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Size-dependent analysis of functionally graded ultra-thin films

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Abstract. In this paper, the first-order shear deformation theory (FSDT) (Mindlin) for continuum incorporating surface energy is exploited to study the static behavior of ultra-thin functionally graded (FG) plates. The size-dependent mechanical response is very important while the plate thickness reduces to micro/nano scales. Bulk stresses on the surfaces are required to satisfy the surface balance conditions involving surface stresses. Unlike the classical continuum plate models, the bulk transverse normal stress is preserved here. By incorporating the surface energies into the principle of minimum potential energy, a series of continuum governing differential equations which include intrinsic length scales are derived. The modifications over the classical continuum stiffness are also obtained. To illustrate the application of the theory, simply supported micro/nano scaled rectangular films subjected to a transverse mechanical load are investigated. Numerical examples are presented to present the effects of surface energies on the behavior of functionally graded (FG) film, whose effective elastic moduli of its bulk material are represented by the simple power law. The proposed model is then used for a comparison between the continuum analysis of FG ultra-thin plates with and without incorporating surface effects. Also, the transverse shear strain effect is studied by a comparison between the FG plate behavior based on Kirchhoff and Mindlin assumptions. In our analysis the residual surface tension under unstrained conditions and the surface Lame constants are expected to be the same for the upper and lower surfaces of the FG plate. The proposed model is verified by previous work.

Keywords: functionally graded plates; surface energy effect; ultra-thin films; size-dependent analysis; finite element analysis

1. Introduction

Atoms at a free surface experience a different local environment than do atoms in the bulk of a material. As a result, the energy associated with these atoms will be different from that of the atoms in the bulk. The excess energy associated with surface atoms is called surface free energy. In classical continuum mechanics, such surface free energy is typically neglected because it is associated with only a few layers of atoms near the surface and the ratio of the volume occupied by the surface atoms and the total volume of material of interest is extremely small. However, for micro/nano-size particles, wires and films, the surface to volume ratio becomes significant, and so

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does the effect of surface free energy.

Ultra-thin plate structures with submicron thicknesses have attracted much attention due to their potential as sensitive, high frequency devices for applications in Micro-electromechanical Systems (MEMS) and Nano-electromechanical Systems (NEMS) (Evoy *et al.* 1999, Lavrik *et al.* 2004, Craighead 2000). For structures with submicron sizes, due to the increasing surface-to-bulk ratio, surface effects are likely to be significant and can considerably modify macroscopic properties.

When the thickness of ultra-thin plates-like films reduces to submicron scale, the surface effect, which is usually neglected in classical thin plate elasticity theory, becomes significant with the increasing of surface-to-bulk ratio (Cammarata 1994, Muller and Saul 2004). It is noticed obviously that there exists a size-dependent mechanical behavior of ultra-thin elastic films with nano-scale thickness (Cammarata and Sieradzki 1989, Wolf 1991, Miller and Shenoy 2000, Liang *et al.* 2002).The understanding and modeling of such size-dependence due to surface effects is currently of particular interest (He *et al.* 2004, Sharma and Ganti 2004, He and Li 2006).

Atomistic simulations results have shown that elastic constants of ultra-thin films can be larger or smaller than their bulk counterparts due to the effect of surface elasticity (Zhou and Huang 2004, Shim *et al.* 2005). In addition, the atomistic lattice model further demonstrates that the values of elastic constants of ultra-thin films are thickness dependent and approach the bulk value as the film thickness increases (Sun and Zhang 2003, Zhang and Sun 2004, Guo and Zhao 2005). However, systematic atomistic studies of mechanical response of thin films need a tremendous computational source and hence they are limited in practical applications.

Gurtin and Murdoch (1975a, b, 1978) formulated a generic continuum model of surface elasticity, where the surface of solids can be viewed as a two dimensional elastic membrane with different material constants adhering to the underlying bulk material without slipping. It is found that the continuum by incorporating surface elasticity can predict the same accurate elastic response of thin films as the case of atomistic modeling if the proper surface constitutive constants are used (Miller and Shenoy 2000). Recently, He *et al.* (2004) proposed a rigorous continuum surface elasticity model and successfully analyzed the size-dependent deformation of nano-films. The surface effects on the large deflection of ultra-thin films are investigated by incorporating surface elasticity into the Von Karman plate theory without consideration of the non-zero normal stress along the thickness direction (Lim and He 2004).The continuum model proposed by Lu *et al.* (2006) takes into account the effect of non-zero normal stress but neglects the effect of nonlinearity. Huang (2008), investigated a modified continuum model of elastic films with nano-scale thickness by incorporating surface elasticity into the conventional nonlinear Von Karman Plate theory. By using Hamilton's principle, the governing equations and boundary conditions of ultra-thin film including surface effects are derived with the Kirchhoff's assumptions, where the effect of non-zero normal stress and large deflection are taken into account simultaneously. The proposed model is then applied to study the bending, buckling and free vibration of simply supported micro/nano-scale thin film in-plane strains and explicit exact solutions are obtained for these three cases.

For the surface of solids, Gibbs (1961) pointed out that surface energy and surface stress are not identical; meaning that, a different amount of reversible work is required to form a unit surface than to increase a large surface by unit area through reversible stretching it. Shuttleworth (1950) derived the relations between surface stress and surface strain for small deformations, which are interpreted from an atomistic viewpoint (Nix and Gao 1998). Gurtin and Murdoch (Gurtin and Murdoch 1975a, Murdoch 2005) established the theoretical framework of the surface elasticity under the classical theory of membrane. Steigmann and Ogden (Steigmann 1999, Steigmann and Ogden 1997) generalized the Gurtin-Murdoch theory to incorporate flexural stiffness of the free surface directly into the constitutive response of surface. Dingreville and Qu (2008) investigated the influence of Poisson's ratio effect on the surface properties under general loading conditions. The effect of surface tension on the elastic properties of nano structures is studied by Wang *et al.* (2010). Where in the absence of external loading, surface tension will induce a residual stress field in the bulk of nano structures. Based on the elastic behavior of nano-sized structural elements such as nano-particles, nano-wires and nano-films, Dingreville *et al.* (2005) investigated an approach for the size dependency of the overall elastic behavior of such nano-sized structural elements. For more review the reader can refer to (Wang *et al.* 2011).

Functionally graded materials (FGMs) are microscopically inhomogeneous composite materials, in which the volume fraction of the two or more materials is varied smoothly and continuously as a continuous function of the material position along one or more dimension of the structure. These materials are mainly constructed to operate in high temperature environments. The concept of functionally graded material (FGM) was proposed in 1984 by the material scientists in Japan (Koizumi 1997). Alieldin *et al.* (2011) suggested three approaches to transform the laminated composite plate, with stepped material properties, to an equivalent functionally graded (FG) plate with a continuous property function across the plate thickness. Such transformations are used to determine the details of a functional graded plate equivalent to the original laminated one. FGMs are usually made of a mixture of ceramic and metals. The ceramic constituent of the material provides a high temperature resistance due to its low thermal conductivity, while the ductile metal constituent, on the other hand, prevents the fracture caused by thermal stress due to high temperature gradient in a very short period of time.

The FGM is suitable for various applications, such as thermal coatings of barrier for ceramic engines, gas turbines, nuclear fusions, optical thin layers, biomaterial electronics, etc. Alibeigloo (2010) derived an exact solution for thermo-elastic response of functionally graded rectangular plates subjected to thermo-mechanical loads. A finite element analysis of thermo-elastic field in a rotating FGM circular disk is studied by Afsar and Go (2010). This study focuses on the finite element analysis of thermo-elastic field in a thin circular functionally graded material disk subjected to a thermal load and an inertia force due to rotation of the disk. Tung and DinhDuc (2010) derived a simple analytical approach to investigate the nonlinear stability of functionally graded plates under mechanical and thermal loads. Equilibrium and compatibility equations for FG plates are derived by using the classical plate theory.

Functionally graded materials are used in many applications, owing to their stability in high thermal environments. To this aim, many approaches are developed to study the thermo-elastic behavior of functionally graded materials. One of these approaches is finite element analysis of such material type.

A generalized refined theory including surface effects is developed by Lü *et al.* (2009a, b) for functionally graded ultra-thin films with different surface properties. The classical generalized shear deformable theory is adopted to model the film bulk, while the bulk stresses along the surfaces of the bulk substrate are required to satisfy the surface balance equations of the continuum surface elasticity. As a result, the shape function also shows size-dependence on the film thickness. It is established that the proposed FGM thin films exhibit significant size-dependence when the thickness approaches to micro-scale. The theory is then used to investigate a simply supported thin film in cylindrical bending. Numerical examples are presented to clarify the effects of surface energies on the bending behavior of FGM films, whose effective elastic moduli are predicted using the Mori–

Tanaka method. The nature of intrinsic length scales and the effects of gradient index and aspect ratio on the displacements are also discussed.

In this paper, the first-order shear deformation theory (FSDT) (Mindlin) for continuum incorporating surface energy is exploited to study the behavior of ultra-thin functionally graded (FG) plates. By incorporating the surface energies into the principle of minimum potential energy, a series of continuum governing differential equations which include intrinsic length scales are derived. The modifications over the classical continuum stiffness are also obtained. Simply supported micro/nano scaled rectangular films subjected to a transverse mechanical load are investigated. Also, a parametric study is provided to investigate the effect of surface energy and transverse shear strain on the FG plate response. In our analysis the residual surface tension under unstrained conditions and the surface Lame constants are expected to be the same for the upper and lower surfaces of the FG plate.

2. Formulation of a continuum plate model incorporating surface effects

In this section the FSDT incorporating surface effects is presented. The FG plate is expected to consist of two material constituents. In our analysis the residual surface tension under unstrained conditions and the surface Lame constants are expected to be the same for the upper and lower surfaces of the FG plate. A series of continuum governing differential equations which include surface energy and transverse shear strain effects are derived. The obtained modifications over the classical continuum model concern two aspects, the material stiffnesses and governing equations. The formulation is built up based on the model that given by Lu *et al.* (2006).

2.1 Governing equations for classical continuum model

In the first order shear deformation plate theory (FSDT) (Mindlin), the Kirchhoff hypothesis is relaxed by making the assumption; the transverse normals do not remain perpendicular to the mid surface after deformation. This amounts to including transverse shear strains in the theory. The inextensibility of transverse normals requires that the vertical deflection ω not be a function of the thickness coordinate z.

Under the same assumptions and restrictions of the classical laminate theory, the displacement field of the first order theory is of the form

$$\begin{aligned} u(x,y,z) &= u_0(x,y)+z\O_x(x,y) \\ v(x,y,z) &= v_0(x,y)+z\O_y(x,y) \\ \omega(x,y,z) &= \omega_0(x,y) \end{aligned} \tag{1}$$

where $(u_0, v_0, \omega_0, \O_x, \O_y)$ are unknown functions to be determined, and (u_0, v_0, ω_0) denotes the displacements of the mid plane ($z = 0$). Note that $\partial u/\partial z = \O_x$ and $\partial v/\partial z = \O_y$ which indicate that $\O_x$ and $\O_y$ are the rotations of a transverse normal about the y-axis and x-axis respectively.

The strain is the variation of the continuum deformation with respect to its volume. So, the linear Green-Lagrange strains components for small deformations and moderate rotations (10º-15º) can be determined from the displacement field given in Eq. (1) as follow

$$\begin{aligned}\varepsilon_{xx} &= u_{0,x} + z\varnothing_{x,x} \\ \varepsilon_{yy} &= v_{0,y} + z\varnothing_{y,y} \\ \gamma_{xy} &= u_{0,y} + v_{0,x} + z(\varnothing_{x,y} + \varnothing_{y,x}) \\ \gamma_{xz} &= \omega_{0,x} + \varnothing_x \\ \gamma_{yz} &= \omega_{0,y} + \varnothing_y\end{aligned} \tag{2}$$

where ε_{ij} are the strain components.

The governing equations of the first order theory will be derived using the principle of minimum potential energy

$$\delta\Pi = \delta U - \delta V = 0 \tag{3}$$

where the virtual strain energy δU, and the virtual work done by applied forces δV are given by

$$\delta U = \int_A \left(\int_z \sigma_{ij} \delta\varepsilon_{ij} dz \right) dxdy \tag{4}$$

$$\delta V = \int_A q\, \delta\omega_0 dxdy \tag{5}$$

where q is the distributed force at the upper surface of the plate. So by integrating through the plate thickness, the minimum total potential energy in terms of the nodal displacements is given by substituting for strain components into Eq. (4) and Eq. (5).

$$\begin{aligned}\delta\Pi = {} & N_{xx,x}\delta u_0 + N_{yy,y}\delta v_0 + N_{xy,y}\delta u_0 + N_{xy,x}\delta v_0 + M_{xx,x}\delta\varnothing_x + M_{yy,y}\delta\varnothing_y + M_{xy,y}\delta\varnothing_x \\ & + M_{xy,x}\delta\varnothing_y + N_{xz,x}\delta\omega_0 + N_{xz}\delta\varnothing_x + N_{yz,x}\delta\omega_0 + N_{yz}\delta\varnothing_y - q\,\delta\omega_0 = 0\end{aligned} \tag{6}$$

where (N_{ij} and M_{ij}) are the stress resultants and ($\delta u_0, \delta v_0, \delta\omega_0, \delta\varnothing_x, \delta\varnothing_y$) are the virtual displacements. So the Euler-Lagrange equations of the theory are obtained by setting the virtual displacements over domain to zero separately

$$\begin{aligned}\delta u_0&: \; N_{xx,x} + N_{xy,y} = 0 \\ \delta v_0&: \; N_{xy,x} + N_{yy,y} = 0 \\ \delta\omega_0&: \; N_{xz,x} + N_{yz,y} - q = 0 \\ \delta\varnothing_x&: \; M_{xx,x} + M_{xy,y} - N_{xz} = 0 \\ \delta\varnothing_y&: \; M_{xy,x} + M_{yy,y} - N_{yz} = 0\end{aligned} \tag{7}$$

The stress resultants can be obtained by integration of the stress components through the plate thickness for classical continuum model neglecting surface effects as follow

$$N_{ij} = \int_{-h/2}^{h/2} \sigma_{ij} dz, \quad M_{ij} = \int_{-h/2}^{h/2} z\,\sigma_{ij} dz \tag{8}$$

The stress resultants are related to the generalized displacements $(u_0, v_0, \omega_0, \varnothing_x, \varnothing_y)$ by the following relations for plates neglecting surface effects

$$
\begin{aligned}
N_{xx} &= A_{11}u_{0,x}+A_{12}v_{0,y}+B_{11}\text{Ø}_{x,x}+B_{12}\text{Ø}_{y,y} \\
N_{yy} &= A_{22}v_{0,y}+A_{12}u_{0,x}+B_{22}\text{Ø}_{y,y}+B_{12}\text{Ø}_{x,x} \\
N_{xy} &= A_{66}(u_{0,y}+v_{0,x})+B_{66}(\text{Ø}_{y,x}+\text{Ø}_{x,y}) \\
N_{xz} &= A_{44}(\omega_{0,x}+\text{Ø}_{x,x}) \\
N_{yz} &= A_{55}(\omega_{0,y}+\text{Ø}_{y,y}) \\
M_{xx} &= D_{11}\text{Ø}_{x,x}+D_{12}\text{Ø}_{y,y}+B_{11}u_{0,x}+B_{12}v_{0,y} \\
M_{yy} &= D_{11}\text{Ø}_{y,y}+D_{12}\text{Ø}_{x,x}+B_{22}v_{0,y}+B_{12}u_{0,x} \\
M_{xy} &= D_{66}(\text{Ø}_{x,y}+\text{Ø}_{y,x})+B_{66}(u_{0,y}+v_{0,x})
\end{aligned} \tag{9}
$$

where for the classical continuum model

$$
A_{ij} = \int_{-h/2}^{h/2} \overline{Q}_{ij}dz\,, \quad D_{ij} = \int_{-h/2}^{h/2} z^2\overline{Q}_{ij}dz\,, \quad B_{ij} = \int_{-h/2}^{h/2} z\overline{Q}_{ij}dz \tag{10}
$$

where $\overline{Q}_{ij}$ are the equivalent material property stiffnesses (Alieldin *et al.* 2011).

Finally, the equilibrium equations for isotropic FG plate for the classical continuum model where is no body forces are included neglecting surface effects in terms of the nodal displacements can be written as

$$
\begin{gathered}
A_{11}u_{0,xx}+A_{12}v_{0,xy}+B_{11}\text{Ø}_{x,xx}+B_{12}\text{Ø}_{y,xy}+A_{66}u_{0,yy}+A_{66}v_{0,xy}+B_{66}\text{Ø}_{x,yy}+B_{66}\text{Ø}_{y,xy} = 0 \\
A_{66}u_{0,xy}+A_{66}v_{0,xx}+B_{66}\text{Ø}_{x,xy}+B_{66}\text{Ø}_{y,xx}+A_{12}u_{0,xy}+A_{22}v_{0,yy}+B_{12}\text{Ø}_{x,xy}+B_{22}\text{Ø}_{y,yy} = 0 \\
A_{44}\omega_{0,xx}+A_{44}\text{Ø}_{x,x}+A_{55}\omega_{0,yy}+A_{55}\text{Ø}_{y,y}-q = 0 \\
B_{11}u_{0,xx}+B_{12}v_{0,xy}+D_{11}\text{Ø}_{x,xx}+D_{12}\text{Ø}_{y,xy}+B_{66}u_{0,yy}+B_{66}v_{0,xy}+D_{66}\text{Ø}_{x,yy}+D_{66}\text{Ø}_{y,xy}+A_{44}(\omega_{0,x}+\text{Ø}_x) = 0 \\
B_{12}u_{0,xy}+B_{22}v_{0,yy}+D_{12}\text{Ø}_{x,xy}+D_{22}\text{Ø}_{y,yy}+B_{66}u_{0,xy}+B_{66}v_{0,xx}+D_{66}(\text{Ø}_{x,xy}+\text{Ø}_{y,xx})+A_{55}(\omega_{0,y}+\text{Ø}_y) = 0
\end{gathered} \tag{11}
$$

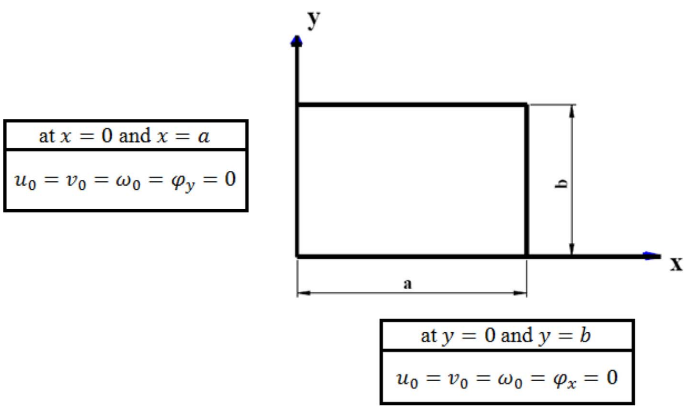

Fig. 1 A simply supported boundary conditions used in the FG plate solution

The illustrated governing equations are then solved and used to study the mechanical behavior of FG plates subjected to sinusoidal distributed load. Fig. 1 shows a set of simply supported boundary conditions that are used in the solution of the FG plate. So, based on the illustrated boundary conditions, the terms that have B_{ij}, A_{11}, A_{12} and A_{66} can be eliminated from the governing equations.

2.2 Governing equations for continuum model incorporating surface energy

In this section a mathematical continuum model incorporating surface effects for plates subjected to mechanical loads is formulated based on the first-order shear deformation theory. The plates are expected to be functionally graded through the plate thickness according to the simple power law.

Consider a thin plate structure with thickness h. The upper and lower surfaces S^+ and S^- of the plate are defined by $z = \pm h/2$, respectively. The governing equations for the body of the plate, where there is no body forces are included, are given by

$$\sigma_{ij,j} = 0 \tag{12}$$

where σ_{ij} denotes the stress components. The surface stresses on the surfaces S^+ and S^- of the plate are denoted by $\overset{+}{\tau}_{i\alpha}$ and $\overset{-}{\tau}_{i\alpha}$, respectively, and satisfied the equilibrium relations (Gurtin and Murdoch 1975a, 1978)

$$\begin{aligned} \overset{+}{\tau}_{i\alpha,\alpha} - \overset{+}{\sigma}_{iz} &= 0 \\ \overset{-}{\tau}_{i\alpha,\alpha} + \overset{-}{\sigma}_{iz} &= 0 \qquad i = x, y, z; \quad \alpha = x, y \end{aligned} \tag{13}$$

where $\overset{\pm}{\sigma}_{iz} = \sigma_{iz}(z = \pm h/2)$ are the bulk stresses at $z = \pm h/2$, respectively.

Since the plate is thin, the stress component σ_{zz} is small comparing to the in-plane stress components, which is simply assumed to be zero in the classical plate theories. However, the surface condition Eq. (13) will not be satisfied with the assumption. To improve the weakness, it is assumed here that the stress component σ_{zz} varies linearly through the thickness and satisfies the balance conditions on the surfaces. With the assumption, σ_{zz} can be written as

$$\sigma_{zz} = 0.5(\overset{+}{\tau}_{\beta z,\beta} - \overset{-}{\tau}_{\beta z,\beta}) + \frac{z}{h}(\overset{+}{\tau}_{\beta z,\beta} + \overset{-}{\tau}_{\beta z,\beta}) \tag{14}$$

The resultant forces N_{ij} and resultant moments M_{ij} are defined in Eq. (8). So, the governing equations for continuum plates incorporating surface energy take the form

$$\begin{aligned} N_{xx,x} + N_{xy,y} + \overset{+}{\tau}_{xx,x} + \overset{-}{\tau}_{xx,x} + \overset{+}{\tau}_{xy,y} + \overset{-}{\tau}_{xy,y} &= 0 \\ N_{yy,y} + N_{xy,x} + \overset{+}{\tau}_{yy,y} + \overset{-}{\tau}_{yy,y} + \overset{+}{\tau}_{xy,x} + \overset{-}{\tau}_{xy,x} &= 0 \\ N_{xz,x} + N_{yz,y} + \overset{+}{\tau}_{yz,y} + \overset{-}{\tau}_{yz,y} + \overset{+}{\tau}_{xz,x} + \overset{-}{\tau}_{xz,x} - q &= 0 \\ M_{xx,x} + M_{xy,y} - N_{xz} + \frac{h}{2}(\overset{+}{\tau}_{xx,x} - \overset{-}{\tau}_{xx,x}) + \frac{h}{2}(\overset{+}{\tau}_{xy,y} - \overset{-}{\tau}_{xy,y}) &= 0 \\ M_{yy,y} + M_{xy,x} - N_{yz} + \frac{h}{2}(\overset{+}{\tau}_{yy,y} - \overset{-}{\tau}_{yy,y}) + \frac{h}{2}(\overset{+}{\tau}_{xy,x} - \overset{-}{\tau}_{xy,x}) &= 0 \end{aligned} \tag{15}$$

If the surface stresses are neglected, Eq. (15) are reduced to the classical continuum plates governing equations (Eq. (7)). The generalized resultant forces and resultant moments can be defined for continuum plate incorporating surface effects as

$$N^*_{i\alpha} = N_{i\alpha} + \overset{+}{\tau}_{i\alpha} + \overset{-}{\tau}_{i\alpha}; \quad i = x, y, z$$

$$M^*_{\alpha\beta} = M_{\alpha\beta} + \frac{h}{2}(\overset{+}{\tau}_{\alpha\beta} - \overset{-}{\tau}_{\alpha\beta}); \quad \alpha, \beta = x, y \tag{16}$$

where the constitutive relations of the surface layers S^+ and S^- as given by Gurtin and Murdoch (1975a, 1978) can be expressed as follow, where the top and bottom layers have same material properties

$$\overset{\pm}{\tau}_{\alpha\beta} = \tau_0\delta_{\alpha\beta} + (\mu_0 - \tau_0)(\overset{\pm}{u}_{\alpha,\beta} + \overset{\pm}{u}_{\beta,\alpha}) + (\lambda_0 + \tau_0)\overset{\pm}{u}_{\gamma,\gamma}\delta_{\alpha\beta} + \tau_0\overset{\pm}{u}_{\alpha,\beta}$$

$$\overset{\pm}{\tau}_{\alpha z} = \tau_0\overset{\pm}{u}_{z,\alpha} \tag{17}$$

Where τ_0 is the residual surface tensions under unconstrained conditions, λ_0 and μ_0 are the surface Lame constants on the upper and lower surfaces, and $\overset{\pm}{u}_i(i = x, y, z = \pm h/2)$ denotes the displacement fields as

$$\overset{\pm}{u}_x = u(x, y, \pm h/2) = u_0(x, y) \pm \frac{h}{2}\text{Ø}_x(x, y)$$

$$\overset{\pm}{u}_y = v(x, y, \pm h/2) = v_0(x, y) \pm \frac{h}{2}\text{Ø}_y(x, y)$$

$$\overset{\pm}{u}_z = \omega(x, y) = \omega_0(x, y) \tag{18}$$

The resultant forces $N^*_{i\alpha}$ and the resultant moments $M^*_{\alpha\beta}$ for the FSDT can be obtained by substituting the displacement fields and the strain components into Eq. (14) and then into Eq. (8) and Eq. (16). If the top and bottom surface layers have considered having the same material properties, the resultant forces and moments can be obtained as

$$N^*_{xx} = 4\tau_0 + (A_{11} + 2(2\mu_0 + \lambda_0))u_{0,x} + (A_{12} + 2(\lambda_0 + \tau_0))v_{0,y} + B_{11}\text{Ø}_{x,x} + B_{12}\text{Ø}_{y,y}$$

$$N^*_{yy} = 4\tau_0 + (A_{22} + 2(2\mu_0 + \lambda_0))v_{0,y} + (A_{12} + 2(\lambda_0 + \tau_0))u_{0,x} + B_{22}\text{Ø}_{y,y} + B_{12}\text{Ø}_{x,x}$$

$$N^*_{xy} = 4\tau_0 + (A_{66} + 2\mu_0)(u_{0,y} + v_{0,x}) + B_{66}(\text{Ø}_{y,x} + \text{Ø}_{x,y})$$

$$N^*_{xz} = A_{44}(\omega_{0,x} + \text{Ø}_{x,x}) + 2\tau_0\omega_{0,x}$$

$$N^*_{yz} = A_{55}(\omega_{0,y} + \text{Ø}_{y,y}) + 2\tau_0\omega_{0,y}$$

$$M^*_{xx} = \left(D_{11} + \mu_0 h^2 + \frac{\lambda_0 h^2}{2}\right)\text{Ø}_{x,x} + \left(D_{12} + \frac{(\lambda_0 + \tau_0)h^2}{2}\right)\text{Ø}_{y,y} + \left(\frac{\upsilon\tau_0 h^2}{6(1-\upsilon)}\right)(\omega_{0,xx} + \omega_{0,yy}) + B_{11}u_{0,x} + B_{12}v_{0,y}$$

$$M^*_{yy} = \left(D_{11} + \mu_0 h^2 + \frac{\lambda_0 h^2}{2}\right)\emptyset_{y,y} + \left(D_{12} + \frac{(\lambda_0 + \tau_0)h^2}{2}\right)\emptyset_{x,x} + \left(\frac{\upsilon\tau_0 h^2}{6(1-\upsilon)}\right)(\omega_{0,xx} + \omega_{0,yy}) + B_{22}v_{0,y} + B_{12}u_{0,x}$$

$$M^*_{xy} = \left(D_{66} + \frac{\mu_0 h^2}{2}\right)(\emptyset_{x,y} + \emptyset_{y,x}) + B_{66}(u_{0,y} + v_{0,x}) \tag{19}$$

So, the equilibrium equations for the continuum model incorporating surface effects for FG plates, take the form

$$A^*_{11}u_{0,xx} + A^*_{12}v_{0,xy} + B_{11}\emptyset_{x,xx} + B_{12}\emptyset_{y,xy} + A^*_{66}u_{0,yy} + A^*_{66}v_{0,xy} + B_{66}\emptyset_{x,yy} + B_{66}\emptyset_{y,xy} = 0$$

$$A^*_{66}u_{0,xy} + A^*_{66}v_{0,xx} + B_{66}\emptyset_{x,xy} + B_{66}\emptyset_{y,xx} + A^*_{12}u_{0,xy} + A^*_{22}v_{0,yy} + B_{12}\emptyset_{x,xy} + B_{22}\emptyset_{y,yy} = 0$$

$$A_{44}(\omega_{0,xx} + \emptyset_{x,x}) + 2\tau_0\omega_{0,xx} + A_{55}(\omega_{0,yy} + \emptyset_{y,y}) + 2\tau_0\omega_{0,yy} = q$$

$$B_{11}u_{0,xx} + B_{12}v_{0,xy} + D^*_{11}\emptyset_{x,xx} + D^*_{12}\emptyset_{y,xy} + B_{66}u_{0,yy} + B_{66}v_{0,xy} + D^*_{66}\emptyset_{x,yy} + D^*_{66}\emptyset_{y,xy}$$

$$+ A_{44}(\omega_{0,x} + \emptyset_x) + \left(\frac{\upsilon\tau_0 h^2}{6(1-\upsilon)}\right)(\omega_{0,xxx} + \omega_{0,yyx}) = 0$$

$$B_{12}u_{0,xy} + B_{22}v_{0,yy} + D^*_{12}\emptyset_{x,xy} + D^*_{22}\emptyset_{y,yy} + B_{66}u_{0,xy} + B_{66}v_{0,xx} + D^*_{66}(\emptyset_{x,xy} + \emptyset_{y,xx}) + A_{55}(\omega_{0,y} + \emptyset_y)$$

$$+ \left(\frac{\upsilon\tau_0 h^2}{6(1-\upsilon)}\right)(\omega_{0,xxy} + \omega_{0,yyy}) = 0 \tag{20}$$

where the material parameters for isotropic FG plate will be

$$A^*_{11} = A^*_{22} = A_{11} + 2(2\mu_0 + \lambda_0)$$

$$A^*_{12} = A_{12} + 2(\lambda_0 + \tau_0)$$

$$A^*_{66} = A_{66} + 2\mu_0$$

$$D^*_{11} = D^*_{22} = D_{11} + \mu_0 h^2 + \frac{\lambda_0 h^2}{2}$$

$$D^*_{12} = D_{12} + \frac{(\lambda_0 + \tau_0)h^2}{2}$$

$$D^*_{66} = D_{66} + \frac{\mu_0 h^2}{2} \tag{21}$$

By comparing Eq. (20) with Eq. (11), some terms have been added to the governing equations of the classical model to incorporate surface effects. The additional terms appeared as a result of the plate surface tension, which provides additional stiffnesses to the classical form of the governing equations. For zero surface tensions, Eq. (20) reduces to the classical form of governing equations.

Also, in Eq. (21) some terms have been added to the classical plate stiffnesses to incorporate surface effects.

2.3 Effect of surface energy on the constitutive relations of the bulk material of the plate

In this section the effect of the surface energy on the bulk of the isotropic FG plate is focused. This effect appears on the plate continuum strains and stresses. The obtained modifications over the classical continuum model affect the plate deformation, and hence the continuum strains. The plate continuum stresses are related to its continuum strains and material stiffnesses.

$$\sigma_{xx}^{*} = \bar{Q}_{11}(u_{0,x}+z\varnothing_{x,x})+\bar{Q}_{12}(v_{0,y}+z\varnothing_{y,y})+\left(\frac{\upsilon}{(1-\upsilon)}\left(\frac{2\tau_0 z}{h}\right)\right)(\omega_{0,xx}+\omega_{0,yy})$$

$$\sigma_{yy}^{*} = \bar{Q}_{22}(v_{0,y}+z\varnothing_{y,y})+\bar{Q}_{12}(u_{0,x}+z\varnothing_{x,x})+\left(\frac{\upsilon}{(1-\upsilon)}\left(\frac{2\tau_0 z}{h}\right)\right)(\omega_{0,xx}+\omega_{0,yy})$$

$$\sigma_{xy}^{*} = \bar{Q}_{66}[(u_{0,y}+v_{0,x})+z(\varnothing_{y,x}+\varnothing_{x,y})]$$

$$\sigma_{xz}^{*} = \bar{Q}_{44}[(\omega_{0,x}+\varnothing_{x})]$$

$$\sigma_{yz}^{*} = \bar{Q}_{55}[(\omega_{0,y}+\varnothing_{y})] \tag{22}$$

3. Effective mechanical properties of FG plates

Mechanical properties of a FG plate include Young's modulus E, shear modulus G, Poisson's ratio υ, and mass density ρ, and thermal properties include the coefficient of thermal expansion α, the thermal conductivity γ, and the specific heat capacity C. ρ and υ are usually linear functions of material volume ratios, but others are nonlinear functions of material volume ratios because they depend on material microstructures (Aboudi 1991). The effective property P of a FG plate will be estimated using the simple, Voigt arithmetic method where the distributions of volume fractions through the plate thickness are assumed to follow the simple power law

$$P = P_1+(P_2-P_1)\left(\frac{z-z_i}{z_{i+1}-z_i}\right)^n$$

where P_1 and P_2 are the properties of the first and second constituent materials (metal and ceramic), n can be any non-negative real number and $z_{i+1} \geq z \geq z_i$. Where the plate thickness $h = z_{i+1}-z_i$. For detailed modeling of effective material properties of FGMs the reader has to refer to Aboudi (1991), Suresh and Mortensen (1998), Muliana (2009).

4. Numerical results

In this section, some numerical examples are simulated to verify the proposed model and to study the effect of surface energy on the continuum plate response.

4.1 Verification of the model stiffnesses

The following numerical examples are used to verify the proposed continuum model incorporating surface effect. Consider an isotropic homogenous simple supported square plate of thickness h and side length a. The isotropic plate is expected to be made of two sets of material parameters are given in Table 1.

From the proposed continuum model incorporating surface effects, the bending stiffness for the isotropic homogenous plate for Material I and Material II are found to be

For Material I

$$D_{11}^{*} = 5 * 10^{9}\, h^{3} + 11500\, h^{2}$$

For Material II

$$D_{11}^{*} = 1.5936 * 10^{10}\, h^{3} - 1.5\, h^{2}$$

while, from the model proposed by Lu *et al.* (2006), the bending stiffness of the same plate materials are found to be

For Material I

$$D_{LH}^{*} = 5 * 10^{9}\, h^{3} + 11499.99995\, h^{2}$$

For Material II

$$D_{LH}^{*} = 1.5936 * 10^{10}\, h^{3} - 1.5\, h^{2}$$

also, from the model proposed by Lü *et al.* (2009a, b), the bending stiffness of the same plate materials are found to be

For Material I

$$D_{C.F}^{*} = 5 * 10^{9}\, h^{3} + 11500\, h^{2}$$

For Material II

$$D_{C.F}^{*} = 1.5936 * 10^{10}\, h^{3} - 1.5\, h^{2}$$

Table 1 Material I and Material II material properties

Material parameter	Material I	Material II
Young's Modulus	$E = 5.625 \times 10^{10}$ N/m^2	$E = 17.73 \times 10^{10}$ N/m^2
Poisson's ratio	$\upsilon = 0.25$	$\upsilon = 0.27$
Lama Constants	$\lambda_0 = 7000$ N/m, $\mu_0 = 8000$ N/m	$\lambda_0 = -8$ N/m, $\mu_0 = 2.5$ N/m
Surface tension	$\tau_0 = 110$ N/m	$\tau_0 = 1.7$ N/m

The bending stiffness based on the classical continuum model is given by for both materials

For Material I

$$D_{11} = 5 * 10^{9}\ h^{3}$$

For Material II

$$D_{11} = 1.5936 * 10^{10}\ h^{3}$$

Fig. 2 and Fig. 3, show the non-dimensional difference between the plate bending stiffness predicted by the continuum model incorporating surface effects and the classical continuum model for both Material I and Material II respectively $(D^{*}_{11} - D_{11}/D_{11})$. For Material I, the size effect becomes significant when the thickness of the plate is smaller than 10 μm (Fig. 2), while for Material II it is significant when the thickness of the plate is of order of 1 nm (Fig. 3). The results

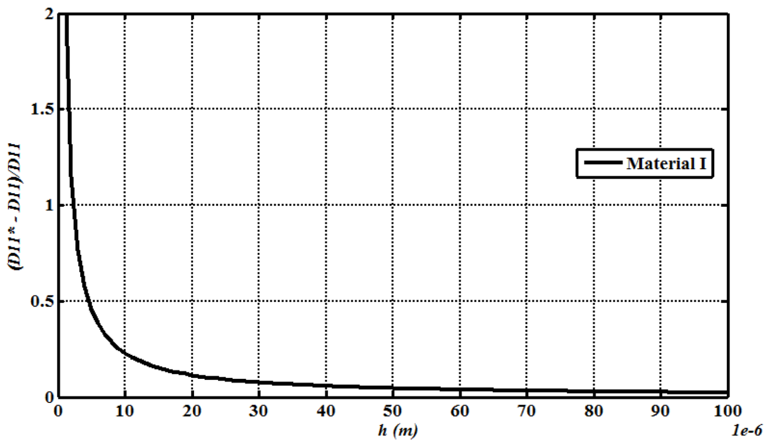

Fig. 2 Non-dimensional difference between plate bending stiffness predicted by continuum model incorporating surface effects and the classical continuum model for Material I

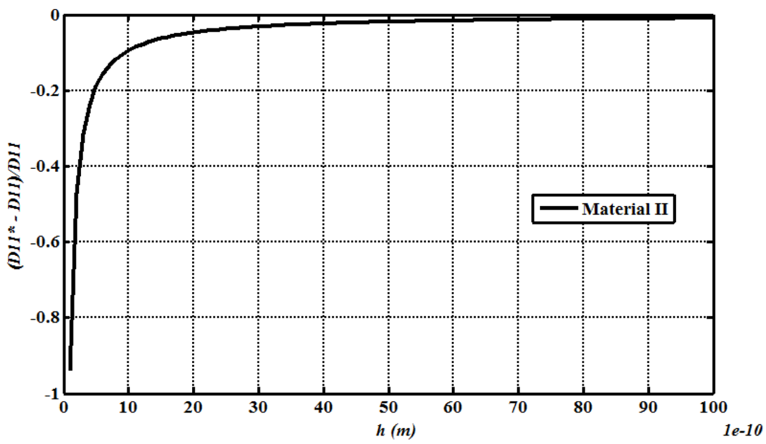

Fig. 3 Non-dimensional difference between plate bending stiffness predicted by continuum model incorporating surface effects and the classical continuum model for Material II

agree with the discussions of Lu *et al.* (2006) and Lim and He (2004) for the same problems. It is also noted that the bending stiffness increases for Material I, while decreases for Material II, when the plate thickness is reduced. It shows that surface effects could stiffen or soften the material properties (Zhou and Haung 2004). The significant difference of the thickness order on the influence of the size effects for Material I and Material II is due to the surface elastic properties defined for the two materials.

4.2 Size-dependent analysis of ultra-thin FG plates

In this section, some numerical examples are performed for infinitely wide isotropic FG plate of length a and of thickness h ($a/h = 10$). The FG plate is expected to be made of Silicon (Si) at the upper surface and Aluminum (Al) at the lower surface and functionally graded according to the simple power law. Table 2, shows the bulk material properties and the surface properties of Al and Si. The plate is subjected to a sinusoidal transverse distributed mechanical load of intensity $q = 1$ kPa.

Table 2 Material properties of the FG plate

Property	Aluminum	Silicon
Young's modulus	$E_L = 68.5$ GPa	$E_U = 210$ GPa
Poisson's ratio	$\upsilon_L = 0.35$	$\upsilon_U = 0.35$
Surface tension	$\tau_{0L} = 0.7575$ N/m	$\tau_{0U} = 0.7575$ N/m
Surface Lama constants	$\mu_{0L} = -1.575$, $\lambda_{0L} = 1.177$ N/m	$\mu_{0U} = -1.575$, $\lambda_{0U} = 1.177$ N/m

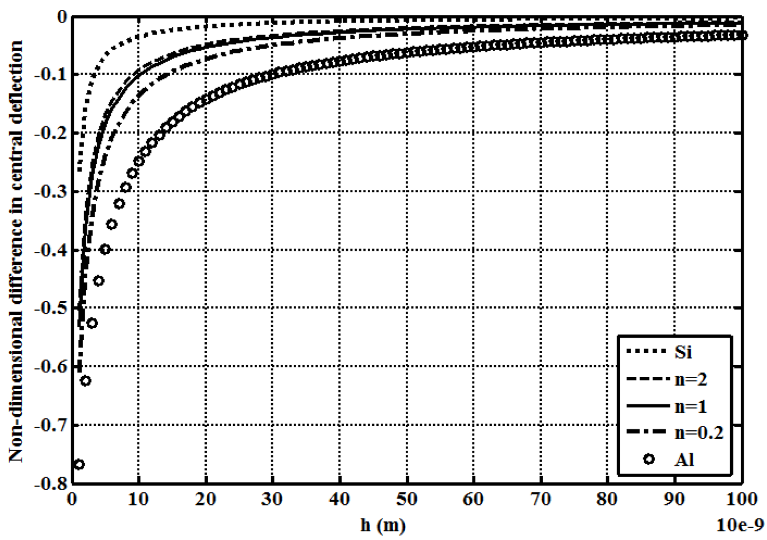

Fig. 4 Non-dimensional differences between deflection predicted by continuum model incorporating surface effects and classical continuum model for $a/h = 10$

Fig. 4, shows the non-dimensional difference between central deflection predicted by the size-dependent model (continuum model incorporating surface effects) and the classical continuum model (neglecting surface effects) ($\omega^* - \omega_0/\omega_0$) for different grading material parameter *n*. The results are agreed with that obtained at (Lü *et al.* 2009b). The figure shows that the FG and homogenous films provides negative non-dimensional differences in deflection, which means that surface tensions stiffen the films. Also, the surface tension of Al is higher than that of Si, so the non-dimensional differences for Al are higher than that of Si. Increasing the grading parameter *n* reducing the non-dimensional differences in deflection, for FG plates whose upper surface tension is smaller than its lower surface tension.

Figs. 5-7, show the non-dimensional stress distribution through the FG plate thickness for different grading parameter *n* and plate thicknesses. The non-dimensional axial stresses ($\bar{\sigma}_{xx} = \sigma_{xx}/q$) are calculated at the center point ($x = a/2$) for difference plate thicknesses. Fig. 5 shows a linear distribution for axial stress. For grading parameters more than zero, a nonlinear distribution is presented. The axial stresses are shown to be compressive at both surfaces of the

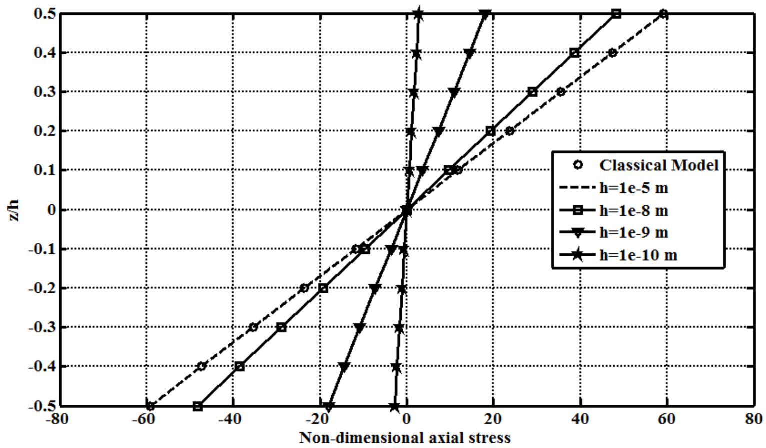

Fig. 5 Non-dimensional axial stress distribution through plate thickness ($n = 0$)

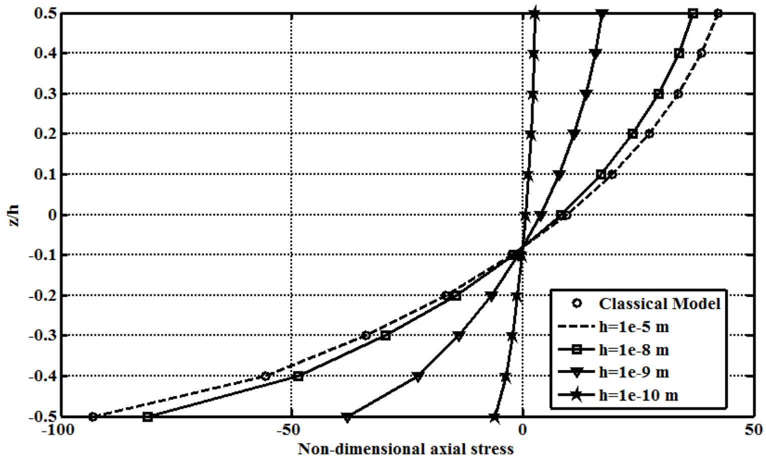

Fig. 6 Non-dimensional axial stress distribution through plate thickness ($n = 0.5$)

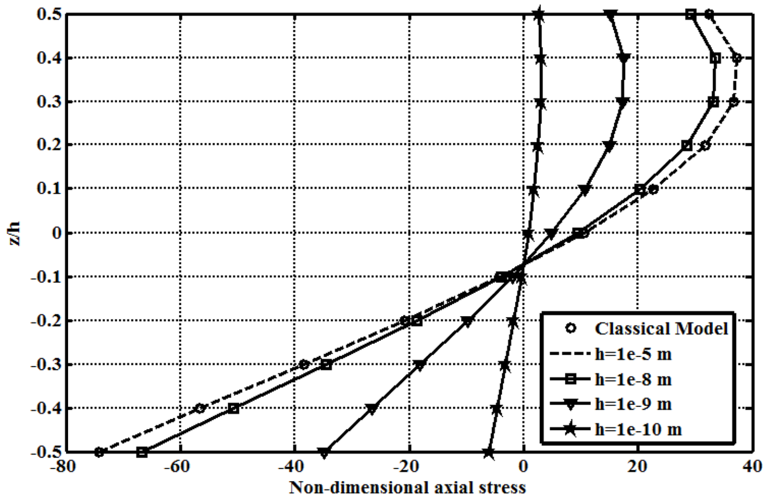

Fig. 7 Non-dimensional axial stress distribution through plate thickness ($n = 2$)

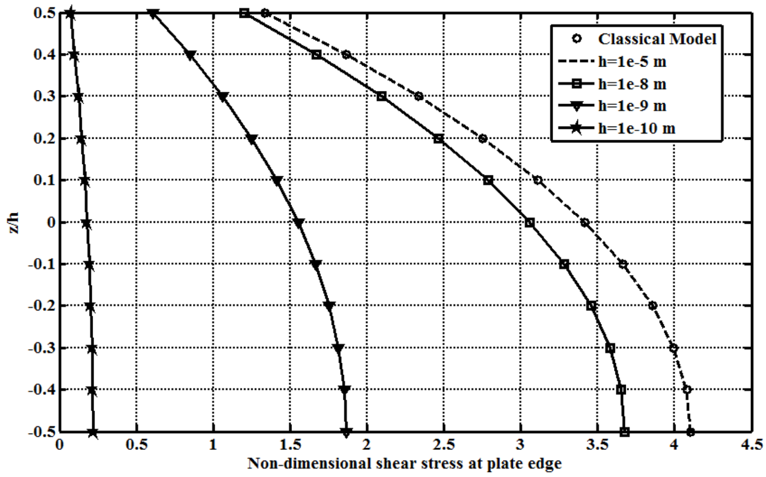

Fig. 8 Non-dimensional shear stress distribution through plate thickness at plate outer edge ($n = 2$)

plate, and this compressive stress increases gradually by decreasing the plate thickness.

Fig. 8, shows the non-dimensional shear stress distribution through the plate thickness for grading parameter n = 2 at the outer edge of the plate (x = 0).

4.3 Transverse shear strain effect

The incoming numerical examples are performed for a cylindrical bending of an infinitely wide isotropic FG plate of length $a = 1 \times 10^{-8}$ m and of thickness h. The ratio a/h varies from 1 to 50. Fig. 9 shows the non-dimensional deflection $\overline{\omega} = \omega h^3/a^4$ versus a/h ratio for both Kirchhoff and Mindlin plate theories considering and neglecting surface effects. Kirchhoff plate theory shows a constant deflection for various a/h ratios for classical continuum model, while a nearly constant

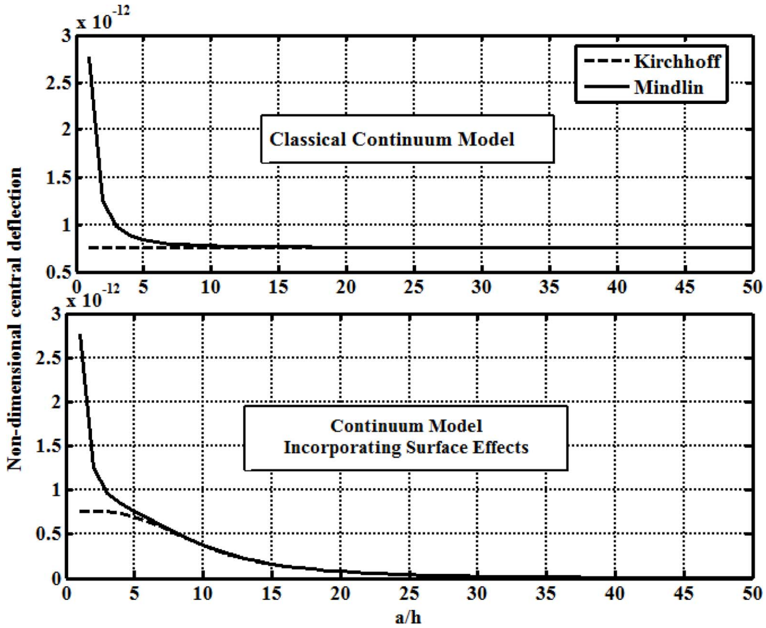

Fig. 9 Non-dimensional deflections versus length-to-thickness ratio for classical continuum model and continuum model incorporating surface effects ($n = 2$)

deflection is provided for *a/h* ratios that higher than 30 for incorporating surface effects. The figure shows that, for Mindlin plate theory the transverse shear effect vanishes for *a/h* ratios that higher than 10 for both classical models and incorporating surface effects models. We can conclude that, for macro/nano scale thicknesses the transverse shear effects are dominant for *a/h* ratios that less than 10. Also, the effect of varying *a/h* ratio on the plate deflection vanishes for ratios higher than 10 for macro-scale thicknesses, while for nano-scale thicknesses the effect vanishes for ratios higher than 30.

5. Conclusions

In this paper, the first-order shear deformation theory (FSDT) for continuum incorporating surface energy is investigated to study the response of ultra-thin functionally graded (FG) plates. By incorporating the surface energies into the principle of minimum potential energy, a series of continuum governing differential equations which include intrinsic length scales are derived. The modifications over the classical continuum model stiffnesses are also obtained. A simply supported micro/nano scaled films subjected to a transverse mechanical load are investigated. Also, a parametric study is provided to investigate the effect of surface energy on the FG plate response. In our analysis the residual surface tension under unstrained conditions and the surface Lame constants are expected to be the same for, both, the upper and lower surfaces of the FG plate. The numerical results leads to the following conclusions:

1. For ultra-thin plates, the size-dependent analysis considering the effect of surface energy of such ultra-thin plates has shown a predictable variation rather than that given by the classical

continuum analysis which neglecting surface effects.

2. The size effect becomes significant as the plate thickness decreases and approaches its intrinsic thickness.
3. The surface energy could stiffen or soften the plate material stiffness depending on surface material properties.
4. The ultra-thin plate response depends noticeably on the surface energy, which requires a precise measurement technique or efficient atomistic computational models to predict the material constants of the surface material.
5. For macro/nano scale thicknesses the transverse shear effects are dominant for length-to-thickness ratios that less than 10. Also, the effect of varying of length-to-thickness ratio on the plate deflection vanishes for ratios higher than 10 for macro-scale thicknesses, while for nano-scale thicknesses the effect vanishes for ratios higher than 30.
6. Larger plate thicknesses require larger length-to thickness ratios to consider surface effects. But for smaller thicknesses surface effects become significant for smaller ratios.

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