Structural Engineering and Mechanics, Vol. 45, No.3 (2013) 277-309 277
DOI: http://dx.doi.org/10.12989/sem.2013.45.3.277

Geometrically nonlinear analysis of plane frames composed of flexibly connected members

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(Received October 9, 2012, Revised January 3, 2013, Accepted January 6, 2013)

Abstract. Beam-to-column connections behaviour plays an important role in the analysis and design of steel and precast concrete structures. The paper presents a computer-based method for geometrically nonlinear frames with semi-rigid beam-to-column connections. The analytical procedure employs modified stability functions to model the effect of axial force on the stiffness of members. The member modified stiffness matrix, and the modified fixed end forces for various loads were found. The linear and nonlinear analyses were applied for two planar steel structures. The method is readily implemented on a computer using matrix structural analysis techniques and is applicable for the efficient nonlinear analysis of frameworks.

Keywords: geometrically nonlinear analysis; semi-rigid connection; plane frames

1. Introduction

It is customary in conventional analysis and design of steel and precast concrete frameworks to represent the actual joint behaviour by two extreme kinds of idealized models, i.e., the fully rigid joint model and the pinned joint model. The notions of either pinned or rigid joints are, however, simply extreme cases of true joint behaviour, and experimental investigations, many of which are referred to in (Jones *et al*. 1983), show clearly that actual joints exhibit characteristics over a wide spectrum between these extremes. The models with ideal connections simplify analysis procedure, but often cannot represent real structural behaviour. This discrepancy is reported in numerous experimental investigations of steel frames with different types of connections (Jones *et al*. 1983). The rigid connection idealization indicates that relative rotation of the connection does not exist and the end moment of the beam is entirely transferred to the columns. In contrast to the rigid connection assumption, the pinned connection idealization indicates that any restraint does exist for rotation of the connection and the connection moment is zero. Although these idealizations simplify the analysis and design process, the predicted response of the frame may be different from its real behaviour. Therefore, this idealization is not adequate as all types of connections are more or less, flexible or semi-rigid. It is proved by numerous experimental investigations that have been carried out in the past (Nethercot 1985, Davisson *et al*. 1987, Moree *et al*. 1993). The term semi-rigid is used to express the real connection behaviour. Therefore, beam-to-column

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connections in the analysis/design of steel and precast concrete frames should be described as semi-rigid connections.

Generally, nodal connections of plane frames are subjected to influence of bending moments, axial forces and shear forces. The effects of axial and shear forces can usually be ignored, and only the influence of bending moments is of practical interest. The constitutive moment-relative rotation relation, M-ϕ, depends on the particular type of connection. Most experiments have shown that the M-ϕ curve is nonlinear all the whole domain and for all types of connections. Therefore, modelling of the nodal connection is very important for the analysis and design of frame structure.

Based on experimental work due to static monotonic loading tests carried out for various types of beam-to column connections, many models have been suggested to approximate the connection behaviour. The simplest and the most common one is the linear model that has been broadly used for its simplicity (Monforton and Wu 1963, Aksogan and Akkaya 1991, Gorgun 1992, Gorgun *et al.* 2012, Gorgun and Yilmaz 2012). This approach is based on modelling the connection as a lengthless rotational spring. This method is widely used in semi-rigid analysis of frames, and the implementation of this approach requires small modifications in the existing analysis programs. This modification doe not considerably increase the computational time. Therefore, each element of the frame consists of a finite length element with a lengthless rotational spring. However, this model is good only for the low level loads, when the connection moment is quite small. In each other case, when the connection rigidity decrease compared with its initial value, a nonlinear model is necessary. Several mathematical models to describe the nonlinear behaviour of connections have been formulated and widely used in research practice (Wu and Chen 1990). Often, many authors use the so called corrective matrices to modify the conventional stiffness matrices of the beams with fully fixity at both ends (Romstad and Subramanian 1970, Frye and Morris 1975). Elements of the corrective matrices are functions of the particular nondimensional parameters-fixity factors, or rigidity index.

In addition to the linear behaviour, many studies have been developed to the nonlinear analysis of the static and dynamic behaviour of frames with semi-rigid connections using different models of geometric nonlinearity of elements and nodal connections (Xu *et al.* 2005, Aristizabal-Ochoa 2007, Liu 2009). In most studies, the effect of shear deformation and axial force on elastic behaviour has been ignored as being of little consequence. However, there are steel frameworks for which shear effects may be significant (e.g., those that have deep transfer girders (Aksogan and Dincer 1991, Aristizabal-Ochoa 2012, Gorgun *et al.* 2012). Also, in the analysis of structural systems the members forming the planar frames are general1y assumed to be rigidly connected among each other. However, more often than not the assumption of pin connections is also employed in such cases where the rigidity of the connection cannot be provided to a dependable degree. In fact, both of the foregoing assumptions are unrealistic when one is treating steel frames and especially, nowadays, widely used precast reinforced concrete structures. In such structures beams and columns behave as if they are semi-rigidly, or flexibly, connected among themselves, as far as the rotations of the ends are concerned. Hence, experimentally determined effective rotational spring constants for those connections should be used in the analyses of such structures. This paper presents a computer-based method for geometrically nonlinear analysis of planar frameworks with semi-rigid connections to explicitly account for the influence of axial force on elastic behaviour. Stability functions are employed to model the effect of axial force on the elastic bending stiffness of members (Livesley and Chandler 1956, Majid 1972, Chen and Lui 1991), and the influence of semi-rigid connections is taken into account. The shear-stiff stability functions presented in (Livesley and Chandler 1956, Majid 1972, Chen and Lui 1991) are modified to take shear deformability into account for comparison. The history of the stability functions for shear-flexible members is given in (Al-Sarraf 1986, Mottram 2008).

The geometrically nonlinear elastic analysis procedure is a direct extension of the conventional matrix displacement method of linear-elastic analysis. The nonlinear analysis method is verified for three example benchmark steel structures from the literature (Aksogan and Dincer 1991, Aksogan and Akkaya 1991, Gorgun 1992, Aristizabal-Ochoa 2012, Gorgun *et al.* 2012).

The present study is an attempt to prepare a computer program that treats the aforementioned type of structures elegantly, taking into consideration the behaviour of the flexible connections on elastic behaviour along with the effect of geometric nonlinearity due to the axial forces in the members. As is well known, the upper limit of the load in any structure is the critical value of the load, the buckling load, which is found by taking geometric nonlinearity into consideration. Hence, the results of the present study will constitute the basis of the stability analysis of the same type of structures.

The method used in the present study is the well-known stiffness method of structural analysis. First, the stiffness matrix of a member elastically supported against rotation at both ends is obtained using the second order analysis. Then, the fixed end forces are found for a member elastically supported at the two ends by rotational springs for a uniformly distributed load, a concentrated load, a linearly distributed load, a symmetrical trapezoidal distributed load and an unsymmetrical triangular distributed load. For the latter analysis, the second order theory was employed once again, along with the use of differential equations which yielded trigonometric functions for the case of axial compressive force and hyperbolic functions for the case of axial tensile force.

The computer program that was prepared can be used to solve static problems of plane frames composed of members that are semi-rigidly connected at the joints.

2. Analysis model

This study concerns planar steel and precast concrete frameworks discretized as an assembly of beam-column members that beams flexibly connected to columns taking into account the effect of axial deformations. It is assumed that there are no out-of-plane actions, and bending, shearing or axial deformation (ϕ, γ or δ) under the action of moment, shear or axial force (M, V or P) is concentrated at member sections.

The present study is mainly composed of two parts. The first part is comprised of the analytical study that employs the matrix method which is commonly used in structural analysis. In this part, the stiffness matrix of the structure of concern is obtained, the contributions of different types of loads to the loading vector are found and the formulation of the equilibrium equations for the determination of the unknown displacements is explained. Actually, besides the more complicated type of functions compared to linear analysis, there is also a need for separate analyses for compressive and tensile axial forces which doubles the analytical work. In the second part of the study the pertinent computer program was prepared.

In the present study, the method used being the matrix stiffness method the main concern is to set up the relation between the loading and the displacement vectors of a given structure.

To accomplish this, the first thing to be done is to find the relation between the end forces and the end deflections for a prismatic planar beam-column member. The terms “force” and “deflection” are taken to be general expressions signifying direct forces and moments, and linear deflections and rotations respectively. Towards this end we must first define the sign convention and notation which is done in Fig. 1 where positive senses of the entities at the two ends in the axial, transverse and rotational directions are shown with the arrows numbered from one to six. The left and the right ends of the member are also shown along with the corresponding spring constants, which express the ratio of flexural stiffness of connection to flexural stiffness of beam to which it is attached. The lengths of the springs are supposed to be zero. The physical properties of the member are designated in the conventional manner-E, G, L, I, A and A_s denote Young's modulus, shear modulus, length, cross-sectional moment of inertia, cross-sectional area and equivalent shear area respectively; while p_i, d_i and f_i (i = 1, 2, …, 6) are local axis member-end forces, deformations and fixed end forces, respectively. k_1 and k_2 are the constants of the rotational springs at the left and the right ends, respectively, The member is perfectly straight, and uniform in cross-section throughout its length. The material of the member is linearly elastic.

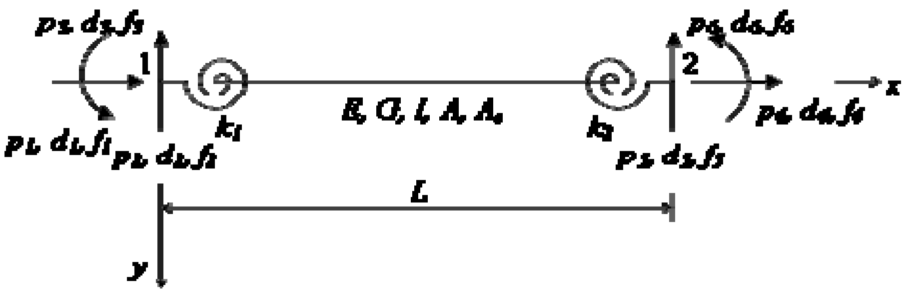

Fig. 1 Beam-column member model

2.1 Modified stiffness matrix of a flexibly connected member

In order to obtain a force-displacement relationship of a beam-column member with semi-rigid connections, the superposition method cannot be applied. The force-deformation relationship for the beam-column member in Fig. 1 is

$$p = kd + f \tag{1}$$

where the vectors of end-section forces $p = [p_1, p_2, \ldots p_6]^T$, deformations $d = [d_1, d_2, \ldots d_6]^T$ and fixed end forces due to intermediate loads between joints $f = [f_1, f_2, \ldots f_6]^T$ are referenced to the local-axis system for the member, and the local-axis stiffness matrix k for the member is a six by six matrix.

The shear contribution in the entire deflection of a beam element as treated in the ordinary small deflection elastic theory is very simple; and, it is very small compared with the flexural deflection.

Letting $y = y_m + y_s$ show the entire downwards deflection of a beam-column member in Fig. 2, the deflection due to bending only is shown by y_m and that due to shear is shown by y_s, and x show the distance from the left end of the member, one can find the different elements of the stiffness matrix by taking each and every end displacement to be unity at a time, when the others are zero and solving the differential equation.

$$y'' = y''_m + y''_s = -\frac{M}{EI\left(1 - P/GA_s\right)} \tag{2}$$

where a prime shows a derivative with respect to x and EI is the flexural rigidity of the member.

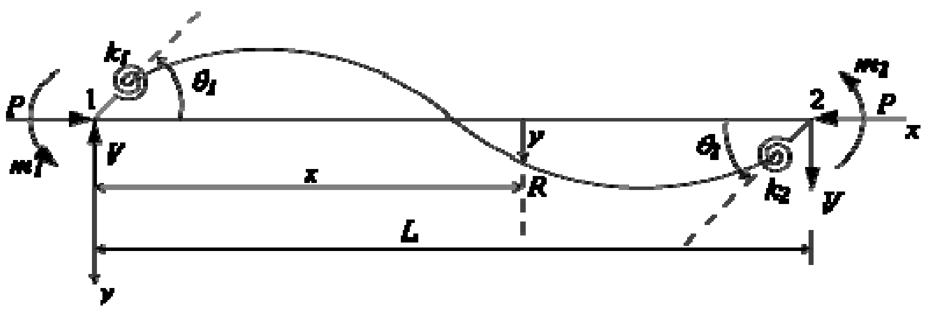

Fig. 2 Notation for beam-column member with axial force

When there is an axial force P, the bending moment M at some representative point R in Fig. 2, distant x from the left-hand end

$$M = \pm Py + Vx - m_1 \tag{3}$$

where P is the absolute value of the axial force in the member and the sign in front of it in Eq. (3) is positive for compression and negative for tension, V is the end shear force, m_1 is the modified fixed end moment at $x = 0$, Defining

$$\alpha = \begin{cases} \sqrt{\dfrac{P/EI}{\left(1 - P/GA_s\right)}} & P < 0 \\ \sqrt{\dfrac{P/EI}{\left(1 + P/GA_s\right)}} & P > 0 \end{cases} \tag{4a}$$

It is readily possible to conduct the same analysis using Euler-Bernoulli beam theory, which ignores the effect of shear deformation on elastic behaviour, by setting the beam-column member shear stiffness $Ga_s = \infty$ in Eqs. (2) and (4a). Now

$$\alpha = \sqrt{P/EI} \tag{4b}$$

the general solution of Eq. (2) is

$$y = A\sin\left(\alpha x\right) + B\cos\left(\alpha x\right) - \frac{V}{P}x + \frac{m_1}{P} \tag{5}$$

for axial compressive force.

When the axial force is tensile and the first term in the bending moment expression in Eq. (3) changes sign, then the general solution of Eq. (2) is again given by Eq. (5) only changing the signs of the last two terms and the trigonometric functions to their corresponding hyperbolic ones. Assigning the unit end displacements to the outer ends of the springs, each at a time and using the equilibrium equations for the free body diagrams of the members along with Eq. (5) and the suitable boundary conditions for the displacements and slopes at the inner ends of the springs, the local–axis stiffness matrix for the member is

$$k = \begin{bmatrix} k_{11} & 0 & 0 & k_{14} & 0 & 0 \\ & k_{22} & k_{23} & 0 & k_{25} & k_{26} \\ & & k_{33} & 0 & k_{35} & k_{36} \\ & & & k_{44} & 0 & 0 \\ & Sym & & & k_{55} & k_{56} \\ & & & & & k_{66} \end{bmatrix} \tag{6}$$

The effects of the flexible connections are included in the stiffness matrix by modifying the stiffness terms of frame member with rigid connections. The stiffness influence coefficients k_{ij} (i = 1, 2, …6 : j = 1, 2, …6) in Eq. (6) take into account the influence that axial force, and semi-rigid connections have on elastic bending stiffness and are defined as follows

$$k_{11} = k_{11}^{a} = k_{44} = -k_{14} = -k_{41} \tag{7a}$$

$$k_{22} = k_{22}^{r} = k_{55} = -k_{25} = -k_{52} \tag{7b}$$

$$k_{23} = k_{23}^{r} = k_{32} = -k_{35} = -k_{53} \tag{7c}$$

$$k_{26} = k_{26}^{r} = k_{62} = -k_{56} = -k_{65} \tag{7d}$$

$$k_{33} = k_{33}^{r} \tag{7e}$$

$$k_{36} = k_{36}^{r} = k_{63} \tag{7f}$$

$$k_{66} = k_{66}^{r} \tag{7g}$$

In Eq. (7a), $k_{11}^{a} = EA/L$ is elastic axial stiffness. In Eqs. 7(b)-(g), the stiffness influence coefficients; if the axial force in the member is zero (linear solution), $P = 0$

$$k_{22}^{r} = \frac{12EI}{L^3}\frac{(1+\beta_1+\beta_2)}{1+4(\beta_1+\beta_2+3\beta_1\beta_2)} \tag{8}$$

$$k_{23}^{r} = \frac{6EI}{L^2}\frac{(1+2\beta_2)}{1+4(\beta_1+\beta_2+3\beta_1\beta_2)} \tag{9}$$

$$k_{26}^{r} = \frac{6EI}{L^2} \frac{\left(1+2\beta_1\right)}{1+4\left(\beta_1+\beta_2+3\beta_1\beta_2\right)} \tag{10}$$

$$k_{33}^{r} = \frac{4EI}{L} \frac{\left(1+3\beta_2\right)}{1+4\left(\beta_1+\beta_2+3\beta_1\beta_2\right)} \tag{11}$$

$$k_{36}^{r} = \frac{2EI}{L} \frac{1}{1+4\left(\beta_1+\beta_2+3\beta_1\beta_2\right)} \tag{12}$$

$$k_{66}^{r} = \frac{4EI}{L} \frac{\left(1+3\beta_1\right)}{1+4\left(\beta_1+\beta_2+3\beta_1\beta_2\right)} \tag{13}$$

where, for compressive axial force, $P < 0$

$$k_{22}^{r} = \frac{EI}{L^3\Omega}\psi^3\delta^2\left\{\left(1-\psi^2\beta_1\beta_2\right)\sin\psi + \psi\left(\beta_1+\beta_2\right)\cos\psi\right\} \tag{14}$$

$$k_{23}^{r} = \frac{EI}{L^2\Omega}\psi^2\delta\left(\psi\beta_2\sin\psi - \cos\psi + 1\right) \tag{15}$$

$$k_{26}^{r} = \frac{EI}{L^2\Omega}\psi^2\delta\left(\psi\beta_1\sin\psi - \cos\psi + 1\right) \tag{16}$$

$$k_{33}^{r} = \frac{EI}{L\Omega}\psi\left\{\left(1+\psi^2\delta\beta_2\right)\sin\psi - \psi\delta\cos\psi\right\} \tag{17}$$

$$k_{36}^{r} = \frac{EI}{L\Omega}\psi\left(\psi\delta - \sin\psi\right) \tag{18}$$

$$k_{66}^{r} = \frac{EI}{L\Omega}\psi\left\{\left(1+\psi^2\delta\beta_1\right)\sin\psi - \psi\delta\cos\psi\right\} \tag{19}$$

and for the tensile axial force; $P > 0$

$$k_{22}^{r} = \frac{EI}{L^3\Omega}\psi^3\delta^2\left\{\left(1+\psi^2\beta_1\beta_2\right)\sinh\psi + \psi\left(\beta_1+\beta_2\right)\cosh\psi\right\} \tag{20}$$

$$k_{23}^{r} = \frac{EI}{L^2\Omega}\psi^2\delta\left(\psi\beta_2\sinh\psi + \cosh\psi - 1\right) \tag{21}$$

$$k_{26}^{r} = \frac{EI}{L^2\Omega}\psi^2\delta\left(\psi\beta_1\sinh\psi + \cosh\psi - 1\right) \tag{22}$$

$$k_{33}^{r} = -\frac{EI}{L\Omega}\psi\left\{\left(1-\psi^2\delta\beta_2\right)\sinh\psi - \psi\delta\cosh\psi\right\} \tag{23}$$

$$k_{36}^{r} = -\frac{EI}{L\Omega}\psi\left(\psi\delta - \sinh\psi\right) \tag{24}$$

$$k_{66}^{r} = -\frac{EI}{L\Omega}\psi\left\{\left(1-\psi^{2}\delta\beta_{1}\right)\sinh\psi - \psi\delta\cosh\psi\right\} \tag{25}$$

account for elastic bending stiffness.
In Eqs. (2), 4(a), (8)-(25), the parameters

$$\beta = \frac{EI}{L^{2}GA_{s}},\quad GA_{s} = \infty \rightarrow \beta = 0 \quad \text{(Neglecting effect of shear deformation)} \tag{26}$$

$$\beta_{1} = \frac{1}{4k_{1}} \tag{27}$$

$$\beta_{2} = \frac{1}{4k_{2}} \tag{28}$$

$$\Omega = \begin{cases} \psi\left\{\delta\left(\psi^{2}\beta_{1}\beta_{2}-1\right)+\beta_{1}+\beta_{2}\right\}\sin\psi - \left\{2+\psi^{2}\delta\left(\beta_{1}+\beta_{2}\right)\right\}\cos\psi + 2 & P<0 \\ \psi\left\{\delta\left(\psi^{2}\beta_{1}\beta_{2}+1\right)-\beta_{1}-\beta_{2}\right\}\sinh\psi - \left\{2-\psi^{2}\delta\left(\beta_{1}+\beta_{2}\right)\right\}\cosh\psi + 2 & P>0 \end{cases} \tag{29}$$

in which

$$\psi = \begin{cases} L\sqrt{\dfrac{P/EI}{\left(1-P/GA_{s}\right)}} & P<0 \\ L\sqrt{\dfrac{P/EI}{\left(1+P/GA_{s}\right)}} & P>0 \end{cases},\quad GA_{s} = \infty \rightarrow \psi = L\sqrt{P/EI} \tag{30a}$$

$$\delta = \begin{cases} 1-P/GA_{s} & P<0 \\ 1+P/GA_{s} & P>0 \end{cases},\quad GA_{s} = \infty \rightarrow \delta = 1.0 \tag{30b}$$

are well-known stability functions that account for the influence of axial force on elastic bending stiffness. The effect of axial forces on the deformed shape of the member are included in the stiffness matrix by using modified stability functions.
Finally, in Eqs. (27)-(28), the dimensionless parameters for the ends, 1 and 2, of the member

$$k_{1} = \frac{J_{1}}{4EI/L} \tag{31}$$

$$k_{2} = \frac{J_{2}}{4EI/L} \tag{32}$$

where J_1 and J_2 are the stiffness of the flexible connections at the ends of the member and $4EI/L$ is the stiffness of the member (defined only as the moment required to cause unit rotation of one of its ends).

$$J_1 = \frac{M_{con1}}{\phi} \tag{33}$$

$$J_2 = \frac{M_{con2}}{\phi} \tag{34}$$

where M_{con1} and M_{con2} are the hogging bending moments of the flexible connections at the ends of the member. This assumes a linear moment-rotation relationship and the connection stiffness, J, is the slope of this relationship. The values of k_1 and k_2 depend on the known semi-rigid connection stiffness and the geometrical and elastic properties of the connected member. They vary from zero for a frictionless pin connection to infinity for a perfectly rigid connection. Eq.s (31) and (32) are for the general case of unequal connection stiffness. Usual steel building frames will have identical connections at both girder ends, although exterior and interior connections may act differently, and the analysis will then deal with equal stiffnesses, $J = J_1 = J_2$.
The stiffness influence coefficients k_{ij} (i = 1, 2, …6 : j = 1, 2, …6) in Eq. (6) take into account the influence that axial force, and semi-rigid connections have on elastic bending stiffness including shear effects are given in Gorgun *et al.* (2012).

2.2 Modified fixed end moments

So far only structures loaded at joints have been considered, but in rigid jointed structures this is generally not the case. In order to deal with this problem, the whole solution process must be reviewed. In the analysis of skeletal structures by the stiffness method it was observed that the loading vector might contain fixed-end forces due to loads applied between joints. It is found that the presence of an axial load, shear force, and the influence of semi-rigid connections in a member affects the values of the fixed-end forces, and this is summarised in this section.

Concerning fixed end forces for numerous types of span loadings, although the computations involved are rather tedious, the method of approach is straightforward and simple. What needs to be done in each case is to employ the method used for finding the stiffness matrix, namely apply Eq. (2) where bending moment M given by Eq. (3), is expressed with an additional term or terms due to the span loading and the force V at the left end is found by using the moment equilibrium equation relative to the right end. Moreover, for the case of symmetrical trapezoidal distributed load, by making use of symmetry, the mid-span slope was taken to be zero. The corresponding transverse forces can be found by making use of the two equations of equilibrium for the member. The moments at the elastically restrained ends of a loaded member for some frequently encountered loads found for linear and nonlinear cases are presented as follows with the notation given in the relative figures.

Uniformly distributed load. Fig. 3 shows an elastically restrained member of length L and uniform flexural EI, loaded with a uniformly distributed load of intensity w per unit length over the whole span. The modified fixed end moments on the member ends due to a uniform downward load, w, are

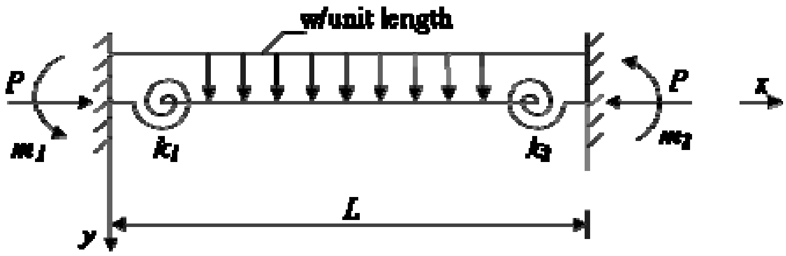

Fig. 3 Uniformly distributed load

$$m_1 = \begin{cases} \dfrac{wL^2}{12}\dfrac{(1+6\beta_2)}{1+4(\beta_1+\beta_2+3\beta_1\beta_2)} & P=0 \\ \dfrac{wL^2}{2\psi^2\Omega}\left\{\psi\left[4+\psi^2\beta_2\right]\sin\psi+\left[4-\psi^2(1-2\beta_2)\right]\cos\psi-\left[4+\psi^2(1+2\beta_2)\right]\right\} & P<0 \\ -\dfrac{wL^2}{2\psi^2\Omega}\left\{\psi\left[4-\psi^2\beta_2\right]\sinh\psi-\left[4+\psi^2(1-2\beta_2)\right]\cosh\psi+\left[4-\psi^2(1+2\beta_2)\right]\right\} & P>0 \end{cases} \qquad (35a)$$

$$m_2 = -\begin{cases} \dfrac{wL^2}{12}\dfrac{(1+12\beta+6\beta_1)}{1+4(\beta_1+\beta_2+3\beta_1\beta_2)} & P=0 \\ \dfrac{wL^2}{2\psi^2\Omega}\left\{\psi\left[4+\psi^2\beta_1\right]\sin\psi+\left[4-\psi^2(1-2\beta_1)\right]\cos\psi-\left[4+\psi^2(1+2\beta_1)\right]\right\} & P<0 \\ \dfrac{wL^2}{2\psi^2\Omega}\left\{\psi\left[4-\psi^2\beta_1\right]\sinh\psi+\left[4+\psi^2(1-2\beta_1)\right]\cosh\psi-\left[4-\psi^2(1+2\beta_1)\right]\right\} & P>0 \end{cases} \qquad (35b)$$

Concentrated load at any point. Modified fixed end moments in the same uniform member of length L by an unsymmetrical point load of W as shown in Fig. 4.

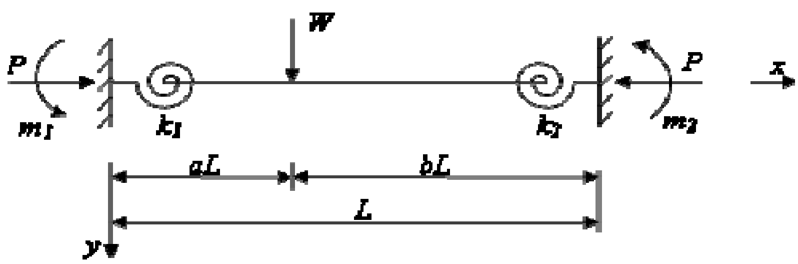

Fig. 4 Single-point load

$$m_1 = \begin{cases} WLa\dfrac{b\left\{b+2\beta_2\left(b+1\right)\right\}}{1+4\left(\beta_1+\beta_2+3\beta_1\beta_2\right)} & P=0 \\ \dfrac{WL}{\psi\Omega}\left\{\left(\left(1+\psi^2 b\beta_2\right)\sin\psi - b\psi\cos\psi - \sin a\psi - a\psi\right)\right. & \\ \left.-\left(1+\psi^2\beta_2\right)\sin b\psi + \psi\cos b\psi\right\} & P<0 \\ -\dfrac{WL}{\psi\Omega}\left\{\left(\left(1-\psi^2 b\beta_2\right)\sinh\psi - b\psi\cosh\psi - \sinh a\psi - a\psi\right)\right. & \\ \left.-\left(1-\psi^2\beta_2\right)\sinh b\psi + \psi\cosh b\psi\right\} & P>0 \end{cases} \tag{36a}$$

$$m_2 = -\begin{cases} WLb\dfrac{a\left\{a+2\beta_1\left(a+1\right)\right\}}{1+4\left(\beta_1+\beta_2+3\beta_1\beta_2\right)} & P=0 \\ \dfrac{WL}{\psi\Omega}\left\{\left(\left(1+\psi^2 a\beta_1\right)\sin\psi - a\psi\cos\psi - \sin b\psi - b\psi\right)\right. & \\ \left.-\left(1+\psi^2\beta_1\right)\sin a\psi + \psi\cos a\psi\right\} & P<0 \\ -\dfrac{WL}{\psi\Omega}\left\{\left(\left(1-\psi^2 b\beta\right)\sinh\psi - a\psi\cosh\psi - \sinh b\psi - b\psi\right)\right. & \\ \left.-\left(1-\psi^2\beta_1\right)\sinh a\psi + \psi\cosh a\psi\right\} & P>0 \end{cases} \tag{36b}$$

When $a = b = 0.5$, Eqs. 36(a)-(b), give the values for the concentrated load at midspan.

linear variation of load. In Fig. 5, for example, the same uniform member is shown loaded by a total load W, which is distributed with an intensity varying linearly from w_1 at the left-hand end to w_2 at the right.

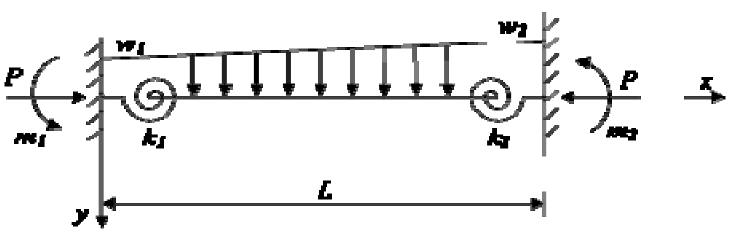

Fig. 5 Linear variation of load

$$m_1 = \begin{cases} \dfrac{L^2}{60}\dfrac{\left\{\left(3w_1+2w_2\right)+2\left(8w_1+7w_2\right)\beta_2\right\}}{1+4\left(\beta_1+\beta_2+3\beta_1\beta_2\right)} & P=0 \\ \dfrac{\left(w_1+w_2\right)L^2}{2\psi^2\Omega}\left\{\psi\left[4+\psi^2\beta_2\right]\sin\psi\right. & \\ \left.+\left[4-\psi^2\left(1-2\beta_2\right)\right]\cos\psi-\left[4+\psi^2\left(1+2\beta_2\right)\right]\right\} & P<0 \\ -\dfrac{\left(w_1+w_2\right)L^2}{2\psi^2\Omega}\left\{\psi\left[4-\psi^2\beta_2\right]\sinh\psi\right. & \\ \left.+\left[4-\psi^2\left(1-2\beta_2\right)\right]\cosh\psi-\left[4+\psi^2\left(1+2\beta_2\right)\right]\right\} & P>0 \end{cases} \tag{37a}$$

$$m_2 = -\begin{cases} \dfrac{L^2}{60}\dfrac{\left\{3w_2+2w_1+2\left(8w_2+7w_1\right)\beta_1\right\}}{1+4\left(\beta_1+\beta_2+3\beta_1\beta_2\right)} & P=0 \\ \dfrac{\left(w_1+w_2\right)L^2}{2\psi^2\Omega}\left\{\psi\left[4+\psi^2\beta_1\right]\sin\psi\right. & \\ \left.+\left[4-\psi^2\left(1-2\beta_1\right)\right]\cos\psi-\left[4+\psi^2\left(1+2\beta_1\right)\right]\right\} & P<0 \\ -\dfrac{\left(w_1+w_2\right)L^2}{2\psi^2\Omega}\left\{\psi\left[4-\psi^2\beta_1\right]\sinh\psi\right. & \\ \left.+\left[4-\psi^2\left(1-2\beta_1\right)\right]\cosh\psi-\left[4+\psi^2\left(1+2\beta_1\right)\right]\right\} & P>0 \end{cases} \tag{37b}$$

Symmetrical trapezoidal load (See Fig. 6):

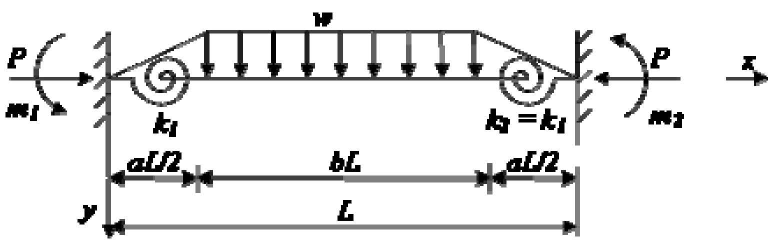

Fig. 6 Symmetrical trapezoidal load

$$m_1 = \begin{cases} \dfrac{wL^2}{96}\dfrac{\left\{8+a^2(a-4)\right\}}{(1+2\beta_1)} & P=0 \\ \dfrac{wL^2}{4a\psi^3}\dfrac{\left\{\left[a\psi^2(a-2)-8\right]\cos(\psi/2)+8\cos b(\psi/2)\right\}}{\sin(\psi/2)+\psi\beta_1\cos(\psi/2)} & P<0 \\ -\dfrac{wL^2}{4a\psi^3}\dfrac{\left\{\left[a\psi^2(a-2)+8\right]\cosh(\psi/2)-8\cosh b(\psi/2)\right\}}{\sinh(\psi/2)+\psi\beta_1\cosh(\psi/2)} & P>0 \end{cases} \tag{38a}$$

$$m_2 = -\begin{cases} \dfrac{wL^2}{96}\dfrac{\left\{8+a^2(a-4)\right\}}{(1+2\beta_2)} & P=0 \\ \dfrac{wL^2}{4a\psi^3}\dfrac{\left\{\left[a\psi^2(a-2)-8\right]\cos(\psi/2)+8\cos b(\psi/2)\right\}}{\sin(\psi/2)+\psi\beta_2\cos(\psi/2)} & P<0 \\ -\dfrac{wL^2}{4a\psi^3}\dfrac{\left\{\left[a\psi^2(a-2)+8\right]\cosh(\psi/2)-8\cosh b(\psi/2)\right\}}{\sinh(\psi/2)+\psi\beta_2\cosh(\psi/2)} & P>0 \end{cases} \tag{38b}$$

$$m_1 = \begin{cases} \dfrac{wL^2}{60}\dfrac{\left\{3a^3-7a^2+3a+3+\left(6a^3-24a^2+16a+16\right)\beta_2\right\}}{1+4\left(\beta_1+\beta_2+3\beta_1\beta_2\right)} & P=0 \\ \dfrac{wL^2}{\psi\Omega}\left\{H1+H2+H3\right\} & P\neq 0 \end{cases} \tag{39a}$$

Triangular load. Determined the fixed-end moments in the uniform member shown in the Fig. 7, when subjected to an unsymmetrical load, with a linear variation of intensity but of total $wL/2$.

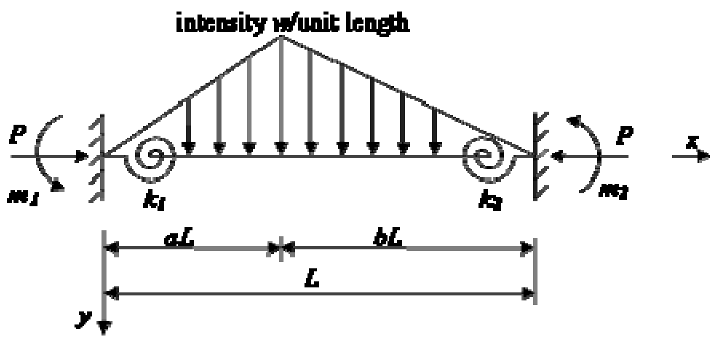

Fig. 7 Triangular load

$$m_2 = -\begin{cases} \dfrac{wL^2}{60}\dfrac{\left\{3b^3 - 7b^2 + 3b + 3 + \left(6b^3 - 24b^2 + 16b + 16\right)\beta_1\right\}}{1 + 4\left(\beta_1 + \beta_2 + 3\beta_1\beta_2\right)} & P = 0 \\ \dfrac{wL^2}{\psi\Omega}\left\{H1 + H2 + H3\right\} & P \neq 0 \end{cases} \tag{39b}$$

in which

$$\left.\begin{aligned} H1 &= \left(\frac{1}{b\psi^2} + \frac{1}{3} - \frac{b}{6}\right)\left[\sin\psi - \psi\right] \\ H2 &= -\frac{1}{ab\psi^2}\left[\sin a\psi + \left(1 + \psi^2\beta_2\right)\sin b\psi - \psi\cos b\psi\right] \\ H3 &= \left(\frac{1}{a\psi^2} + \frac{1}{3} - \frac{a}{6}\right)\left[\left(1 + \psi^2\beta_2\right)\sin\psi - \psi\cos\psi\right] \end{aligned}\right\} \quad P < 0 \tag{40a}$$

$$\left.\begin{aligned} H1 &= \left(\frac{1}{b\psi^2} - \frac{1}{3} + \frac{b}{6}\right)\left[\sinh\psi - \psi\right] \\ H2 &= -\frac{1}{ab\psi^2}\left[\sinh a\psi + \left(1 - \psi^2\beta_2\right)\sinh b\psi - \psi\cosh b\psi\right] \\ H3 &= \left(\frac{1}{a\psi^2} - \frac{1}{3} + \frac{a}{6}\right)\left[\left(1 - \psi^2\beta_2\right)\sinh\psi - \psi\cosh\psi\right] \end{aligned}\right\} \quad P > 0 \tag{40b}$$

The remaining other two nonzero modified fixed end forces, the shear forces and the axial forces, of relevance at the ends are found using static equilibrium equations. The modified fixed end moments at the right ends for above frequently encountered loads for linear and nonlinear cases are found either from symmetry or by an interchange of a and b, β_1 and β_2 or w values at the two ends, and the sign in front of it negative.

3. Analysis procedure

The geometrically nonlinear analysis is an iterative procedure that, for each iteration, involves formulating and solving the equilibrium equations

$$KD = F \tag{41}$$

where K, global stiffness matrix; D, vector of nodal displacements; F, vector of specified (equivalent) nodal loads.

IT = 1 is an initially specified value selected to ensure that first-order linear-elastic behaviour of the structure for the first iteration.

If the structure stiffness matrix K is non-singular at the end of an iteration, Eq. (41) are solved for nodal displacements D. Member end forces p_i and deformations d_i are found. The axial forces for each member are checked to detect the elastic behaviour and applied to modify member stiffness matrices k and fixed end moments m_i and hence the structure stiffness matrix, K, before commencing the next iteration (see Section 2).

The iterative-load analysis procedure continues until either the specified iteration level is reached, or the difference between the axial forces found in two successive iterations is less than 0.1% for each member ($P_{i+1} \cong P_i I$).

The analytical expressions having been prepared for all the quantities of relevance for the problem, it remained only to write down a computer program for numerical applications. That was done and the resulting program contains special differences compared to a linear analysis. The main difference is that there is an iteration which can be stopped when a desired accuracy is reached. The geometric stiffness matrix, as it is called, due to axial force is a relevant feature of this analysis, which actually is the cause of the necessity for the iterative procedure. The computer program analysis starts with zero axial forces in all members, giving the linear solution at the first step. It assumes the axial forces in members to be zero initially. It setups the overall stiffness matrix, analyzes the frame under the external loads, obtains joint displacements and member end forces. Then, at each new load step the axial forces and frame deflections found in the previous step are used in the computations, of both the modified stiffness matrix (calculates the corresponding stability functions) and the modified fixed end forces. The nonlinear analysis terminated when the difference between the axial forces found in two successive iterations is less than 0.1% for each member. When the predetermined precision is attained, the iteration stops and the final displacements and rotations, member end forces, and variations of bending moment along relevant members are determined. The maximum value of the bending moment in each member is given, along with the maximum value and its position on the member.

During these iterations the determinant of the overall structure stiffness matrix is calculated and loss of stability is checked. If the convergence in the axial force is obtained without loss of stability, the joint displacements and member forces obtained in this nonlinear response are used in the computation of fitness values for this individual. It should be noted that in this algorithm the design load is not applied incrementally in the nonlinear analysis. Instead it is applied immediately and iterations are carried out at this load. It should also be pointed out that during the nonlinear analysis the fixed end moments change from one iteration to another due to axial forces in the members and rotational springs attached at the ends of members. The modified fixed end moments are calculated by taking into account the effect of shear deformations and the effect of flexible end connection for a frame member.

The nonlinear analysis procedure is illustrated by the flow chart in Fig. 8. Further computational details are provided through the analysis examples presented in the following section.

4. Analysis examples

The geometrically nonlinear iterative analysis procedure is illustrated in the following for three example structures comprised of steel beam-column members with rigid and semi-rigid connections. The first example is a six-story two-bay steel building framework for which analytical results found using the computer programme are compared with other analytical results (Aksogan and Dincer 1991, Gorgun 1992). The second example is a four-story two-bay steel building framework, the linear and nonlinear analysis of which have also been extensively studied

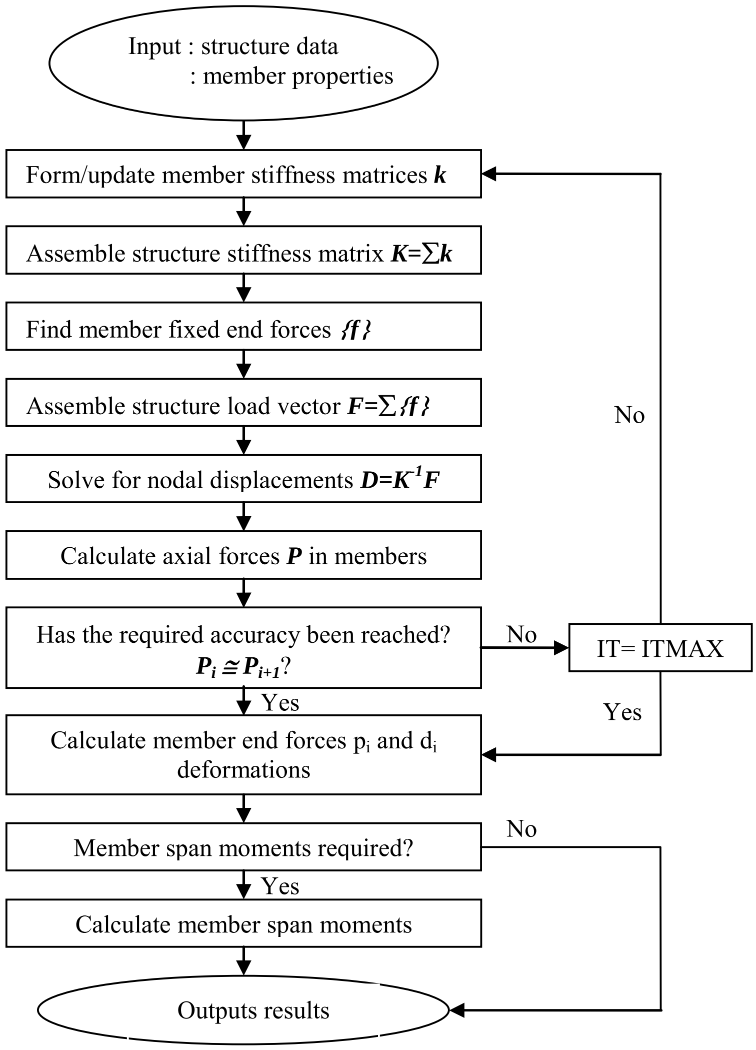

Fig. 8 Flow chart for geometrically nonlinear analysis

in the literature from a variety of different computational viewpoints (Aksogan and Akkaya 1991, Gorgun 1992). The third example is a two-story one-bay frame for which analytical results found using the computer programme are compared with other analytical results (Aristizabal-Ochoa 2012). This frame is made of the pultruded FRP beam-column with bending taking place about the major axis.

The geometrically nonlinear analysis results include the values of the bending, shearing, axial elastic stiffness and semi-rigid connections for member end sections at which elastic deformation occurs.

4.1 Example 1: A six-story two-bay steel building framework

Consider the 6-story by 2-bay steel framework subjected to different kinds of distributed service-level design gravity span loads and the pattern of concentrated point loading shown in Fig. 9. The structure is a building frame that supports loads shown in Fig. 9. All beams have IPN 300, 1st and 2nd floor columns have IPN 360, 3rd and 4th floor columns have IPN 300 and 5th and 6th floor columns have IPN 180-shape sections that are oriented with their webs in the plane of the framework and are assumed to be fully restrained against out-of-plane behaviour. Shape factor f = 5 / 6, Poisson's ratio v = 0.3. The framework has 30 members, 21 nodes and 54 degrees-of-freedom (dof) for nodal displacement (i.e., lateral and vertical translation and rotation dof at each of the eighteen free nodes 4-21). The members and nodes are designated by a square and a circle symbol (□, O) with a number inscribed in it that indicates the member or node number respectively, shown in Fig. 10. Briefly discussed in the following are the results of the study that demonstrate analytically the influence that shear and the geometrically nonlinear have on the behaviour of the member end moments.

The analytical results presented in Tables 1 and 2 account for the combined influence that bending and shearing have on elastic behaviour, and were found using the computer programme to include the effect that shear deformations have on elastic behaviour. It is readily possible to conduct the same analysis using Euler-Bernoulli beam theory, which ignores the effect of shear deformation on elastic behaviour, by setting the beam-column member shear stiffness $Ga_s = \infty$ in Eqs. (2) and (4a).

The analysis results found by this study are given in Tables 1 and 2 and compared with the results of other studies (Aksogan and Dincer 1991, Gorgun 1992, Gorgun *et al.* 2012).

This example frame originally appeared in (Aksogan and Dincer 1991) and, since then, its nonlinear analysis has been studied by a number of researchers from a variety of computational viewpoints. The results for the method proposed herein are in close agreement with those for all other methods. Tables 1 and 2 compare the member end moments of Aksogan and Dincer (1991) who neglected the effect of semi-rigid connections with those obtained from the formulations by Gorgun (1992) who neglected the effect of shear deformation and present study, which incorporates the axial shortening effect, shear deformations, geometrically nonlinear effect and semi rigid connections. It can be seen the results are almost in agreement for v = 0, indicating the negligible influence of shear deformations and v = 0.3, indicating the influence of shear deformations on the member end moments. The extreme moment values obtained for member 6 (a column member). Shear effect has changed the top end moment m_2 for this member by 450% (increases from 0.12 kNm to 0.66 kNm) and 66% (increases from 0.86 kNm to 1.43 kNm), respectively, for linear and nonlinear solutions. The nonlinear effect has changed the same end moment for this member by as much as approximately 617% (increases from 0.12 kNm to 0.86 kNm), while both the nonlinear and the shear effects have changed the end moment by as much as approximately 1092% (increases from 0.12 kNm to 1.43 kNm) for this frame example. It should be noticed that the superposition is not valid here.

4.2 Example 2: a four-story two-bay steel building framework

Consider the 4-story by 2-bay steel framework subjected to different kinds of service-level design gravity span loads and direct loads shown in Fig. 11. The structure is a building frame that supports loads shown in Fig. 11. All member have HE 1000 M-shape sections (Aksogan and

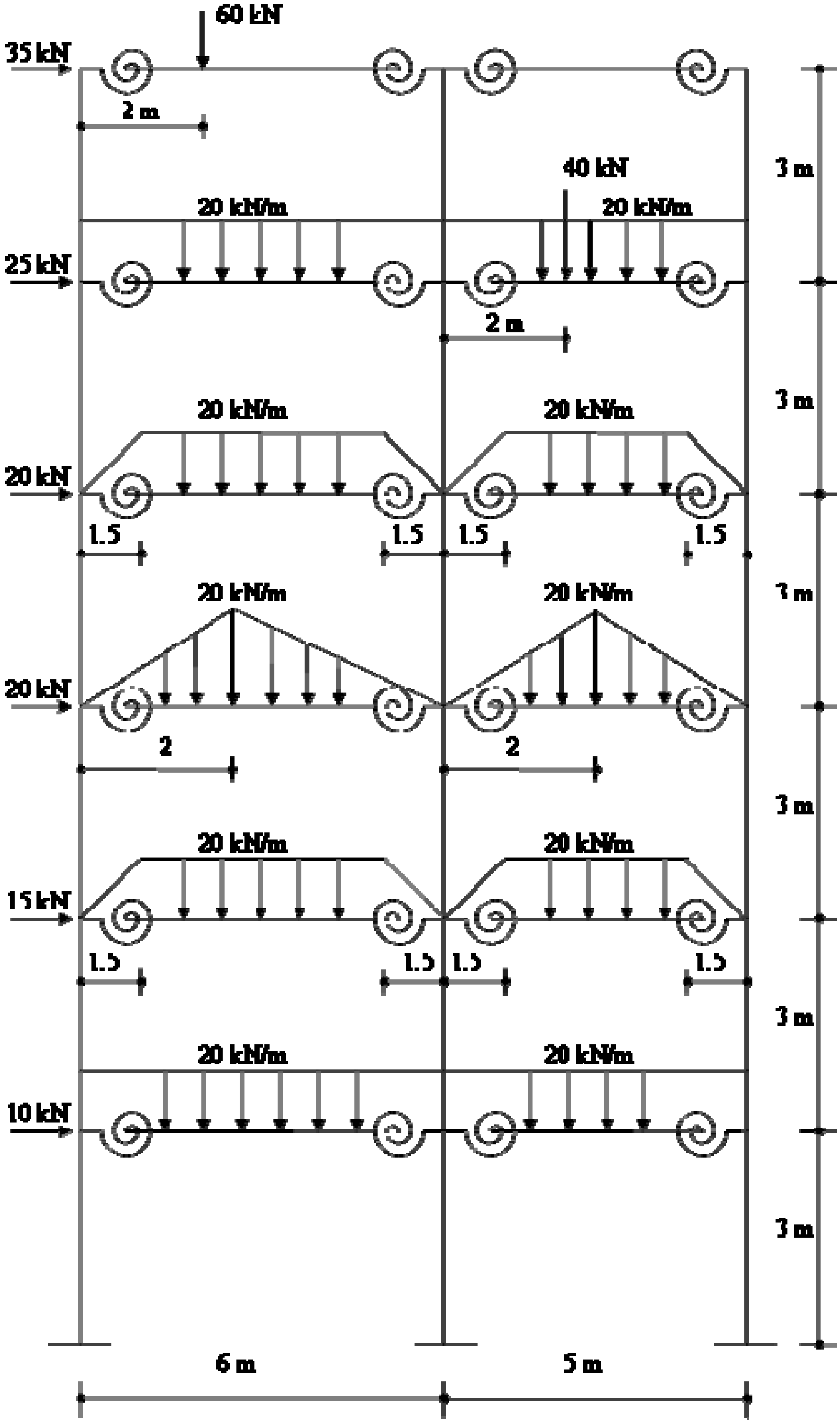

Fig. 9 Geometry and loading of the example 1

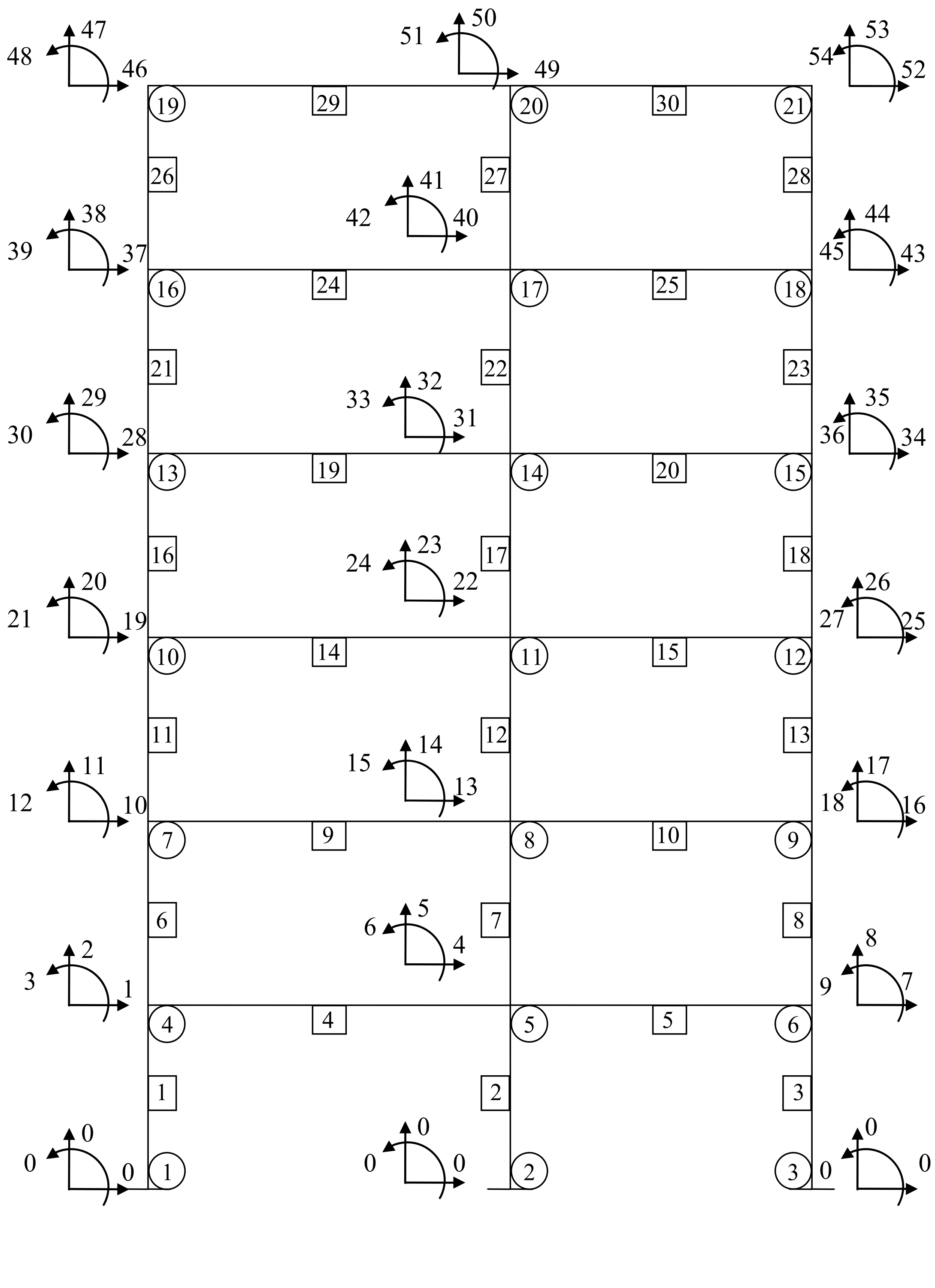

Fig. 10 Coding and numbering of the example 1

Table 1 Example 1: Comparison of member end moments with rigid connections for linear frame analysis

Member	Member end moments (kNm)									
	Neglecting the effect of shear deformation (v = 0)						Considering the effect of shear deformation (v = 0.3)			
	Aksogan and Dincer (1991)		Gorgun (1992)		Present study		Aksogan and Dincer (1991)		Gorgun *et al.* (2012)	
	m_1	m_2	m_1	m_2	m_1	m_2	m_1	m_2	m_1	m_2
1	86.50	-11.12	86.50	-11.12	86.50	-11.12	88.13	-11.80	88.13	-11.79
2	114.37	43.51	114.37	43.51	114.37	43.51	114.39	43.04	114.38	43.04
3	108.96	32.78	108.96	32.78	108.96	32.78	109.19	32.05	109.19	32.05
4	-5.02	-118.33	-5.02	-118.33	-5.02	-118.33	-5.37	-118.23	-5.37	-118.23
5	-24.69	-109.62	-24.69	-109.62	-24.69	-109.62	-23.88	-108.94	-23.87	-108.94
6	16.13	0.12	16.13	0.12	16.13	0.12	17.17	0.66	17.16	0.66
7	99.50	90.51	99.50	90.51	99.50	90.51	99.07	89.77	99.06	89.77
8	76.83	61.91	76.83	61.91	76.83	61.91	76.89	61.45	76.89	61.45
9	-20.68	-119.12	-20.68	-119.12	-20.68	-119.12	-21.22	-119.16	-21.22	-119.15
10	-41.76	-115.71	-41.76	-115.71	-41.76	-115.71	-41.05	-115.21	-41.04	-115.20
11	20.57	25.46	20.57	25.46	20.57	25.46	20.56	25.45	20.56	25.47
12	70.37	73.18	70.37	73.18	70.37	73.18	70.43	73.24	70.42	73.26
13	53.80	56.63	53.80	56.63	53.80	56.63	53.76	56.55	53.75	56.55
14	-26.56	-92.49	-26.56	-92.49	-26.56	-92.49	-27.20	-92.63	-27.23	-92.66
15	-45.82	-100.85	-45.82	-100.85	-45.82	-100.85	-45.23	-100.49	-45.22	-100.48
16	1.11	-5.39	1.11	-5.39	1.11	-5.39	1.75	-4.74	1.76	-4.75
17	65.14	79.88	65.14	79.88	65.14	79.88	64.62	79.51	64.63	79.50
18	44.23	55.03	44.23	55.03	44.23	55.03	43.94	54.93	43.94	54.92
19	-11.21	-100.06	-11.21	-100.06	-11.21	-100.06	-11.86	-100.15	-11.86	-100.14
20	-14.82	-91.13	-14.82	-91.13	-14.82	-91.13	-14.40	-90.91	-14.39	-90.90
21	16.59	14.86	16.59	14.86	16.59	14.86	16.61	14.92	16.61	14.92
22	34.99	35.50	34.99	35.50	34.99	35.50	35.03	35.54	35.03	35.54
23	36.10	41.96	36.10	41.96	36.10	41.96	35.98	41.92	35.98	41.92
24	-12.98	-107.38	-12.98	-107.38	-12.98	-107.38	-13.22	-107.22	-13.22	-107.22
25	45.81	-69.53	45.81	-69.53	45.81	-69.53	45.78	-69.45	45.78	-69.45
26	-1.87	-1.56	-1.87	-1.56	-1.87	-1.56	-1.71	-1.34	-1.71	-1.34
27	26.08	31.80	26.08	31.80	26.08	31.80	25.91	31.66	25.91	31.66
28	27.57	22.99	27.57	22.99	27.57	22.99	27.53	22.95	27.53	22.95
29	1.56	-34.55	1.56	-34.55	1.56	-34.55	1.34	-34.47	1.34	-34.47
30	2.76	-22.99	2.76	-22.99	2.76	-22.99	2.81	-22.95	2.81	-22.95

Akkaya 1991) that are oriented with their webs in the plane of the framework and are assumed to be fully restrained against out-of-plane behaviour with the following properties: section depth h = 1008 mm, flange width b_f = 302 mm, web thickness t_w = 21.0 mm, flange thickness t_f = 40.0 mm, section area A = 44400 mm^2, moment of inertia $I = 7220 \times 10^6$ mm^4, and shape factor f = 5 / 6, Poisson's ratio v = 0.3. The framework has 20 members, 15 nodes and 36 degrees-of-freedom (dof) for nodal displacement (i.e., lateral and vertical translation and rotation dof at each of the twelve free nodes 4-15). The members and nodes are designated by a square and a circle symbol (□, O) with a number inscribed in it that indicates the member or node number respectively, shown in Fig. 12. The spring constants are given for the respective beams being 0.5 for the outer ends and 0.6 for the inner ends.

The analysis results found by this study are illustrated in Figs. 13 and 14 and compared with the results of other studies in Table 3 and the results of linear analysis compared with nonlinear analysis with (v = 0.3)/without (v = 0) the effect of shear deformation in Tables 4-5.

The nonlinear analysis terminated when the difference between the axial forces found in two successive iterations is less than 0.1% for each member.

To give an idea about the effect of spring constants, on the displacements, the variations of the horizontal sway deflections of four nodes (joint nodes 4, 7, 10 and 13) of the frame with varying spring constants for all the springs in the structure are plotted in Fig. 12. Values are given at the joints each floor level for all the nonlinear analyses with shear effects for $k_1 = k_2 = 0$ (pin), 0.5, 1.0 and 10^9 (rigid). The difference between the linear and nonlinear deflections for both the semi-rigid and rigid connections is less than 1mm over the full height of the structure.

Table 2 Example 1: Comparison of member end moments with rigid connections for nonlinear frame analysis

Member	Member end moments (kNm)									
	Neglecting the effect of shear deformation (v = 0)						Considering the effect of shear deformation (v = 0.3)			
	Aksogan and Dincer (1991)		Gorgun (1992)		Present study		Aksogan and Dincer (1991)		Gorgun *et al.* (2012)	
	m_1	m_2	m_1	m_2	m_1	m_2	m_1	m_2	m_1	m_2
1	88.40	-10.99	88.40	-10.99	88.40	-10.99	90.12	-11.67	90.11	-11.67
2	116.28	44.24	116.28	44.24	116.28	44.24	116.38	43.78	116.38	43.78
3	110.75	33.01	110.75	33.01	110.75	33.01	111.07	32.27	111.07	32.27
4	-6.54	-119.74	-6.54	-119.74	-6.54	-119.74	-6.94	-119.68	-6.94	-119.68
5	-26.35	-111.35	-26.35	-111.35	-26.35	-111.35	-25.58	-110.72	-25.58	-110.72
6	17.52	0.86	17.52	0.86	17.52	0.86	18.61	1.43	18.61	1.43
7	101.85	92.53	101.85	92.53	101.85	92.53	101.48	91.84	101.47	91.84
8	78.35	62.90	78.35	62.90	78.35	62.90	78.46	62.46	78.45	62.47
9	-22.49	-120.89	-22.49	-120.89	-22.49	-120.89	-23.09	-120.99	-23.08	-120.98
10	-43.79	-117.89	-43.79	-117.89	-43.79	-117.89	-43.12	-117.44	-43.11	-117.43
11	21.63	26.62	21.63	26.62	21.63	26.62	21.66	26.65	21.65	26.69
12	72.14	75.01	72.14	75.01	72.14	75.01	72.26	75.14	72.25	75.15
13	54.98	57.88	54.98	57.88	54.98	57.88	54.97	57.83	54.96	57.83
14	-28.30	-94.10	-28.30	-94.10	-28.30	-94.10	-29.00	-94.29	-29.03	-94.32
15	-47.69	-102.85	-47.69	-102.85	-47.69	-102.85	-47.13	-102.54	-47.12	-102.53
16	1.68	-4.94	1.68	-4.94	1.68	-4.94	2.35	-4.25	2.36	-4.25
17	66.78	81.65	66.78	81.65	66.78	81.65	66.28	81.29	66.29	81.28
18	44.97	55.69	44.97	55.69	44.97	55.69	44.70	55.63	44.70	55.62
19	-12.80	-101.52	-12.80	-101.52	-12.80	-101.52	-13.50	-101.66	-13.50	-101.65
20	-16.38	-93.00	-16.38	-93.00	-16.38	-93.00	-15.99	-92.82	-15.99	-92.81
21	17.74	16.07	17.74	16.07	17.74	16.07	17.75	16.11	17.75	16.11
22	36.25	36.81	36.25	36.81	36.25	36.81	36.36	36.96	36.36	36.96
23	37.31	43.11	37.31	43.11	37.31	43.11	37.20	43.06	37.20	43.06
24	-14.18	-108.22	-14.18	-108.22	-14.18	-108.22	-14.42	-107.83	-14.42	-107.83
25	45.06	-70.73	45.06	-70.73	45.06	-70.73	44.68	-70.55	44.68	-70.55
26	-1.89	-1.46	-1.89	-1.46	-1.89	-1.46	-1.70	-1.05	-1.70	-1.05
27	26.36	32.09	26.36	32.09	26.36	32.09	26.19	31.89	26.19	31.89
28	27.62	23.15	27.62	23.15	27.62	23.15	27.49	23.06	27.49	23.06
29	1.46	-34.80	1.46	-34.80	1.46	-34.80	1.05	-34.57	1.05	-34.57
30	2.71	-23.15	2.71	-23.15	2.71	-23.15	2.68	-23.06	2.68	-23.06

A drift factor of sway deflection = height/500 of the frame is defined by the continuous line in Fig.13. It can be seen that only where the semi-rigid joints are considered are the deflections less than height/500.

Horizontal sway deflections of four nodes (joint nodes 4, 7, 10 and 13) of the frame with rigid connections only are plotted in Fig. 14 for all the linear and nonlinear analyses with (ν = 0.3) and without (ν = 0) shear effects. The difference between the linear and nonlinear deflections for rigid connections is less than 1 mm while the difference between the linear or nonlinear deflections with and without shear effects for rigid connections is less than 2mm over the full height of the structure. It can be seem that the effect of the shear deformations is greater than the effect of the geometric nonlinearity on the deflections for this frame example.

The extreme moment values obtained for members 7 (a column member) and 17 (a beam member). Shear effect has increased the top end moment of member 7 by 146% (increases from -0.84 kNm to 0.39 kNm), the nonlinear effect has reduced the left end moment for member 17 by as much as approximately 22% (decreases from 1.41 kNm to 1.11 kNm), while both the nonlinear and the shear effects have changed the above mentioned moments by as much as approximately 155% (from 1.41 kNm to -0.77 kNm) for this frame example.

This example frame originally appeared in (Aksogan and Akkaya 1991) and, since then, its nonlinear analysis has been studied by a number of researchers from a variety of computational viewpoints. Gorgun (1992) conducted nonlinear analysis of the frame with semi-rigid connections neglecting shear deformations (ν = 0).The lateral and vertical deflections and rotation behaviour found for these various analyses are given in Table 5. The results for the method proposed herein are in close agreement with those for all other methods. The slight discrepancies between the methods are likely mainly due to different ways in which the member fixed end forces are considered. It is worth noting that the structural model for the proposed method involved significantly the nonlinear geometric effects, shear effects, and flexible beam-to-column connections than the other methods.

4.3 Example 3: Second order analysis of a plane frame made of beams and columns with semi-rigid connections

Determine the second-order member forces of each member of the frame shown in Fig. 15 (Aristizabal-Ochoa 2012). This frame is made of the pultruded FRP beam-column with bending taking place about the major axis. Assume that: A = 5800mm$_2$, $EI = 7.85 \times 10^8$kN–mm^2; GA_S = 5340 kN; elastic moduli E = 18.863 kN/mm^2, and G = 2.671 kN/mm^2. Include the effects of shear deformations and also the effects of the flexural moments on the axial stiffness in the analysis. The framework has 6 members, 6 nodes and 12 degrees-of-freedom (dof) for nodal displacement (i.e., lateral and vertical translation and rotation dof at each of the four free nodes C-F). The spring constants are given for the respective members being 1.75 for the beam ends and 6.75 for the column down ends.

To facilitate comparison with other published results for this example (Aristizabal-Ochoa 2012), the analysis results found by this study are given in Tables 6-8 and compared with the published results of other study (Aristizabal-Ochoa 2012) for the first-and second-order elastic analysis for the first and second iterations.

The member end moments of each member of the frame showing the final end actions obtained from the first-order elastic analysis are given in Table 6. In the second-order elastic analysis, both the applied axial loads and the axial loads resulting from frame action given in Table 6 are

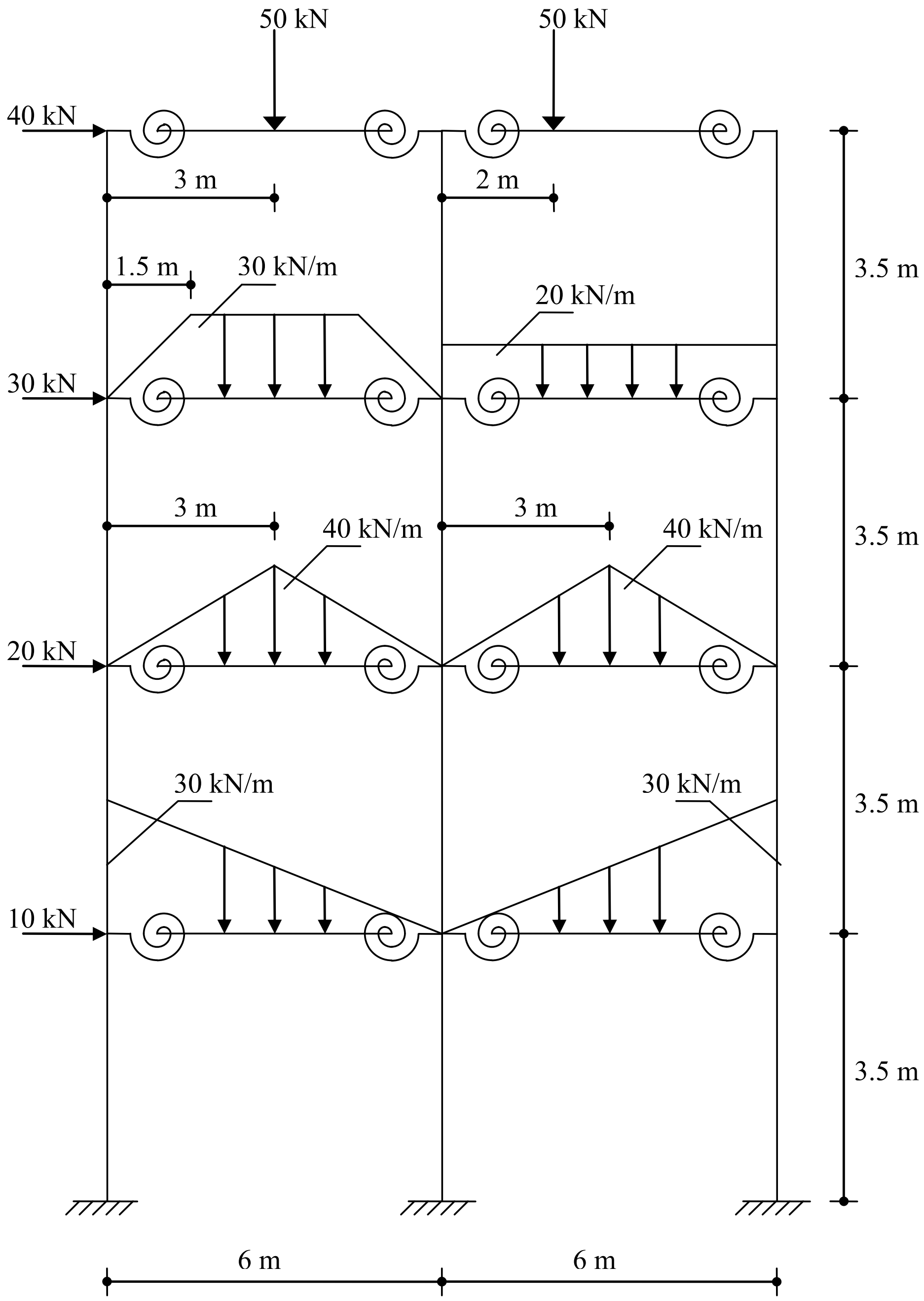

Fig. 11 Geometry and loading of the example 2

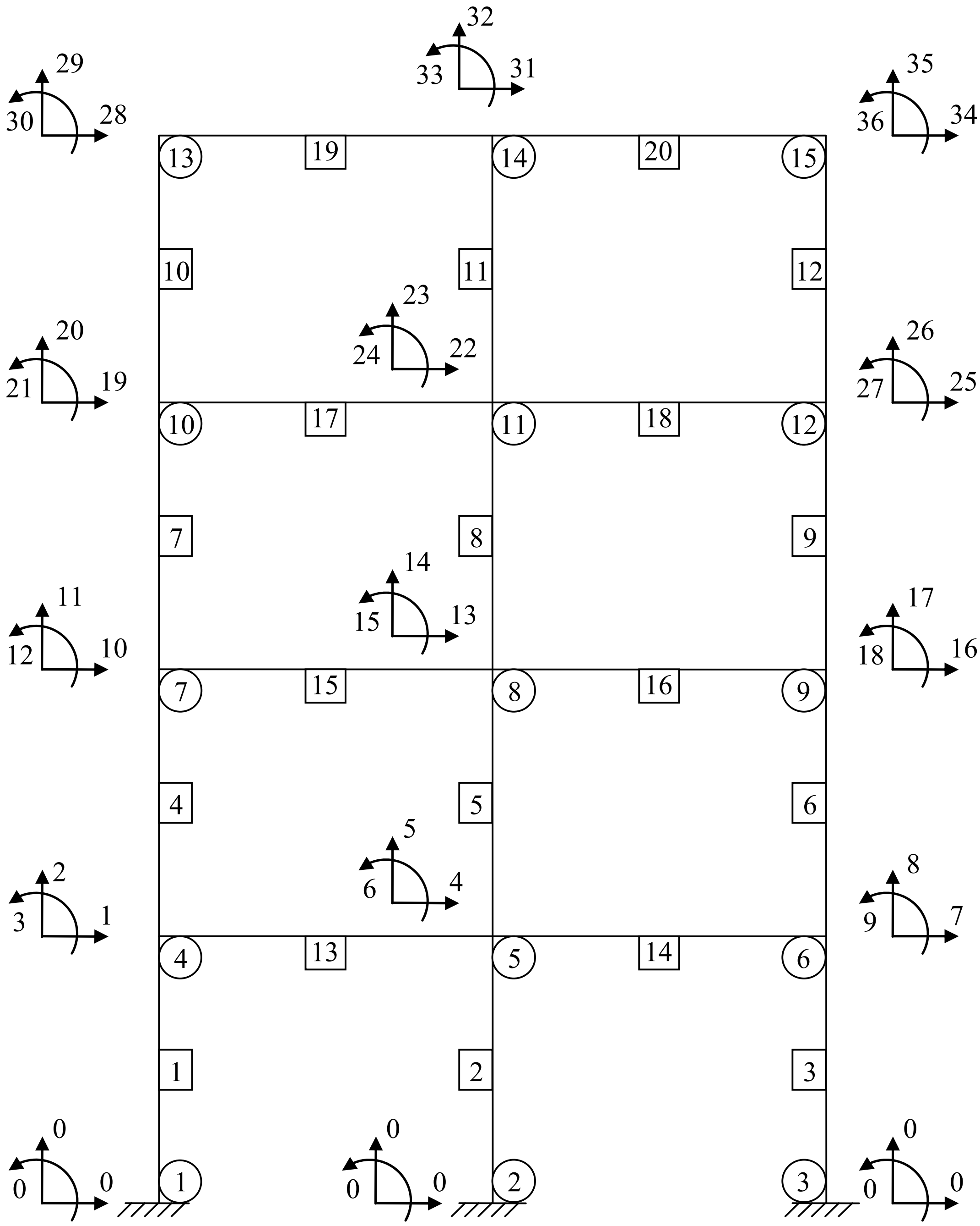

Fig. 12 Coding and numbering of the example 2

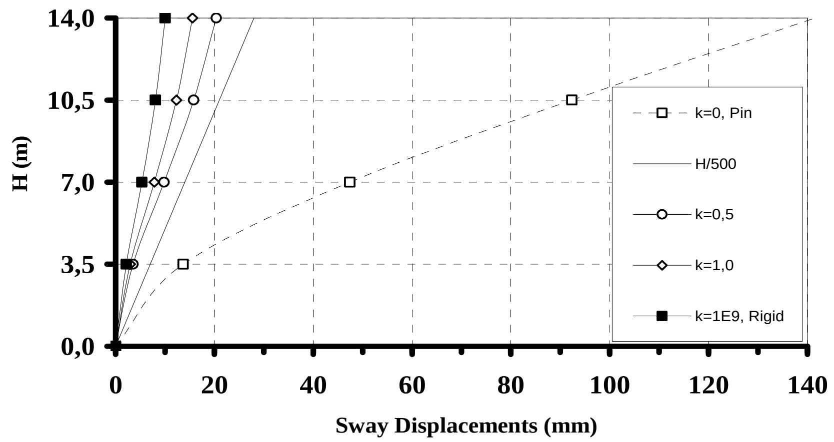

Fig. 13 Sway displacements at each floor level in the example problem with varying spring constants *k*

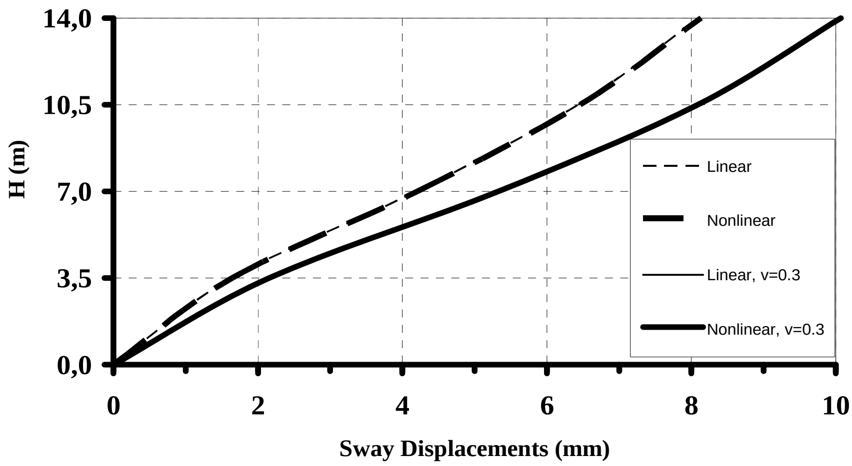

Fig. 14 Sway displacements at each floor level in the example problem with rigid connections

considered. The member final moments of the 12 degrees-of-freedom (dof) system of each iteration are summarised (kiloNewton and meters are utilized throughout) and given in Table 7. Since the maximum difference in the displacements between the first- and second-order elastic analysis is small (2.69%, the lateral translation of the top right corner (joint node F) of the frame) further iteration were consider unnecessary. Notice that the rotations, vertical and lateral deflections, and end bending moments increased significantly (over 17%, 3% and 21%, and 14% of their primary values, respectively) caused by the geometric nonlinear effects mentioned in the introduction.

This example frame originally appeared in Aristizabal-Ochoa (2012). The lateral and vertical deflections, and rotation behaviour found for these various analyses are given in Table 8. The results for the method proposed herein are in close agreement with those for all other methods. The slight discrepancies between the methods are likely mainly due to different ways in which the modified stiffness coefficients and member fixed end forces are considered. It is worth noting that the structural model for the proposed method involved significantly fewer beam -column elements and nodes than the other methods.

Table 3 Example 2: Comparison of member end moments with semi-rigid connections for linear frame analysis

Member	Member end moments (kNm)					
	Neglecting the effect of shear deformation ($\nu = 0$)					
	Aksogan and Akkaya (1991)		Gorgun (1992)		Present study	
	m_1	m_2	m_1	m_2	m_1	m_2
1	-13.57	103.70	-13.67	103.88	-13.67	103.88
2	18.16	119.80	18.02	119.98	18.02	119.98
3	7.48	114.40	7.31	114.48	7.31	114.48
4	15.78	35.78	15.45	36.06	15.45	36.06
5	63.29	80.12	63.29	80.54	63.29	80.54
6	53.55	66.48	52.94	66.73	52.94	66.73
7	21.18	-1.66	20.00	-0.84	20.00	-0.84
8	71.71	49.50	69.97	50.26	69.97	50.26
9	62.67	41.59	63.02	42.59	63.02	42.59
10	6.40	-20.36	7.05	-21.41	7.05	-21.41
11	58.82	27.47	59.74	26.00	59.74	26.00
12	50.26	17.40	50.80	17.83	50.80	17.83
13	-22.21	-68.35	-22.38	-68.55	-22.38	-68.55
14	-29.93	-73.96	-30.02	-74.04	-30.02	-74.04
15	-14.12	-91.25	-14.60	-91.72	-14.60	-91.72
16	-21.55	-95.14	-21.83	-95.52	-21.83	-95.52
17	-0.83	-78.06	1.41	-73.55	1.41	-73.55
18	-21.12	-80.07	-22.41	-80.85	-22.41	-80.85
19	-6.40	-40.65	-7.05	-41.25	-7.05	-41.25
20	-18.17	-50.26	-18.49	-50.80	-18.49	-50.80

Table 4 Example 2: Comparison of member end moments with semi-rigid connections for linear and nonlinear frame analysis

Member	Member end moments (kNm)							
	Neglecting the effect of shear deformation (v = 0)				Considering the effect of shear deformation (v = 0.3)			
	Nonlinear Gorgun (1992)		Nonlinear Present study		Linear Gorgun *et al.* (2012)		Nonlinear Gorgun *et al.* (2012)	
	m_1	m_2	m_1	m_2	m_1	m_2	m_1	m_2
1	-13.75	104.68	-13.75	104.68	-15.76	108.46	-15.89	109.46
2	18.15	120.79	18.15	120.79	16.15	119.35	16.29	120.25
3	7.24	115.23	7.24	115.23	5.98	115.82	5.89	116.69
4	15.85	36.56	15.85	36.56	15.39	38.67	15.84	39.26
5	63.97	81.30	63.97	81.30	61.61	79.67	62.34	80.48
6	53.35	67.23	53.35	67.23	52.63	67.02	53.08	67.58
7	20.44	-0.78	20.44	-0.78	21.19	0.39	21.67	0.45
8	70.56	50.55	70.56	50.55	69.40	49.44	70.04	49.74
9	63.46	42.64	63.46	42.64	62.68	41.91	63.16	41.97
10	7.20	-21.55	7.20	-21.55	8.95	-20.75	9.14	-20.91
11	60.07	26.01	60.07	26.01	58.97	24.98	59.32	24.98
12	50.97	17.69	50.97	17.69	50.48	17.37	50.67	17.22
13	-22.81	-68.99	-22.81	-68.99	-22.91	-67.40	-23.38	-67.87
14	-30.46	-74.47	-30.46	-74.47	-28.43	-73.00	-28.90	-73.47
15	-15.07	-92.20	-15.07	-92.20	-15.78	-90.68	-16.29	-91.20
16	-22.32	-96.00	-22.32	-96.00	-20.36	-94.53	-20.88	-95.05
17	1.11	-73.84	1.11	-73.84	-0.44	-73.19	-0.77	-73.50
18	-22.73	-81.15	-22.73	-81.15	-21.18	-80.05	-21.53	-80.38
19	-7.20	-41.42	-7.20	-41.42	-8.91	-41.23	-9.14	-41.40
20	-18.65	-50.97	-18.65	-50.97	-17.74	-50.48	-17.92	-50.67

Table 5 Example 2: Comparison of joint displacements with semi-rigid connections for linear and nonlinear frame analysis

Displacement no.	Joint displacements lateral and vertical translations (mm), rotations (radians)			
	Neglecting the effect of shear deformation (v = 0) Present study		Considering the effect of shear deformation (v = 0.3) Gorgun *et al.* (2012)	
	Linear	Nonlinear	Linear	Nonlinear
1	2.982	3.007	3.443	3.475
2	-0.599	-0.597	-0.597	-0.595
3	-0.001357	-0.001368	-0.001434	-0.001447

Table 5 Continued

3	-0.001357	-0.001368	-0.001434	-0.001447
4	2.989	3.014	3.450	3.482
5	-1.326	-1.326	-1.332	-1.333
6	-0.001177	-0.001187	-0.001191	-0.001203
7	2.985	3.010	3.446	3.478
8	-0.985	-0.986	-0.980	-0.981
9	-0.001237	-0.001249	-0.001268	-0.001281
10	8.494	8.568	8.477	9.567
11	-1.029	-1.026	-1.025	-1.022
12	-0.001595	-0.001608	-0.001702	-0.001717
13	8.425	8.499	8.408	9.499
14	-2.434	-2.434	-2.447	-2.447
15	-0.0013760	-0.001388	-0.001400	-0.001413
16	8.399	8.473	9.380	9.471
17	-1.679	-1.682	-1.671	-1.674
18	-0.001396	-0.001409	-0.001434	-0.001449
19	13.783	13.898	15.233	15.371
20	-1.301	-1.298	-1.295	-1.291
21	-0.001354	-0.001363	-0.001462	-0.001472
22	13.652	13.767	15.101+	15.240
23	-3.099	-3.099	-3.116	-3.116
24	-0.001148	-0.001156	-0.001169	-0.0012
25	13.584	13.699	15.033	15.172
26	-2.075	-2.078	-2.064	-2.068
27	-0.001160	-0.001169	-0.001194	-0.001204
28	17.851	17.990	19.632	19.798
29	-1.365	-1.361	-1.358	-1.353
30	-0.001026	-0.001031	-0.001119	-0.001125
31	17.567	17.706	19.353	19.519
32	-3.305	-3.305	-3.323	-3.324
33	-0.000759	-0.000763	-0.000777	-0.000782
34	17.441	17.580	19.228	19.395
35	-2.181	-2.184	-2.170	-2.174
36	-0.000780	-0.000785	-0.000812	-0.000818

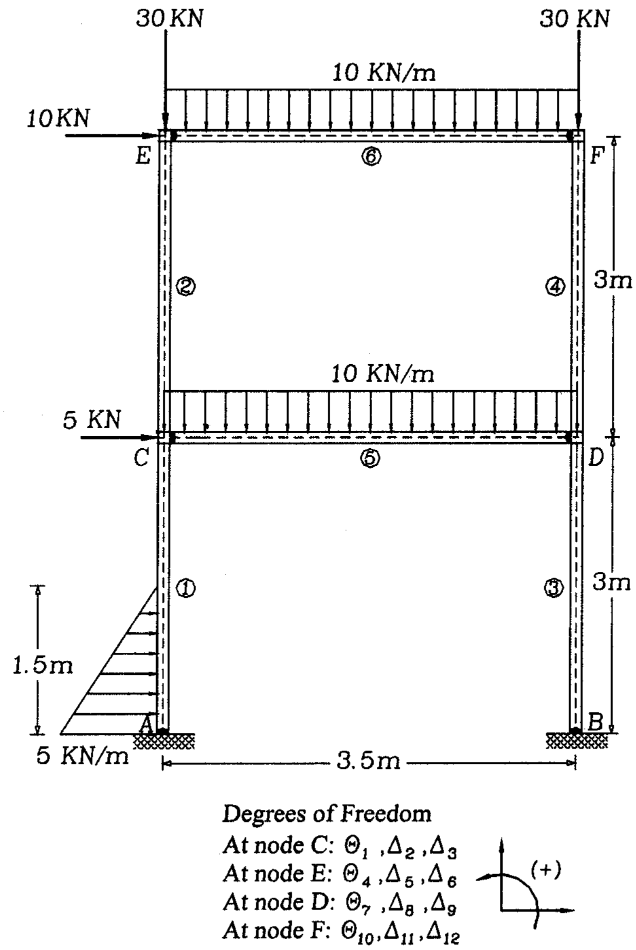

Fig. 15 Example 3: First- and second-order analysis of an unbraced frame with semi-rigid connections: (a) Structural model; (b) Degrees of freedom (adapted from Aristizabal Ochoa 2012)

Table 6 Example 3: Comparison of member end moments with semi-rigid connections for the first-order elastic analysis

Member	Member end moments (kNm)			
	First-order elastic analysis			
	Aristizabal-Ochoa (2012)		Present study	
	m_1	m_2	m_1	m_2
1	15.9560	4.6638	16.4602	4.6059
2	1.2732	2.7759	1.2827	2.8219
3	16.7500	9.5043	16.5340	9.2749
4	10.7720	15.1790	10.7774	15.1180
5	-5.9371	-14.5950	-5.8886	-20.0523
6	-2.7759	-15.1790	-2.8219	-15.1180

Table 7 Example 3: Comparison of member end moments with semi-rigid connections for the second-order elastic analysis

Member	Member end moments (kNm)							
	Second-order elastic analysis (First Iteration)				Second-order elastic analysis (Second Iteration)			
	Aristizabal-Ochoa (2012)		Present study		Aristizabal-Ochoa (2012)		Present study	
	m_1	m_2	m_1	m_2	m_1	m_2	m_1	m_2
1	19.3610	5.9407	19.5320	6.0656	19.3610	5.9277	19.5564	6.0591
2	3.1824	5.2304	2.8463	4.9060	3.1927	5.2348	2.8500	4.9074
3	20.076	10.950	19.2536	10.6313	20.060	10.962	19.2325	10.6343
4	12.254	17.315	12.4380	17.2401	12.245	17.308	12.4357	17.2367
5	-9.1231	-23.204	-8.9119	-23.0693	-9.1205	-23.207	-8.9090	-23.0700
6	-5.2304	-17.315	-4.9060	-17.2401	-5.2348	-17.308	-4.9074	-17.2367

Table 8 Example 3: Comparison of joint displacements with semi-rigid connections for the first- and second-order elastic analysis

Degrees of Freedom	Joint displacements Lateral and vertical translations (m), Rotations (radians)					
	First-order elastic analysis		Second-order elastic analysis (First Iteration)		Second-order elastic analysis (Second Iteration)	
	Aristizabal Ochoa (2012)	Present study	Aristizabal Ochoa (2012)	Present study	Aristizabal Ochoa (2012)	Present study
Θ_1	-0.020630	-0.021129	-0.026878	-0.025486	-0.026867	-0.025479
Δ_2	-0.002	-0.001439	-0.002	-0.001358	-0.0016	-0.001358
Δ_3	0.053	0.054484	0.073	0.066206	0.073	0.066203
Θ_4	-0.017756	-0.018188	-0.022796	-0.021382	-0.022803	-0.021385
Δ_5	-0.003	-0.002601	-0.003	-0.002487	-0.003	-0.002487
Δ_6	0.115	0.117700	0.157	0.144576	0.157	0.144572
Θ_7	-0.016215	-0.016211	-0.021721	-0.020550	-0.021724	-0.020553
Δ_8	-0.002	-0.002126	-0.003	-0.002207	-0.003	-0.002206
Δ_9	0.053	0.054484	0.073	0.066214	0.073	0.066213
Θ_{10}	-0.007795	-0.007917	-0.011130	-0.010881	-0.011514	-0.010874
Δ_{11}	-0.004	-0.003569	-0.004	-0.003683	-0.004	-0.003683
Δ_{12}	0.115	0.117424	0.157	0.144304	0.157	0.144300

5. Conclusions

The first and second order modified stiffness matrix and fixed end moments of an Euler-Bernoulli beam-column member with semi-rigid beam-to-column end connections including the combined effects of bending plus shear deformations and shear component of the applied axial forces are derived in a classical manner. The proposed method herein is based on the modified stability functions for beam-columns with semi-rigid connections. The validity and effectiveness of the modified proposed equations are verified against well documented solutions on plane frames (Aksogan and Akkaya 1991, Aksogan and Dincer 1991, Gorgun 1992, Aristizabal-Ochoa 2012)

The main advantages of the proposed method: (1) the method attempts at including/neglecting shear deformation into a beam-column element in order to then analyse plane frames with the effects of semi-rigid connections. The effects of semi-rigid connections are condensed into the stiffness matrix coefficients and into the modified fixed end moments of each element for zero (the first-order elastic analysis), compression and tension axial force (the second-order elastic analysis) without introducing any additional degrees of freedom. It can be understood that using such elements would severely reduce the computational time when analysis large frame structures. The second-order elastic analysis of structures made of Timoshenko beam-columns is cumbersome. This is due to the combined effects of shear distortions and shear forces induced by the axial forces along each beam-column element as they deflect laterally along their span that must be taken into account in the second order analysis. Current Finite Element Methods and computer programs do not take into account these two effects. However, commercially available finite element software has the capability to deal with: shear deformation in beams, deep beam analysis and nonlinear analysis. (2) the matrices are defined in terms of the elastic axial stiffness and the “modified” stability functions. (3) the modified stiffness matrices and fixed end forces for various span loadings can be incorporated into computer programs without major difficulties making the method practical and versatile. Different types of span loadings are considered and most of the span loadings not being found in the literature for zero, compression and tension axial forces. (4) the proposed method is more accurate than any other method available and capable of capturing the phenomena of buckling under axial forces with the above mentioned effects.

The modified stiffness matrices are limited to the elastic stability and second-order analyses of framed structures with semi-rigid connections made of Timoshenko beam-columns of various cross sections having different shape factor. In framed structures in which the external loads are applied along their beam-column members, the process of determining the induced axial forces in each beam-column member in a second-order static analysis is iterative requiring more than one set of calculations and checks. The validity of both matrices and fixed end moments is verified against available solutions of stability analysis and nonlinear geometric elastic behaviour of framed structures with semi-rigid connections using a single segment for each beam and column member without introducing additional degrees of freedom. Three examples are included to demonstrate the effectiveness of the proposed matrices and fixed end forces.

The analytical results indicate that the stability and the nonlinear response of framed structures are not only affected by the magnitude of the axial force in its members, the magnitude and location of the restraints against horizontal drift, and the degree of the semi-rigidity of the connections, but also by the reduction in the axial stiffness of each member caused by the bending moments and shear deformations along their spans. Shear deformations and the flexibility of the semi-rigid connections increase the lateral deflections of the framed structures and reduce their critical axial loads. The effects of shear deformations, semi-rigid connections and second-order P-Δ effects must be considered in the analysis of the beam-column with relatively low effective shear areas or low shear modulus resulting in members with shear stiffness GA_s of the same order of magnitude as EI/L^2. Significant increases in the modified end bending moments and in the horizontal deflections are caused by the geometric nonlinear effects. The second-order effects should not be neglected, particularly in slender framed structures.

It is noticed from the design examples that semi-rigid connection flexibility affects the distribution of forces in the frame and causes increase in the drift of the frame. This in turn necessitates the consideration of $P-\Delta$ effect in the frame analysis. It required three to five iterations in the design examples considered to obtain the nonlinear response of frame which clearly indicates the significance of geometric nonlinearity in the analysis and design of semi-rigid steel frames. It is also noticed that consideration of $P-\Delta$ effect and shear deformation yields a heavier frame in the case of semi-rigid as well as rigid frame. The analysis examples demonstrate that the proposed nonlinear analysis method based on bending, shearing and axial stiffness approximately simulates the elastic behaviour of steel structures. Comparisons with results found by other methods for the frame examples determined that the proposed method can effectively predict the member end forces of steel frameworks, achieve more accurate results than the conventional method.

Compared to other approaches, the primary advantages of the proposed method are its simplicity, practicality and efficiency. The proposed stiffness coefficients simplify the means to account for geometric nonlinearity, effect of shear deformation, and semi-rigid connections. Finally, studies have shown that the proposed method can be readily and effectively implemented for the advanced analysis and design of steel frames and especially, nowadays, widely used precast reinforced concrete structures.

Acknowledgements

The research presented in this paper was carried out at the University of Dicle. Department of Civil Engineering. The author wishes to express his appreciation to the Department of Civil Engineering of Dicle University for encouragement and providing computer facilities during the investigation.

References

Aksogan, O. and Akkaya, F. (1991), “A computer program for the analysis of flexibly connected frames”, *Cukurova Univ. J. Fac. Eng. Arch*., **6**(2), 25-41.

Aksogan, O. and Dincer, R. (1991), “Nonlinear analysis of planar frames with linear prismatic members having rigid end sections taking shear deformation into consideration”, *Cukurova Univ. J. Fac. Eng. Arch*., **6**(1), 125-137.

Al-Sarraf, S.Z. (1986), “Shear effect on the elastic stability of frames”, *Struct. Eng.*, **64B**(2), 43-47.

Aristizabal-Ochoa, D.J. (2007), “Large deflection and postbuckling behavior of Timoshenko beam–columns with semi-rigid connections including shear and axial effects”, *Eng. Struct*., **29**, 991-1003.

Aristizabal-Ochoa, D,J. (2012), “Matrix method for stability and second-order analysis of Timoshenko beam-column structures with semi-rigid connections”, *Eng. Struct*., **34**, 289-302.

Chen, W.F. and Lui, E.M. (1991), *Stability design of steel frames*, CRC Press, Boca Raton, Florida.
Davisson, J.B., Kirby, P.A. and Nethercot, D.A. (1987), "Rotational stiffness characteristics of steel beam-to-column connections", *J. Construct Steel Res*., **8**, 17-54.
Dhillon, B.S. and Abdel-Majid, S. (1990), "Interactive analysis and design of flexibility connected frames", *J. Comput. Struct*., **36**(2), 189-202.
Frye, M.J. and Morris, G.A. (1975), "Analysis of flexibly connected steel frames", *Can. J. Civil. Eng*., **2**(3), 280-91.
Gorgun, H. (1992), "The nonlinear analysis of planar frames composed of flexibly connected members", *Cukurova Univ. J. Fac. Eng. Arch.*, **8**(2), 117-129.
Gorgun, H., Yilmaz, S., and Karacan, S.S. (2012), "Nonlinear analysis of frames composed of flexibly connected members with rigid end sections accounting for shear deformations", *Sci. Res. Essays.*, **7**(21), 2390-2408.
Gorgun, H., Yilmaz, S. (2012), "Geometrically nonlinear analysis of plane frames with semi-rigid connections accounting for shear deformations", *Struct. Eng. Mech.*, **44**(4), 539-569.
Jones, S.W., Kirby, P.A. and Nethercot, D.A. (1983), "The analysis of frames with semi-rigid connections-a state of the art report", *J. Construct Steel Res*., **3**(2), 2-13.
Liu, Y., Xu, L. and Grierson, D.E. (2008), "Compound-element modeling accounting for semi-rigid connections and member plasticity", *Eng. Struct*., **30**, 1292-1307.
Liu, Y. (2009), "Hybrid-member stiffness matrix accounting for geometrical nonlinearity and member inelasticity in semi-rigid frameworks", *Eng. Struct*., **31**, 2880-2895.
Livesley, R.K. and Chandler. D.B. (1956), *Stability functions for structural frameworks*, Manchester University Press, Manchester.
Majid, K.I. (1972), *Non-linear structures*, Butterworths, London.
Monforton, G.R. and Wu, T.S. (1963), "Matrix analysis of semi-rigidly connected frames," *ASCE J. Struct. Div*., **89**(ST6), 13-42.
Moree, D.B., Nethercot, D.A. and Kirby, P.A. (1993), "Testing steel frames at full scale: appraisal of results and implications for design", *Struct. Eng*., **71**, 428-35.
Mottram, J.T. and Aberle, M. (2002), "When should shear-flexible stability functions be used in elastic structural analysis?", *Proceedings of the Institution of Civil Engineers: Structures and Buildings*, **152**(1), 31-40.
Mottram, J.T. (2008), "Stability analysis for pitched portal frames of fibre reinforced polymer", *Proceedings of the 4th International Conference on FRP Composites in Civil Engineering (CICE 2008)*, Empa, Dübendorf.
Nethercot, D,A. (1985), *Steel beam-to-column connections-a review of test data*, CIRIA, London.
Romstad, K,M. and Subramanian, C,V. (1970), "Analysis of frames with partial connection rigidity", *ASCE J. Struct. Eng*., **96**(ST11), 2283-2300.
Timoshenko, S.P. and Gere, J,M. (1961), *Theory of elastic stability*, 2nd Edition, McGraw-Hill, New York.
Wu, F.S. and Chen, W.F. (1990), "A design model for semi-rigid connections", *Eng. Struct*., **12**(2), 88-97.
Xu, L., Liu, Y. and Grierson, D,E. (2005), "Nonlinear analysis of steel frameworks through direct modification of member stiffness properties", *Adv. Eng. Software*, **36**, 312-324.