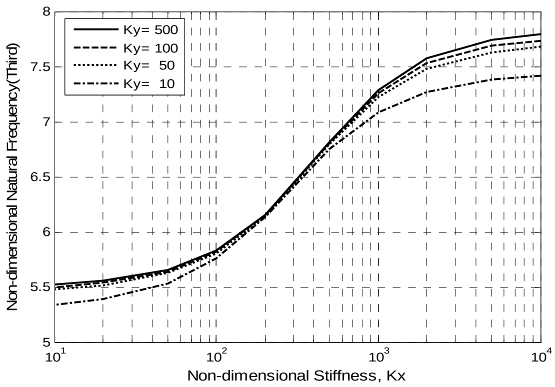
These estimates are obtained with data without any measurement error. However, for practical measurement of natural frequencies, effect of measurement noise also needs to be considered. Natural frequencies are generally measured using FFT analyzers. The major source of error with these instruments is spectral resolution error, which depends on data block size for FFT processing. For a typical 2048 data block size, 800 spectral lines are displayed in the given frequency range f, which means two adjacent frequency data will be separated by f /800. This is known as resolution error as any frequency falling within two adjacent frequency lines cannot be displayed at actual position but at one of these two frequency positions only. The error in the measurement then becomes f/2*800 = 0.0625%. With a data block size of 4096, there will be 1600 spectral lines in the frequency range and measurement error will be f/2*1600 = 0.03125%. Over and above this resolution error, there may be random noise also. Hence the frequency values considered in earlier simulation are perturbed by $\pm0.05\%$ and $\pm0.1\%$ to test the effect of measurement error on the estimation procedure. Tables 2 and 3 present the estimates of joint stiffness parameters for various combinations of K_x = 500, 100 and K_y = 100, 50, 20, 10 under $\pm0.05\%$ and $\pm0.1\%$ measurement error.



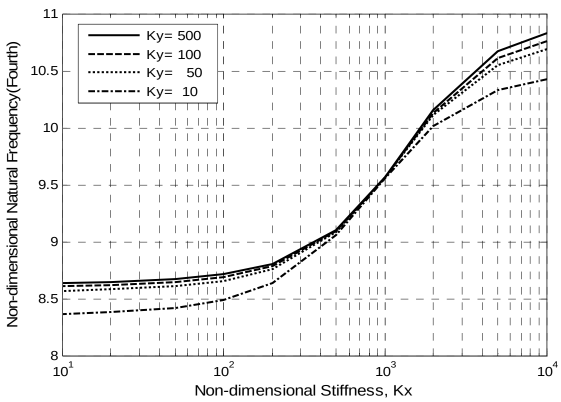
(a) Sensitivity of first natural frequency



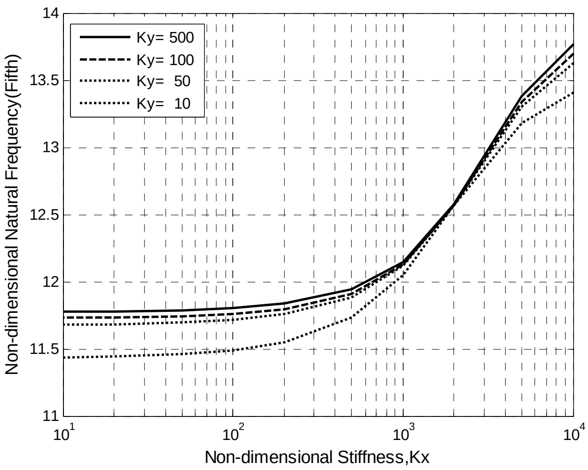
(b) Sensitivity of second natural frequency



(c) Sensitivity of third natural frequency



(d) Sensitivity of fourth natural frequency



(e) Sensitivity of fifth natural frequency

Fig. 3(a-e) Sensitivity of natural frequencies on support stiffness parameters for the cantilever beam

Table 1a Computed non-dimensional frequencies λ_i of a cantilever beam for different joint parameters using FEM

Joint Parameters	λ_1	λ_2	λ_3	λ_4	λ_5
K_x = 100 K_y = 100	1.8228	3.7746	5.8178	8.687	11.7608
K_x = 500 K_y = 100	1.8500	4.4368	6.8022	9.0929	11.9156
K_x = 1000 K_y = 100	1.8534	4.5442	7.2550	9.5648	12.1335
K_x = 1000 K_y = 500	1.8679	4.5755	7.2837	9.5658	12.1464
K_x = 2000 K_y = 1000	1.8715	4.6352	7.5772	10.1576	12.5728

Table 1b Estimates of joint stiffness parameters in absence of measurement error using Sub-Structure Synthesis (SSS) model

Joint parameters specified in FE model	Estimate of K_x with SSS model	% Error in estimation of K_x	Estimate of K_y with SSS model	% Error in estimation of K_y
K_x = 100; K_y = 100	99.8639	-0.136%	100.199	0.19%
K_x = 500; K_y = 100	500.165	0.031%	100.278	0.27%
K_x = 1000; K_y =100	1000.15	0.015 %	100.167	0.16%
K_x = 1000; K_y = 500	1015.87	1.587 %	515.831	3.16%
K_x = 2000; K_y = 1000	2132.19	6.61 %	1087.98	8.79%

Table 2a Estimates and estimation errors for K_x = 100 with 0.05% frequency perturbation

Exact values	Estimation of K_x			Estimation of K_y		
	+0.05% Freq. perturbation	-0.05% Freq. Perturbation	Average % error	+0.05% Freq. perturbation	-0.05% Freq. Perturbation	Average % error
K_x = 100 K_y = 10	100.5948	99.3243	0.64%	10.2923	9.7270	2.83%
K_x = 100 K_y = 20	101.5015	98.546	1.48%	20.7382	19.3149	3.56%
K_x = 100 K_y = 50	103.127	97.0636	3.03%	52.667	47.3155	5.35%

Table 2b Estimates and estimation errors for K_x = 100 with 0.1% frequency perturbation

Exact values	Estimation of K_x			Estimation of K_y		
	+0.1% Freq. perturbation	-0.1% Freq. Perturbation	Average % error	+0.1% Freq. perturbation	-0.1% Freq. Perturbation	Average % error
K_x = 100 K_y = 10	101.3257	98.807	1.26%	10.5864	9.4551	5.65%
K_x = 100 K_y = 20	103.1432	97.2183	2.96%	21.5059	18.6537	7.13%
K_x = 100 K_y = 50	106.745	94.37	6.18%	55.3564	44.6693	10.68%

Following observations can be made from the estimation results presented in Tables 2, 3.

1) Estimation error is more under $\pm 0.1\%$ measurement error compared to that under $\pm 0.05\%$ measurement error for any combination of K_x and K_y.

2) When the stiffness parameters are low, say K_x = 100 and K_y =10, error is very small, i.e., only 0.64% in K_x and 2.83% in K_y for $\pm 0.05\%$ measurement error. However for higher stiffness values

such as for K_x = 500 and K_y = 100, error is rather much higher, i.e., 5.80% and 10.95% respectively under ±0.05% measurement error. From Fig. 3(a), it is observed that first frequency becomes insensitive to K_x above 400 and so estimated error is more. Since lower modes are important in estimation of support stiffness parameters, accuracy in measurement of first natural frequency improves the result as shown in Table 3b.
3) At higher stiffness values, frequency sensitivity reduces and joint becomes stiff.

Table 3a Estimates and estimation errors for K_x = 500 with 0.05% frequency perturbation

Exact values	Estimation of K_x			Estimation of K_y		
	+0.05% Freq. perturbation	-0.05% Freq. Perturbation	Average % error	+0.05% Freq. perturbation	-0.05% Freq. Perturbation	Average % error
K_x = 500 K_y = 10	506.73	493.47	1.33%	10.243	9.765	2.39%
K_x = 500 K_y = 20	509.348	491.054	1.83%	20.628	19.4	3.07%
K_x = 500 K_y = 50	517.174	484.349	3.28%	53.204	47.328	5.88%
K_x = 500 K_y = 100	531.97	473.99	5.80%	112.34	90.44	10.95%

Table 3b Estimates and estimation errors for K_x = 500 and K_y =100 with 0.05% frequency perturbation in frequencies 2 to 5 only

Exact values	Estimation of K_x			Estimation of K_y		
	+0.05% Freq. perturbation	-0.05% Freq. Perturbation	Average% error	+0.05% Freq. perturbation	-0.05% Freq. Perturbation	Average % error
K_x = 500 K_y = 100	502.799	494.68	0.81%	99.08	100.34	0.74%

Thus although the estimates of joint stiffness parameters are very good in the frequency sensitive region in presence of measurement error, these estimates do get affected by measurement error in the stiff region. Although, the stiff region can be characterized by values of K_x and K_y above 1000 and 500 respectively, one gets to know this only after the parameters are estimated. From Table 1a, it can be seen that for stiffness parameters K_x and K_y equal to or above 1000 and 500, first three natural frequencies are very close to those of the ideal cantilever beam. Thus a preliminary idea of joint stiffness can be obtained from the percentage deviation of lower natural frequencies from those of the theoretical values of an ideal cantilever beam.

4. Experimental case study

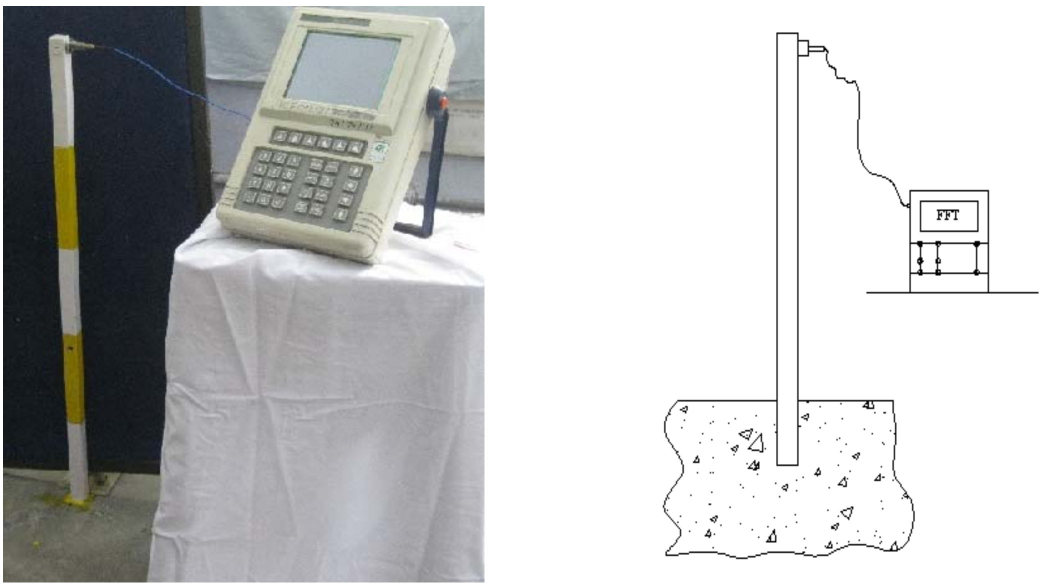
Fig. 4(a) and Fig. 4(b) shows the test set up for measuring the vibration response of a cantilever beam embedded in concrete. The beam was excited through impulse and free vibration response was measured using B&K 4370 (sensitivity 100 mV/g) accelerometer and FFT analyzer (DI-22, Diagnostic Instruments, UK).

A data block size of 2048 samples with 800 spectral lines was used to measure the natural frequencies of beam. The geometrical and material properties of beam are listed in Table 4. The material properties in terms of ratio (E/ρ) used in FE model is updated through the testing of a beam in free-free condition.

The vibration response of cantilever beam embedded in concrete is measured using FFT analyzer and is shown in Fig. 5.

The measured natural frequencies at the resonance peaks of frequency spectrum is given in Table 5**;** to improve the accuracy in measurement, different frequency bands were selected in FFT analyzer e.g. for first natural frequency a band of 0-100 Hz and for second 0-200 Hz was selected. However a frequency spectrum with 0-500 Hz frequency band is shown here.

It is observed that the measured natural frequencies are less by more than 5% compared to natural frequencies of ideal cantilever beam; so one can predict that the joint stiffness will be in sensitive region (non stiff region). The linear parameters of the joint, i.e. translational stiffness K_1 and rotational stiffness K_2 are identified using first four natural frequencies given in Table 5. Linear equation in two unknowns K_x and K_y were obtained from Eq. (14) using first four measured natural frequencies; Eq. (20) was then used for best estimate of non dimensional stiffness parameters using non dimensional frequencies. The estimated non dimensional stiffness parameters and corresponding linear joint parameters are given in Table 6.



(a) (b)

Fig. 4 (a) Experimental test set up (b) schematic diagram of beam embedded in concrete

Table 4 Dimensions and material properties of beam

Dimensions	Material Properties
Cross sectional area: (0.025 x 0.025) m^2	Elastic modulus, E : 2.075×10^{11} N/m^2
Length of beam, L : 0.797 m	Density, ρ : 7800 Kg/m^3

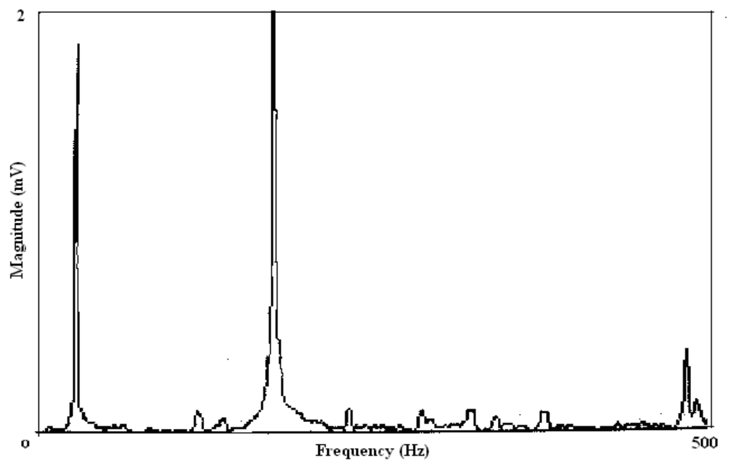


Fig. 5 Vibration response of cantilever beam using real time FFT analyzer (1 mV = 10^{-2} g)

Table 5 Measured natural frequencies of cantilever beam embedded in concrete

Mode	Natural Frequency (Hz)	Non-dimensional Natural Frequency
1	27.5	1.7183
2	176.5	4.3532
3	490.625	7.2579
4	928.5	9.9846
5	1512.5	12.7434

Table 6 The identified joint parameters

Non-dimensional joint stiffness parameters	Joint stiffness parameters
$K_x = 1902.813$	$K_1 = 2.538 \text{x} 10^7$ (N/m)
$K_y = 9.58023$	$K_2 = 8.11 \text{x} 10^4$ (N-m/rad)

Table 7 Comparison of natural frequencies measured experimentally and computed with updated FE model

Mode	f_{r1}	f_{r2}	f_{r3}	f_{r4}	f_{r5}
Measured natural frequencies	27.5 Hz	176.5 Hz	490.625 Hz	928.5 Hz	1512.5 Hz
Natural frequencies with updated FE model	27.436 Hz	176.39 Hz	490.56 Hz	928.4 Hz	1473.6 Hz

The identified stiffness parameter K_x lies outside the sensitive zone of first two modes and it is in sensitive zone of next two modes. Since lower modes are important in boundary parameter estimation, accuracy in the measurement of first two modes can improve the estimation results. Next, the identified joint parameters were incorporated in FE model and natural frequencies are computed; these frequencies are compared with experimentally measured natural frequencies,

Table 7 shows the comparison. The FE model has not been presented here for the sake of brevity.

The results of FE model agrees well with the proposed model of cantilever beam. Although only first four measured natural frequencies are considered in regression analysis, the fifth measured natural frequency is also close to the one computed with FE model. This shows that the higher natural frequencies, not considered in regression analysis, also validate the proposed algorithm.

5. Conclusions

A new procedure for joint stiffness identification has been proposed in this work. The procedure is based on natural frequency measurement and hence is very much convenient in practical applications. Using method of sub-structure synthesis, a frequency equation in terms of the joint stiffness parameters is developed for two parameter structural joint. With the measured natural frequencies, one can obtain an over determined set of equations, which is then processed through multi-linear regression to obtain the best estimates of the joint parameters. It is shown that the procedure gives accurate estimates for a wide range of stiffness values. The identification procedure is demonstrated with an experiment on cantilever beam embedded in concrete. It is seen that the experimentation results agree well with the corresponding FE model.

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