Structural Engineering and Mechanics, Vol. 44, No. 2 (2012) 139-161
DOI: http://dx.doi.org/10.12989/sem.2012.44.2.139

Static response of 2-D functionally graded circular plate with gradient thickness and elastic foundations to compound loads

A. Behravan Rad*

Department of Mechanical Engineering, Karaj Branch, Islamic Azad University, Karaj, Iran

(Received January 23, 2012, Revised June 30, 2012, Accepted September 24, 2012)

Abstract. In this paper, the static behavior of bi-directional functionally graded (FG) non-uniform thickness circular plate resting on quadratically gradient elastic foundations (Winkler-Pasternak type) subjected to axisymmetric transverse and in-plane shear loads is carried out by using state-space and differential quadrature methods. The governing state equations are derived based on 3D theory of elasticity, and assuming the material properties of the plate except the Poisson's ratio varies continuously throughout the thickness and radius directions in accordance with the exponential and power law distributions. The stresses and displacements distribution are obtained by solving state equations. The effects of foundation stiffnesses, material heterogeneity indices, geometric parameters and loads ratio on the deformation and stress distributions of the FG circular plate are investigated in numerical examples. The results are reported for the first time and the new results can be used as a benchmark solution for future researches.

Keywords: FG circular plate; gradient elastic foundation; elasticity; state - space; DQ method

1. Introduction

One of the interesting problems in engineering is the static and dynamic analysis of structures such as beams, plates and shells resting on elastic foundations. Components made of FGMs like plates/shells resting on elastic foundations often find application in aerospace, mechanical, nuclear, vehicles and offshore structures. They are in general subjected to various types of mechanical and thermal loads. There are different approaches to analyze the interaction between a structure and an ambient medium. The Winkler-Pasternak approach is a famous model, widely used to describe the structure-foundation interactions by many researchers and scientists during the past decades.

Functionally graded materials (FGMs) have gained considerable attention in recent years. The mechanical behavior of FGMs circular plates with/without resting on elastic foundation, such as bending due to mechanical loads, vibration, stability and buckling, etc., have also been studied by many scientists. For example, (Praveen and Reddy 1998) studied and analyzed the geometrically nonlinear and transient thermo-elastic behavior of rectangular FGM plates. Yang and Shen (2001) dealt with the dynamic response of initially stressed functionally graded rectangular thin plates

*Corresponding author, M.Sc., E-mail: behravanrad@gmail.com

subjected to partially distribute impulsive lateral loads. Nemat-Alla (2003) introduced the concept of adding a third material constituent to the conventional FGMs material in order to significantly reduce the thermal stresses in machine elements that subjected to sever thermal loading, and his investigation on 2D-FGMs has shown that it is more capable of reducing thermal and residual stresses than one-directional FGMs. Nie and Zhong (2007) investigated the axisymmetric bending of 2D-FGM circular and annular plates based on the three-dimensional theory of elasticity using semi-analytical and ANSYS software. Li *et al.* (2008) presented the elasticity solutions for a transversely isotropic FGM circular plate subject to an axisymmetric transverse load in terms of the polynomials of even order. Huang *et al.* (2008) presented an exact solution for FGMs rectangular thick plates resting on elastic foundation based on the three-dimensional theory of elasticity using infinite dual series of trigonometric functions combined with the state- space method. Ayvaz and Burak (2008) analyzed the free vibration of plates embedded on elastic foundations by using modified vlasov model, and presented the effects of the subsoil depth on natural frequencies and corresponding mode shapes. Wang *et al.* (2009) applied direct displacement method to investigate the free axisymmetric vibration of transversely isotropic Circular plate. Malekzadeh (2009) used DQ method to analysis the free vibration of thick FGM rectangular plates supported on two-parameter elastic foundation. Pradhan and Murmu (2009) investigated the Thermo-mechanical vibration of functionally graded (FG) and functionally graded sandwich (FGSW) beams under variable elastic foundations using differential quadrature method. Lu *et al.* (2009) analyzed free vibration of FG thick rectangular plate on elastic foundations by using 3D theory of elasticity. Nie and Zhong (2010) investigated the dynamic behavior of 2D directional FGM annular plates based on the three-dimensional theory of elasticity using the state- space method combined with the one dimensional differential quadrature rule (DQM). Malekzadeh *et al*. (2010) studied the free vibration analysis of FGMs thick annular plates subjected to thermal environment based on the 3D elasticity theory by using DQ method. Hosseini-Hashemi *et al*. (2010) investigated buckling and free vibration behaviors of radially functionally graded circular and annular sector thin plates subjected to uniform in-plane compressive loads resting on the Pasternak elastic foundation by using the DQ method. (Alibeigloo 2010) discussed bending behavior of FGM rectangular plate with integrated surface piezoelectric layers resting on elastic foundation. Yun *et al*. (2010) investigated the axisymmetric bending of FG circular plates as analytically by using direct displacement method. Behravan Rad *et al*. (2010) studied the static behavior of FGM annular plate resting on uniform elastic foundations under axisymmetric transverse load based on the 3D theory of elasticity and using semi-analytical method. Yu and Wang (2010) studied the buckling of a circular plate on partial concentric elastic foundation by using an exact solution. Sburlati and Bardella (2011) obtained elastic solutions for FG thick circular plate subject to axisymmetric conditions. Naderi and Saidi (2011) presented an exact analytical solution for buckling analysis of moderately thick functionally graded (FG) sector plates resting on Winkler elastic foundation based on the first order shear deformation plate theory. Golmakani and Kadkhodayan (2011) analyzed the axisymmetric nonlinear bending of an annular functionally graded plate under mechanical loading based on FSDT and TSDT by using the dynamic relaxation (DR) method combined with the finite difference technique. Mirtalaie and Hajabasi (2011) studied the free vibration of functionally graded thin annular sector plates by using DQ method. The post buckling behavior of elastic beams on gradient foundation is investigated by Challamel (2011). Kacar *et al*. (2011) considered the free vibration of an Euler-Bernoulli beam resting on a variable Winkler foundation by using the differential transform method. The static and dynamic responses of a completely free elastic beam resting on Pasternak type foundation to symmetrically distributed load and concentrated load at its middle are investigated by Celep *et al.* (2011). Akgoz and Civalek (2011) applied the discrete singular convolution method to investigate the nonlinear vibration behavior of geometrically nonlinear thin laminated plates resting on non-linear elastic foundation. Yas and Tahouneh (2012) investigated the free vibration of functionally graded annular plates on elastic foundations based on the three-dimensional theory of elasticity and using the differential quadrature method. Ponnusamy and Selvamani (2012) studied the wave propagations in a thermo elastic homogeneous circular plate embedded in an elastic medium based on generalized two dimensional theory of thermo elasticity.

Reviewing the literature shows that static analysis of 2D functionally graded non-uniform thickness circular plate resting on quadratically gradient elastic foundations in the radius direction has not been considered and this paper deals with this subject. In this work, material properties are assumed to be graded in the thickness and radius directions according to an exponent and power law distributions with the Poisson's ratio, ν, to be constant. The formulations are based on the three-dimensional theory of elasticity. A semi-analytical method, which makes use of the state space method and the one-dimensional differential quadrature rule, is employed. Effects of the gradient indices, the plate geometric parameters, the foundation stiffnesses and the loads ratio on the static behavior of circular and annular plates are investigated.

2. Problem formulation

2.1 Basic equations

Fig. 1 shows a variable thickness bi-directional functionally graded circular plate with radius r_o rested on quadratic type gradient elastic foundations along radius direction and subjected to combined uniform axisymmetric transverse and in-plane shear loads. A cylindrical coordinate system (r, θ, z) with the origin o located at the center of mid surface of the plate is introduced to describe the displacement field. The FG plate is transversely isotropic and material properties of the plate except the Poisson's ratio ν, varies continuously throughout the thickness and radius directions in accordance with the exponential and power law distributions. The geometry of the plate varies with the quadratic form along the radius direction. The Young's moduli and the thickness of the FG plate are as follow

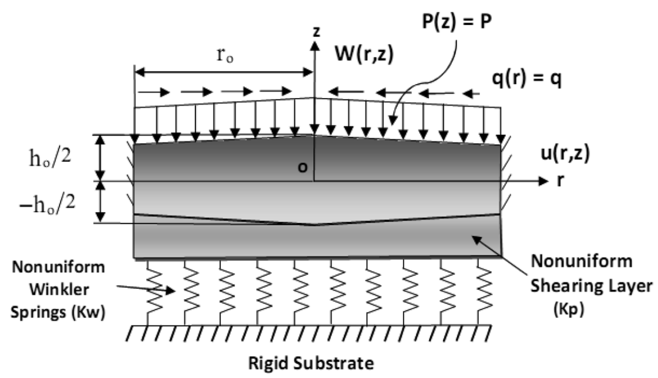


Fig. 1 Geometry of variable thickness 2D-FGMs circular plate resting on gradient elastic foundations

$$E(r,z) = E_b e^{\gamma_1\left(\frac{z}{h}+\frac{1}{2}\right)+\gamma_2\left(\frac{r}{r_o}\right)} \tag{1-a}$$

$$E(r,z) = \left[(E_b - E_t)\left(1-\left(\frac{z}{h}+\frac{1}{2}\right)\right)^{n_1} + E_t\right]\left(1+\left(\frac{r}{r_o}\right)\right)^{n_2} \tag{1-b}$$

$$h(r) = \frac{h_o}{2}\left(1+\alpha_1\left(\frac{r}{r_o}\right)+\alpha_2\left(\frac{r}{r_o}\right)^2\right) \tag{2}$$

Moreover, the plate geometry, the material properties distribution, applied loads, and boundary conditions are independent from circumferential direction, the problem is axisymmetric. For the axisymmetric problem, in the absence of body forces the equilibrium equations are

$$r^{-1}(r\sigma_r)_{,r} + \tau_{rz,z} - r^{-1}\sigma_\theta = 0, \quad r^{-1}(r\tau_{rz})_{,r} + \sigma_{z,z} = 0 \tag{3}$$

the comma denotes differentiation with respect to the indicated variable.

The kinematic equations are

$$\varepsilon_r = u(r,z)_{,r}, \quad \varepsilon_\theta = r^{-1}u(r,z), \quad \varepsilon_z = w(r,z)_{,r}, \quad \gamma_{rz} = u(r,z)_{,z} + w(r,z)_{,r} \tag{4}$$

The linear stress-displacement equations are

$$\sigma_r = \frac{E(r,z)}{(1+\nu)(1-2\nu)}((1-\nu)u(r,z)_{,r} + \nu r^{-1}u(r,z) + \nu w(r,z)_{,z})$$

$$\sigma_\theta = \frac{E(r,z)}{(1+\nu)(1-2\nu)}(\nu u(r,z)_{,r} + (1-\nu)r^{-1}u(r,z) + \nu w(r,z)_{,z})$$

$$\sigma_z = \frac{E(r,z)}{(1+\nu)(1-2\nu)}(\nu u(r,z)_{,r} + \nu r^{-1}u(r,z) + (1-\nu)w(r,z)_{,z})$$

$$\tau_{rz} = \frac{E(r,z)}{2(1+\nu)}(u(r,z)_{,z} + w(r,z)_{,r}) \tag{5}$$

By introducing the following non-dimensional parameters

$$U = \frac{u(r,z)}{h(r)}, \quad W = \frac{w(r,z)}{h(r)}, \quad \eta = \frac{r}{r_o}, \quad \xi = \frac{z}{h(r)}, \quad -0.5 \le \xi \le 0.5, \quad s = \frac{h_o}{2r_o}, \quad \zeta = \xi + 0.5$$

$$\sigma_\eta = \frac{\sigma_r}{p}, \quad \sigma_\Theta = \frac{\sigma_\theta}{p}, \quad \sigma_\xi = \frac{\sigma_z}{p}, \quad \tau_{\eta\xi} = \frac{\tau_{rz}}{p} \tag{6}$$

and considering Eqs. (1)-(5), the normalized form of the governing differential equations in the bottom surface of the plate can be obtained in terms of displacement components as

$$U_{,\xi\xi} = -\left(\frac{2(1-\nu)}{1-2\nu}\right)(s)^2\beta_1^2\left(U_{,\eta\eta} + \left(\frac{1}{\eta}+\Gamma_2+2\beta_2\right)U_{,\eta} + \left(\left(\frac{1}{\eta}+\Gamma_2\right)\beta_2+\beta_3-\frac{1}{\eta^2}\right)U\right) - \left(\frac{2\nu}{(1-2\nu)}\right)(s)^2\beta_1^2\frac{\Gamma_2}{\eta}U$$
$$-\Gamma_1 s\beta_1(W_{,\eta}+\beta_2 W) - \Gamma_1 U_{,\xi} - \left(\frac{1}{1-2\nu}\right)s\beta_1 W_{,\eta\xi} - \left(\frac{2\nu}{(1-2\nu)}\right)\Gamma_2 s\beta_1 W_{,\xi} \tag{7-a}$$

$$W_{,\xi\xi} = -\Gamma_1\left(\frac{\nu}{1-\nu}\right)s\beta_1\left(U_{,\eta}+\left(\beta_2+\frac{1}{\eta}\right)U\right)-\left(\frac{1-2\nu}{2(1-\nu)}\right)s^2\beta_1^2\left(W_{,\eta\eta}+\left(\frac{1}{\eta}+2\beta_2+\Gamma_2\right)W_{,\eta}+\left(\beta_3+\beta_2\left(\frac{1}{\eta}+\Gamma_2\right)\right)W\right)$$
$$-\left(\frac{1}{2(1-2\nu)}\right)s\beta_1\left(U_{,\eta\xi}+\frac{1}{\eta}U_{,\xi}\right)-\left(\frac{1-2\nu}{2(1-\nu)}\right)s\beta_1\Gamma_2 U_{,\xi}-\Gamma_1 W_{,\xi} \quad \text{(7-b)}$$

where

$$\beta_1 = (1+\alpha_1\eta+\alpha_2\eta^2), \quad \beta_2 = (\alpha_1+2\alpha_2\eta)/(1+\alpha_1\eta+\alpha_2\eta^2), \quad \beta_3 = 2\alpha_2/(1+\alpha_1\eta+\alpha_2\eta^2)$$

$\Gamma_1 = \gamma_1$, $\Gamma_2 = \gamma_2$ For exponential distribution of material properties

$\Gamma_1 = n_1(E_t/E_b-1)$, $\Gamma_2 = n_2/(1+\eta)$ For power law distribution of material properties

2.2 The plate-foundation interaction

The foundation is assumed is perfect, frictionless, attached to the plate and separation does not arise. The variable foundation interface pressure p_{zb} for an axisymmetric problem in the referred coordinate system may be expressed as

$$p_{zb} = k_w(r)w_b - \frac{1}{r}\frac{\partial}{\partial r}\left(rk_p(r)\frac{\partial w_b}{\partial r}\right) \quad (8)$$

where p_{zb} is the force per unit area, w_b is the deflection of the bottom surface of the plate. $k_w(r)$, $k_p(r)$ are the quadratic variable Winkler-Pasternak coefficients along the radial direction and can be expressed as

$$k_w(r) = k_{wbo}\left(1+f_1\left(\frac{r}{r_o}\right)+f_2\left(\frac{r}{r_o}\right)^2\right), \quad k_p(r) = k_{Pbo}\left(1+f_1\left(\frac{r}{r_o}\right)+f_2\left(\frac{r}{r_o}\right)^2\right) \quad (9)$$

2.3 Boundary conditions

The edge and boundary conditions for solid circular plate with radius r_o and annular plate with inner radius r_i and outer radius r_o are:

1) solid circular plate

Clamped edge (C)

$$u(r,z) = 0, \quad w(r,z) = 0 \quad \text{at} \quad r = r_o \quad (10)$$

Simply supported edge (S)

$$\sigma_r = 0, \quad w(r,z) = 0 \quad \text{at} \quad r = r_o \quad (11)$$

Free edge (F)

$$\sigma_r = 0, \quad \tau_{rz} = 0 \quad \text{at} \quad r = r_o \quad (12)$$

Regularity conditions on the center of the plate

$$u(r,z) = 0, \quad w(r,z)_{,r} = 0 \quad \text{at} \quad r = 0 \quad (13)$$

2) annular plate
Clamped – clamped edges (C-C)

$$u(r,z) = 0, \quad w(r,z) = 0 \quad \text{at} \quad r = r_i \tag{14-a}$$

$$u(r,z) = 0, \quad w(r,z) = 0 \quad \text{at} \quad r = r_o \tag{14-b}$$

Simply – simply supported edges (S-S)

$$\sigma_r = 0, \quad w(r,z) = 0 \quad \text{at} \quad r = r_i \tag{15-a}$$

$$\sigma_r = 0, \quad w(r,z) = 0 \quad \text{at} \quad r = r_o \tag{15-b}$$

Simply supported – clamped edges (S-C)

$$\sigma_r = 0, \quad w(r,z) = 0 \quad \text{at} \quad r = r_i \tag{16-a}$$

$$u(r,z) = 0, \quad w(r,z) = 0 \quad \text{at} \quad r = r_o \tag{16-b}$$

Clamped – free edges (C-F)

$$u(r,z) = 0, \quad w(r,z) = 0 \quad \text{at} \quad r = r_i \tag{17-a}$$

$$\sigma_r = 0, \quad \tau_{rz} = 0 \quad \text{at} \quad r = r_o \tag{17-b}$$

Boundary conditions at the top and bottom surfaces of the plate are assumed as follows.

at $z = -h/2$

$$\tau_{rz} = 0, \quad \sigma_z = p_{zb} \tag{18}$$

at $z = h/2$

$$\tau_{rz} = -q, \quad \sigma_z = -P \tag{19}$$

3. Solution procedure

It is difficult to analytically solve the governing differentials equations appeared in Eq. (7), if it is not impossible. Hence, one should use an approximate method to find a solution. Here, the semi-analytical approach is employed. This method combines the state space method (SSM) to provide an analytical solution along the thickness direction (z-direction) to express the through-thickness behavior of the plate and the one dimensional differential quadrature method (DQM) to approximate the radial direction effects of the plate. By using this method the governing differentials equations is transformed from physical domain to a normalized computational domain and the special derivatives are discretized by applying the one dimensional differential quadrature method as an efficient and accurate numerical tool. Finally a linear eigenvalue system in terms of the displacements is established and by solving the resulted eigenvalue system, the static response of the plate is obtained.

3.1 DQM procedure and its application

The principle of DQ rule is stated as follow: for a continuous function $\Phi(r)$ defined in an interval $r \in [0, 1]$, its nth order derivative with respect to argument r at an arbitrary given point r_i can be approximated by a linear sum of the weighted function values of $\Phi(r)$ in the whole domain (Shu 2000, Zong and Zhang 2009). This procedure can be expressed mathematically as

$$\frac{\partial^{(n)}\Phi(r_i)}{\partial r^n} = \sum_{j=1}^{N} A_{ij}^{(n)}\Phi(r_j) \quad i = 1, 2, \ldots, N \quad \text{and} \quad n = 1, 2, \ldots, N-1 \tag{20}$$

where $A_{ij}^{(n)}$ are the weighted coefficients determined by the coordinates of the sample points r_i.

It is deduce from this equation, that the important components of DQ approximation are the weighting coefficients and the choice of sample points. In order to determine the weighting coefficients a set of Lagrange polynomials are employed as test functions, and to achieve more accuracy the non-uniform grid spacing is considered. Explicit expressions of the first and second derivatives of the weighted coefficients matrices and also criterions to adopt non-uniformly spaced grid points are presented in appendix 1.

The partial derivatives of the unknown displacements U, W with respect to η appeared in Eq. (7) after applying the DQ rule at an arbitrary sample point η_i can be expressed as

$$(U_{,\eta})_{\eta i} = \sum_{j=1}^{N} A_{ij}U_j, \quad (W_{,\eta})_{\eta i} = \sum_{j=1}^{N} A_{ij}W_j \tag{21-a}$$

$$(U_{,\eta\eta})_{\eta i} = \sum_{j=1}^{N} A_{ij}^{(2)}U_j, \quad (W_{,\eta\eta})_{\eta i} = \sum_{j=1}^{N} A_{ij}^{(2)}W_j \tag{21-b}$$

$$(U_{,\eta\xi})_{\eta i} = \sum_{j=1}^{N} A_{ij}U_{,\xi j}, \quad (W_{,\eta\xi})_{\eta i} = \sum_{j=1}^{N} A_{ij}W_{,\xi j} \tag{21-c}$$

The discretized forms of the edge conditions discussed in Eqs. (10)-(17) can be expressed as follows

Clamped edge (C)

$$U_N = 0, \quad W_N = 0 \quad \text{at} \quad \eta = 1 \tag{22}$$

Simply- supported edge (S)

$$\sigma_{\eta N} = 0, \quad W_N = 0 \quad \text{at} \quad \eta = 1 \tag{23}$$

Free edge (F)

$$\sigma_{\eta N} = 0, \quad \tau_{\eta\xi N} = 0 \quad \text{at} \quad \eta = 1 \tag{24}$$

Regularity conditions in the center of the plate

$$U_1 = 0, \quad W_1 = -\sum_{j=2}^{N} \frac{A_{1j}}{(\alpha_1 + A_{11})}W_j \quad \text{at} \quad \eta = 0 \tag{25}$$

Clamped – clamped edges (C-C)

$$U_1 = 0, \quad W_1 = 0, \quad \text{at} \quad \eta = r_i/r_o \tag{26-a}$$

$$U_N = 0, \quad W_N = 0, \quad \text{at} \quad \eta = 1 \tag{26-b}$$

Simply – simply supported edges (S-S)

$$\sigma_{\eta 1} = 0, \quad W_1 = 0, \quad \text{at} \quad \eta = r_i/r_o \tag{27-a}$$

$$\sigma_{\eta N} = 0, \quad W_N = 0, \quad \text{at} \quad \eta = 1 \tag{27-b}$$

Simply supported – clamped edges (S-C)

$$\sigma_{\eta 1} = 0, \quad W_1 = 0, \quad \text{at} \quad \eta = r_i/r_o \tag{28-a}$$

$$U_N = 0, \quad W_N = 0, \quad \text{at} \quad \eta = 1 \tag{28-b}$$

Clamped – free edges (C-F)

$$U_1 = 0, \quad W_1 = 0, \quad \text{at} \quad \eta = r_i/r_o \tag{29-a}$$

$$\sigma_{\eta N} = 0, \quad \tau_{\eta\xi N} = 0, \quad \text{at} \quad \eta = 1 \tag{29-b}$$

The discretized forms of the boundary conditions at the lower and upper surfaces of the plate, Eqs. (18) and (19) can be written as

At $\xi = -0.5$

$$U_{,\xi} + s\beta_{1i}\left(\sum_{j=2}^{N-1} A_{ij}W_j - \frac{A_{i1}}{(\alpha_1 + A_{11})}\sum_{j=2}^{N-1} A_{1j}W_j\right) + s\beta_{1i}\beta_{2i}W_i = 0 \tag{30-a}$$

$$W_{,\xi} + \frac{s\nu}{1-\nu}\left(\beta_{1i}\sum_{j=1}^{N} A_{ij}U_j + \frac{\beta_{6i}}{\eta_i}U_i\right) = \frac{1}{\Gamma_3}\left(k_1\beta_{4i}\beta_{1i}W_{bi} - k_2\left(\beta_{4i}\beta_{1i}\sum_{j=1}^{N} A_{ij}^{(2)}W_{bj} + \lambda_{1i}\sum_{j=1}^{N} A_{ij}W_{bj} + \lambda_{2i}W_{bj}\right)\right)$$

$$(i = 1, 2, 3, \ldots, N) \tag{30-b}$$

where $k_1 = \dfrac{k_{wb0}(1+\nu)(1-2\nu)h}{2(1-\nu)E_b}$ and $k_2 = \dfrac{k_{pbo}(1+\nu)(1-2\nu)}{2(1-\nu)E_b}\dfrac{s}{r_o}$

$\Gamma_3 = e^{\gamma_2\eta_i}, (1+\eta_i)^{n_2}$ for exponential and power law distributions of materials properties of the plate, respectively.

At $\xi = 0.5$

$$U_{,\xi} + s\beta_{1i}\left(\sum_{j=2}^{N-1} A_{ij}W_j - \frac{A_{i1}}{(\alpha_1 + A_{11})}\sum_{j=2}^{N-1} A_{1j}W_j\right) + s\beta_{1i}\beta_{2i}W_i = \frac{-2(1+\nu)q}{E_b\Gamma_4} \tag{31-a}$$

$$W_{,\xi} + \frac{s\nu}{(1-\nu)}\left(\beta_{1i}\sum_{j=1}^{N} A_{ij}U_j + \frac{\beta_{6i}}{\eta_i}U_i\right) = \frac{-P(1+\nu)(1-2\nu)}{(1-\nu)E_b\Gamma_4} \quad (i = 1, 2, 3, \ldots, N) \tag{31-b}$$

where

$$\beta_{1i} = 1 + \alpha_1\eta_i + \alpha_2\eta_i^2, \quad \beta_{2i} = (\alpha_1 + 2\alpha_2\eta_i)/(1 + \alpha_1\eta_i + \alpha_2\eta_i^2), \quad \beta_{3i} = 2\alpha_2/(1 + \alpha_1\eta_i + \alpha_2\eta_i^2)$$

$$\beta_{4i} = 1 + f_1\eta_i + f_2\eta_i^2, \quad \beta_{5i} = f_1 + 2f_2\eta_i, \quad \beta_{6i} = 1 + 2\alpha_1\eta_1 + 3\alpha_2\eta_i^2$$

$$\lambda_{1i} = \left(\beta_{1i}\left(\frac{1}{\eta_i}\beta_{4i} + \beta_{5i} + 2\beta_{4i}\beta_{2i}\right)\right), \quad \lambda_{2i} = \left(\frac{1}{\eta_i}\beta_{4i}\beta_{2i}\beta_{1i} + \beta_{5i}\beta_{2i}\beta_{1i} + 2\alpha_2\beta_{4i}\right)$$

$\Gamma_4 = e^{\gamma_1 + \gamma_2\eta_i}$, $E_t(1+\eta_i)^{n_2}$ for exponential and power law variations of materials properties of the plate, respectively.

3.2 The state space method

By taking the elements of state vector as $\delta = \{U\ W\ U_{,\zeta}\ W_{,\zeta}\}^T$, the global state space notation of Eq. (7) in discretized points can be written as

$$\{\delta_i(\zeta)\}_{,\zeta} = [D_i]\{\delta_i(\zeta)\} \tag{32}$$

Here, $\{\delta_i(\zeta)\} = \{U_i\ W_i\ U_{,\zeta i}\ W_{,\zeta i}\}^T$ is the global state vector along the plate thickness at the level of ζ and D_i is the coefficient matrix at the sample points. The elements of matrix D_i are given in Appendix 2.

By considering all edge conditions the Eq. (32) can be denoted as follow

$$\{\delta_{ei}(\zeta)\}_{,\zeta} = [D_{ei}]\{\delta_{ei}(\zeta)\} \tag{33}$$

where the subscript 'e' denotes the modified matrix or unknown vector taking account of the edge conditions.

According to the rules of matrix operation, the general solution to Eq. (33) is

$$\delta_{ei}(\zeta) = \exp(\zeta D_{ei})\delta_{ei}(0) \tag{34}$$

Eq. (34) establishes the transfer relations from the state vector on the bottom surface to that at an arbitrary plane ζ of the plate by the exponential matrix of $\exp(\zeta D_{ei})$. Setting $\zeta = 1$ in Eq. (34) gives

$$\delta_{ei}(1) = \exp(D_{ei})\delta_{ei}(0) \tag{35}$$

where $\exp(D_{ei})$ is the global transfer matrix and $\delta_{ei}(1)$, $\delta_{ei}(0)$ are the values of the state variables at the upper and lower planes of the plate, respectively.

By substituting the boundary conditions presented in Eqs. (30) , (31) in to Eq.(35), the following algebraic equations for bending analysis can be obtained

$$GT = Q \tag{36}$$

where G is a $4(N\text{–}2) \times 4(N\text{–}2)$ matrix, Q is a traction force vector and T is

$$T = [U_i(0)\ \ W_i(0)\ \ U_i(1)\ \ W_i(1)]^T, \quad (i = 2, 3, \ldots, N-1) \tag{37}$$

By solving Eq. (36), all state parameters at $\zeta = 0$, $\zeta = 1$ are obtained. We can use Eqs. (34) and (5) to calculate the displacements and the stresses through the thickness of the FGMs circular plate.

Table 1 Mechanical properties of FGMs plate constituents (Praveen and Reddy 1998)

Materials	Aluminum	Zirconia
Young's modulus, E (GPa)	70	151
Poisson's ratio, ν	0.3	0.3

4. The numerical results and discussion

In order to extract the numerical results, Aluminum $E_{Al} = 70$ GPa and Zirconia $E_{zr} = 151$ GPa are considered as the metal and ceramic constituents of the FGMs plate. The material properties of the FGM constituents are taken from (Praveen and Reddy 1998), which are summarized in Table 1. The numerical results are derived for a non-uniform thickness 2D-FGMs clamped solid circular and clamped-clamped annular plates resting on quadratically gradient elastic foundations. The material properties are assumed to have exponential and power law distributions in the thickness and radius directions of the plate according to Eq. (1). The non-linear thickness variation in the radial direction is considered as Eq. (2). To achieve the numerical results non-equally spaced discretization points (Appendix. 1) are considered and the number of discrete points in the radial direction is nine. The plate structural data and the boundary conditions on the lower and the upper surfaces of the plate are

$$E_b = 70 \text{ GPa}, E_t = 151 \text{ GPa}, \nu = 0.3, r_o = 1.0 \text{ m}, r_i = 0.1 \text{ m}, s = 0.015, \alpha_1 = \alpha_2 = 0.1$$

$$\tau_{rz} = 0, \sigma_z = p_{zb} \text{ at } z = -h_o/2 \quad \tau_{rz} = -1 \text{ GPa}, \sigma_z = -1 \text{ GPa at } z = h_o/2 \tag{38}$$

The effects of the plate thickness variability, the plate geometric parameters, the material heterogeneity indices, the loads ratio and the foundation stiffnesses on static behavior of the non-uniform thickness FG circular plate are intensively discussed in the following text. The numerical results for clamped circular plate with exponential and an annular plate with power law distributions of material properties are shown in Figs. 3-9 and Figs. 10-11, respectively.

4.1 Validation of the code

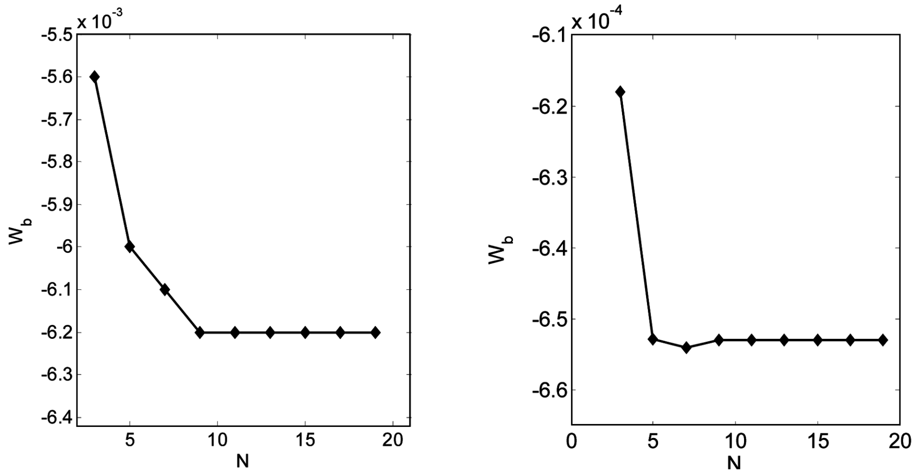
The validity of the prepared code is investigated by solving the small deflection bending of the clamped supported two directional uniform thickness FG circular plate under a uniformly distributed transverse pressure p without elastic foundation. The structural parameters ($r_o = 1.0$ m, $h = 0.1$ m, $\gamma_1 = \gamma_2 = 1$, $E_b = 380$ GPa, $\nu = 0.3$) of the plate and boundary conditions on the bottom and the top surfaces (bottom: $\sigma_z = 0$, $\tau_{rz} = 1$ and top: $\sigma_z = -1$ GPa, $\tau_{rz} = 0$) are considered same as given previously by Nie and Zhong (2007). Non-dimensional transverse deflection of the mentioned plate is determined and the results are presented in Table 1. An excellent agreement is observed between the present results and those are given by Nie and Zhong (2007).

4.2 Convergence of the DQ method

For numerical illustration and to show the effect of the number of the selected discrete points, convergence study of the DQ method is conducted firstly, and is used as an evaluation criterion. The dimensionless transverse deflection W_b vs. the number of discrete points N for clamped edge

Table 2 The non-dimensional deflection of two-directional functionally graded circular plate

ξ		η							
		0.000	0.125	0.250	0.375	0.500	0.625	0.750	0.875
-0.5	Nie and Zhong (2007)	-1.523	-1.462	-1.297	-1.056	-0.776	-0.494	-0.247	-0.073
	present	-1.521	-1.460	-1.295	-1.054	-0.775	-0.493	-0.246	-0.074



(a) Clamped edge circular plate, exponentially variation of material properties

(b) Clamped-Clamped edges annular plate, power law variation of material properties

Fig. 2 Convergence of the DQ method, W_b vs. N

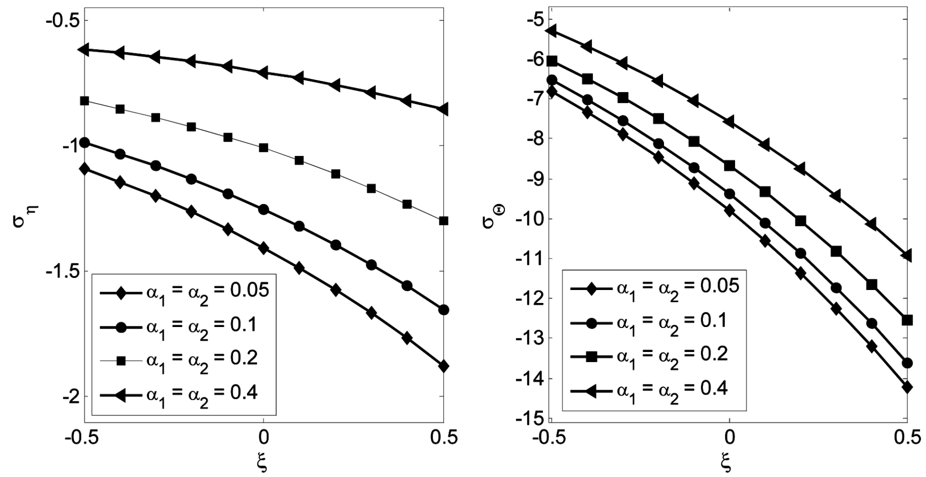
circular plate and clamped-clamped supported annular plate with structural data and boundary conditions appeared in Eq. (38), and $k_1 = k_2 = 0.1$, $\gamma_1 = \gamma_2 = n_1 = n_2 = 0.5$, $f_1 = f_2 = 0.1$ are depicted in Fig. 2. It can be seen from Fig. 2 that the non-dimensional deflection of the plate at the midpoint of radius approaches to a specific value with an increase in the number of the discretization points. Fig. 2 confirms that the convergence of this method is great.

4.3 Effect of thickness variability

The effect of the plate geometry variation coefficients on static response of the plate with conditions shown in Eq. (38) and $\gamma_1 = \gamma_2 = \ln(151/70)$, $f_1 = f_2 = 0.1$, $k_1 = k_2 = 0.1$ at radius midpoint section is plotted in Fig. 3. It is seen from Fig. 3 that in-plane stresses distribution through the thickness of the plate are nonlinear and stresses decreases gradually as α_1 and α_2 increases.

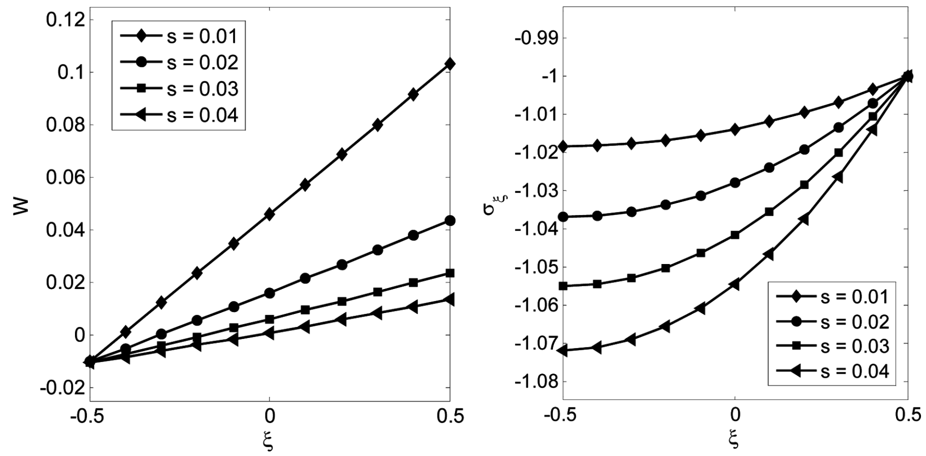
4.4 Effect of the plate aspect ratio

The effect of the plate aspect ratio on static behavior of the plate with conditions discussed in Eq. (38) and $\gamma_1 = \gamma_2 = \ln(151/70)$, $f_1 = f_2 = 0.1$, $k_1 = k_2 = 0.1$ at radius midpoint section is presented in Fig. 4. It can be found from Fig. 4 that W decreases and σ_ξ increases through the thickness of the plate as plate aspect ratio (s) increase.



(a) Radial stress (b) Tangential stress

Fig. 3 Effect of the plate geometry variation coefficients on plane stress components versus the plate thickness

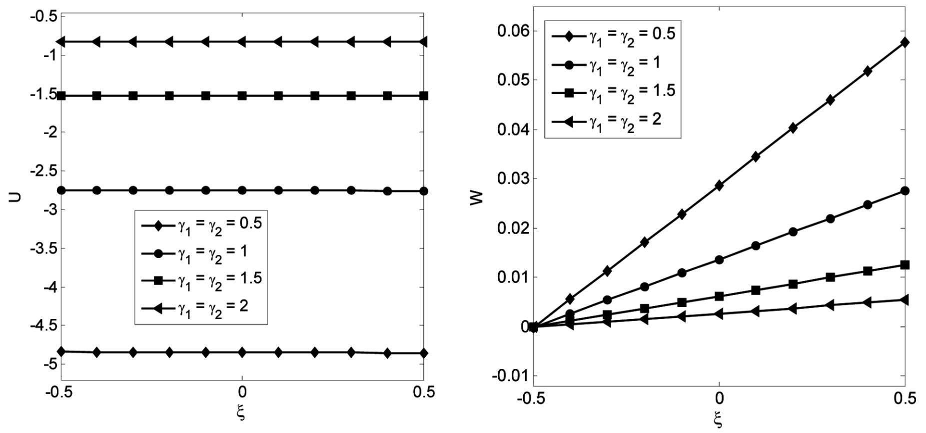


(a) Transverse displacement (b) Transverse normal stress

Fig. 4 Effect of the plate aspect ratio on variation of displacement and stress components versus the plate thickness

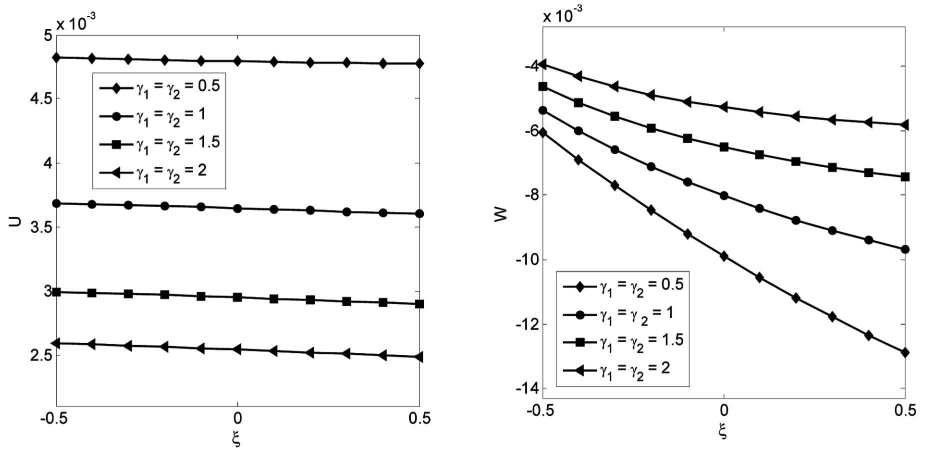
4.5 Effect of the material property heterogeneity

In order to extract the effect of material heterogeneity indices on static response of the plate, the conditions appeared in Eq. (38) and $f_1 = f_2 = 0.1$, $k_1 = k_2 = 0.1$, are chosen and the results are plotted in Figs. 5-7. Fig. 5 illustrates distribution of the displacement components along the thickness at $\eta = 0.5$ due to the distributed in-plane shear load only whereas Fig. 6 includes effect of normal transverse load. The shear traction is exerted on the top surface of the plate has pressed the layers in the radial direction and caused a bending in the layers. For this reason, signs of the displacement components have changed. The radial displacement is affected by the compression due to the shear force much more than that of the bending caused by the transverse loading.



(a) Radial displacement (b) Transverse displacement

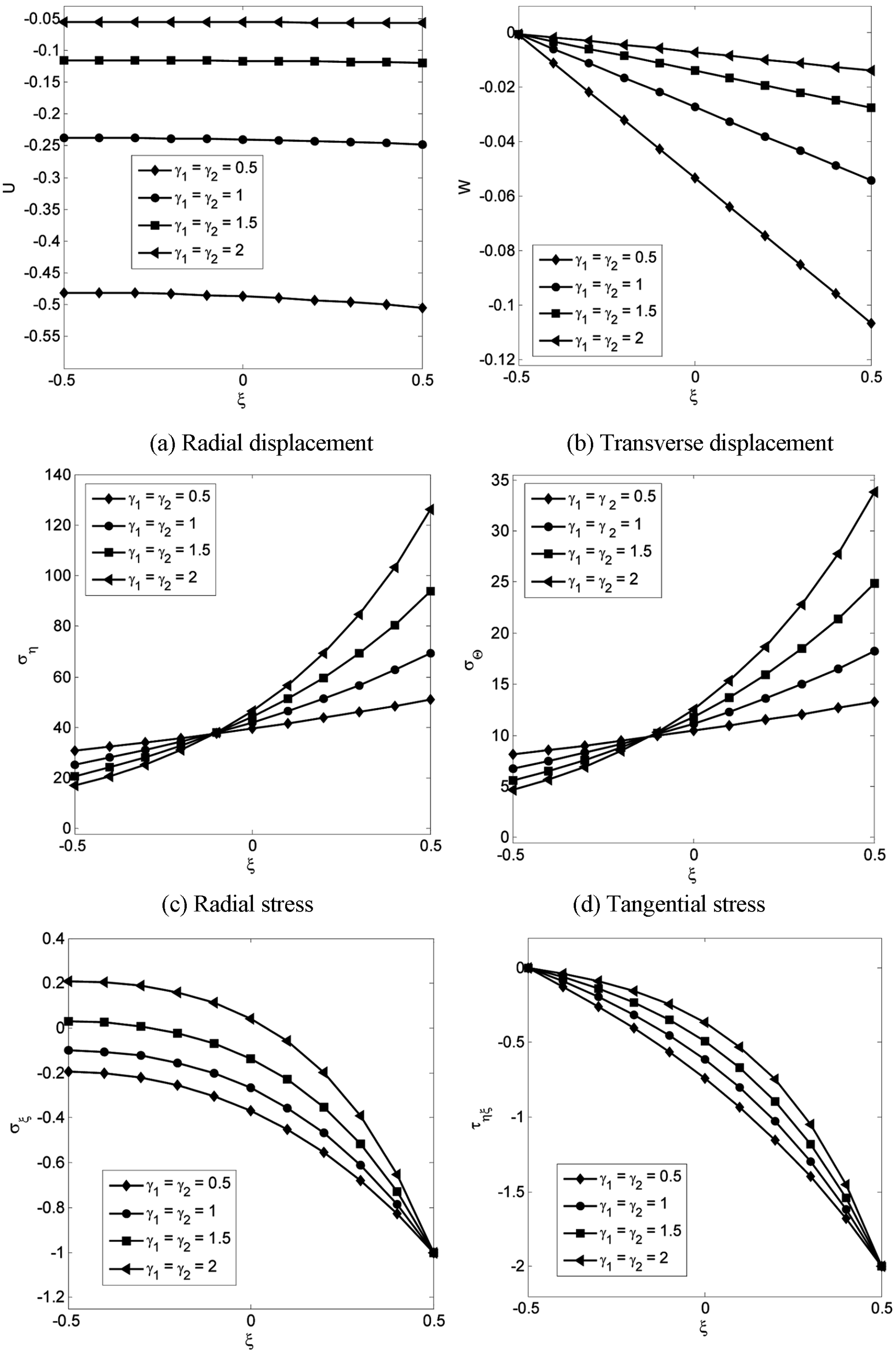
Fig. 5 Effect of the material property graded indices on variation of displacement components versus the plate thickness



(a) Radial displacement (b) Transverse displacement

Fig. 6 Effect of the material heterogeneity indices on non-dimensional displacements across the plate thickness

Fig. 7 illustrates distribution of the displacement and stress components along the thickness at $\eta = 0.96$ due to the compound loads ($p = -1, q = -2$). It can be found from Fig. 7 that the displacement components decreases, σ_η and σ_Θ firstly decreases and then increases, σ_ξ and $\tau_{\eta\xi}$ decreases along the thickness of the plate as γ_1 and γ_2 increases. For the reason of the compression of the layers in the radial direction, the radial and hoop stresses have increased. The distribution of $\tau_{\eta\xi}$ stress through the thickness of the plate converges to the horizontal line with decreasing the graded indices. Decreases of displacements indicate that increasing the material heterogeneity indices will certainly enhance the deformation rigidity of the plate. From Figs. 5-7, it is observed that the deflection of the plate under transverse loading and the radial displacement in shear interaction are more than the other loadings.

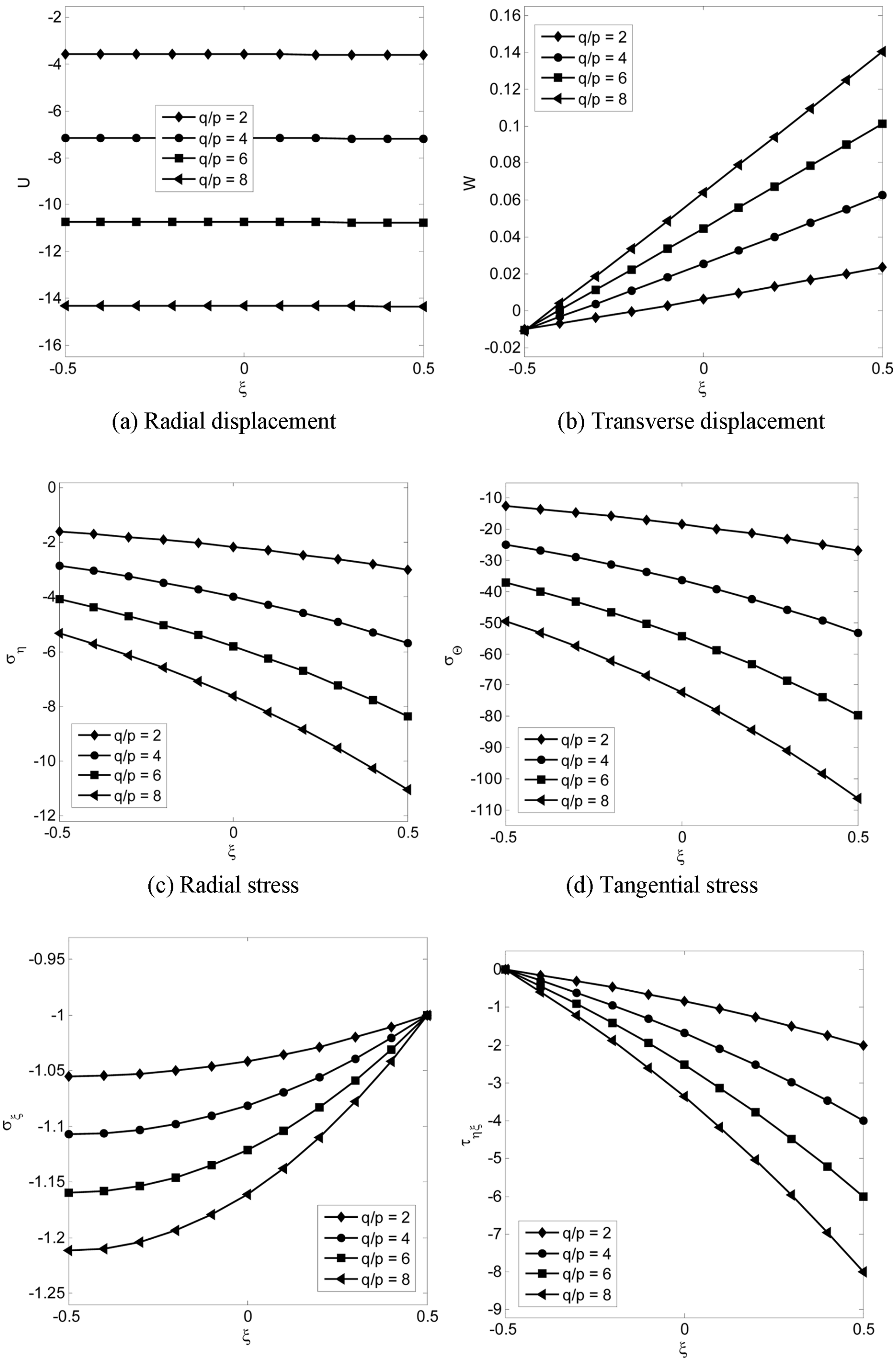


(a) Radial displacement (b) Transverse displacement

(c) Radial stress (d) Tangential stress

(e) Transverse normal stress (f) Transverse shear stress

Fig. 7 Effect of the material heterogeneity indices on mechanical entities versus the plate thickness

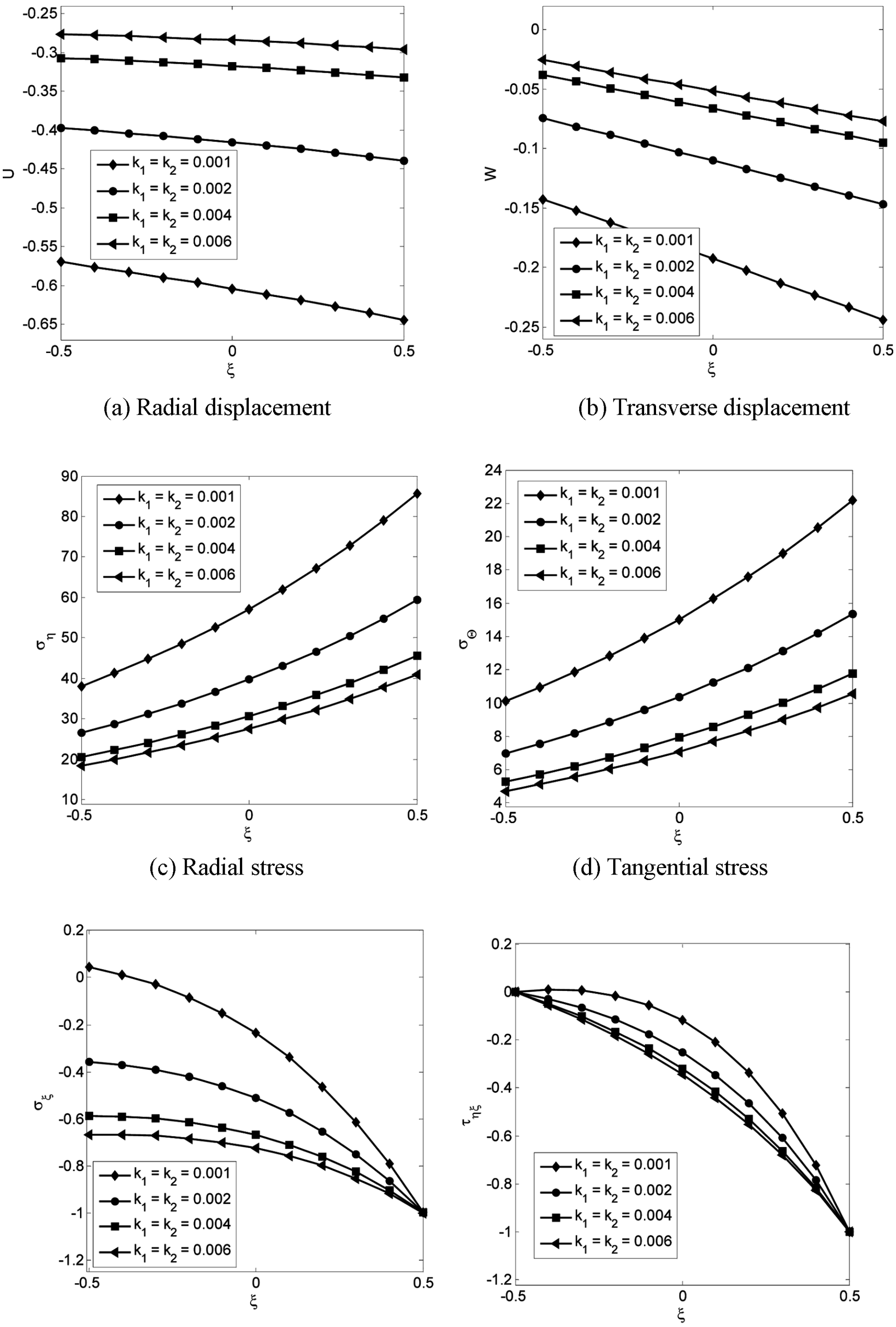


(a) Radial displacement (b) Transverse displacement

(c) Radial stress (d) Tangential stress

(e) Transverse normal stress (f) Transverse shear stress

Fig. 8 Effect of the loads ratio on variation of physical quantities across the plate thickness

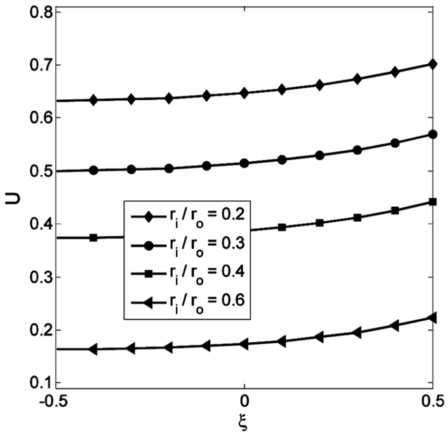


(a) Radial displacement (b) Transverse displacement

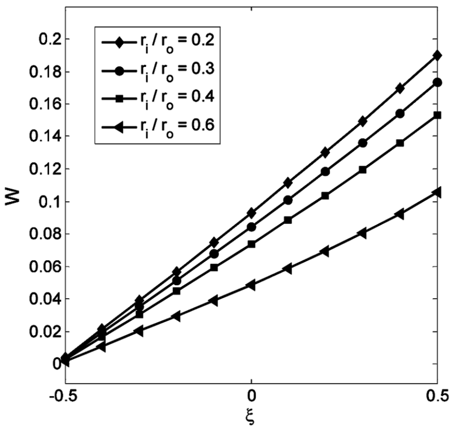
(c) Radial stress (d) Tangential stress

(e) Transverse normal stress (f) Transverse shear stress

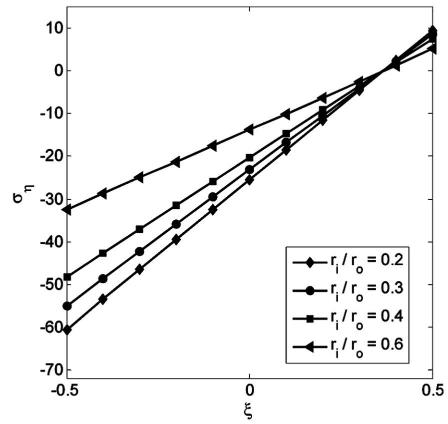
Fig. 9 Effect of the foundation coefficients on variation of mechanical entities across the plate thickness



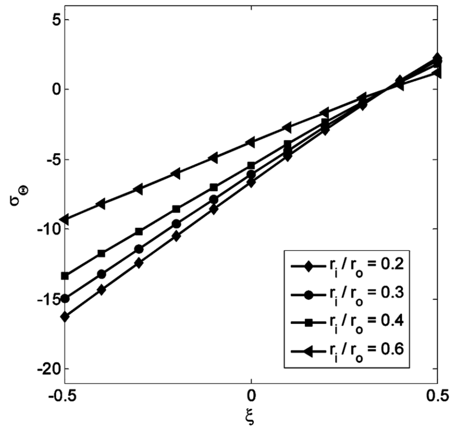
(a) Radial displacement



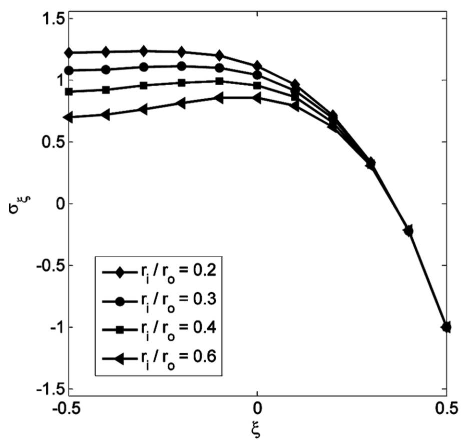
(b) Transverse displacement



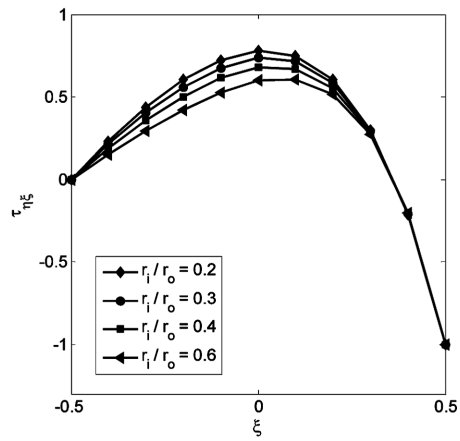
(c) Radial stress



(d) Tangential stress



(e) Transverse normal stress



(f) Transverse shear stress

Fig. 10 Effect of the radiuses ratio on variation of mechanical entities across the plate thickness (C-C annular plate)

4.6 Effect of the loads ratio

With structural parameters discussed in Eq. (38) and $\gamma_1 = \gamma_2 = \ln(151/70)$, $f_1 = f_2 = 0.1$, $k_1 = k_2 = 0.1$ the effect of loads ratio on static response of the plate at $\eta = 0.5$ is plotted in Fig. 8. It is observed from Fig. 8 that all displacements and stresses increases as increase the loads ratio.

4.7 Effect of the foundation stiffnesses

Effect of the foundation stiffnesses on static behavior of the plate whit above mentioned conditions at $\eta = 0.96$ is depicted in Fig. 9. It can be found from Fig. 9 that displacements, σ_η, σ_Θ decreases and σ_ξ, $\tau_{\eta\xi}$ increases as increase k_1, k_2.

4.8 Effect of the radiuses ratio

In this section a variable thickness 2D-FGMs clamped-clamped annular plate is considered. The material properties of the plate vary in thickness and radius directions in accordance with power law distribution as shown in Eq. (1-b). The interactions of the top surface of the plate and the plate structural data are as similar to Eq. (38). The coefficients of plate substrate and the plate material property graded indices are $f_1 = f_2 = 0.1$, $k_1 = k_2 = 0.01$, $n_1 = n_2 = 1$, respectively. The effect of the radiuses ratio on mechanical quantities with coordinate ξ at a location $\eta = 0.97$ is shown in Fig. 10. It is seen from Fig. 10, that the W, σ_η, σ_Θ vary linearly in the thickness direction and the U, σ_ξ, $\tau_{\eta\xi}$ are nonlinear functions of thickness coordinate. It is easy to observe from Fig. 10 that the displacements and stresses become smaller as the radiuses ratio increases.

4.9 Effect of the foundation variability

To demonstrate the effect of the substrate variation coefficients on static response of the plate a

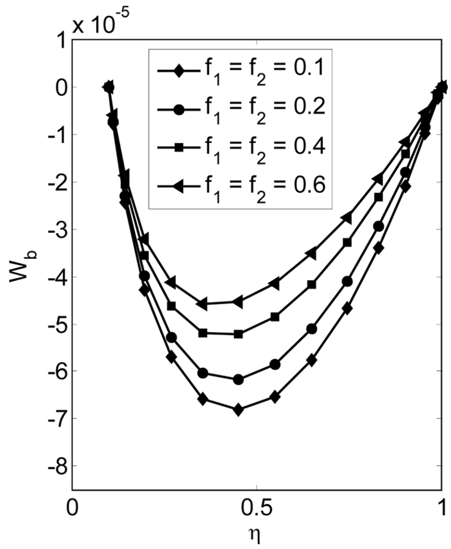


Fig. 11 Effect of the foundation variation coefficients on transverse displacement (C-C annular plate)

clamped-clamped annular plate with conditions shown in Eq. (38) and $n_1 = n_2 = 0.5$, $k_1 = k_2 = 0.1$ is considered and the extracted results is plotted in Fig. 11. It can be seen from Fig. 11 that with increasing foundation variation coefficients the plate deflection decreases.

5. Conclusions

In the present paper, the static behavior of two directional functionally graded circular plate of quadratically varying thickness and resting on gradient two- parameter elastic foundations is investigated based on three dimensional elasticity theory. The plate is assumed to be subjected to both transverse and in-plane shear tractions. The material properties are assumed to vary exponentially and non-exponentially in both thickness and radial directions. The solution is obtained by employing the semi-analytical method. The results confirm the high rate convergence and accuracy of the present method.

By using this method, some results are derived with the most important conclusions that:

1. The presented method is particularly useful to analysis the behavior of heterogeneous plates with a more complicated geometry and boundary conditions.
2. The radial displacement is affected by the compression due to the shear force much more than that of the bending caused by the transverse loading.
3. The additional compression of the layers in the radial direction of the plate due to shear interaction increases the radial and circumferential stresses.
4. Variation scheme of displacements and stresses through the thickness of the plate for exponentially distribution of material properties is different with variation scheme of these quantities to the plate with other distribution of material properties.

Acknowledgements

The author would like to thank the reviewers for their valuable suggestions and comments, which are really helpful in revising the paper.

References

Akgoz, B. and Civalek, O. (2011), "Nonlinear vibration analysis of laminated plates restingon nonlinear two-parameters elastic foundations", *Steel Compos. Struct.*, **11**(5), 403-421.

Alibeigloo, A. (2010), "Three-dimensional exact solution for functionally graded rectangular plate with integrated surface piezoelectric layers resting on elastic foundation", *Mech. Adv. Mater. Struct.*, **17**, 183-195.

Ayvaz, Y. and Burak, O.C. (2008), "Free vibration analysis of plates resting on elastic foundations using modified Vlasov model", *Struct. Eng. Mech.*, **28**(6), 635-658.

Behravan Rad, A., Alibeigloo, A. and Malihi, S.S. (2010), "Static analysis of functionally graded annular plate resting on elastic foundation subject to an axisymmetric transverse load, based on the three dimensional theory of elasticity", *J. Solid Mech.*, **2**(3), 290-304.

Celep, Z., Guler, K. and Demir, F. (2011), "Response of a completely free beam on a tensionless Pasternak foundation subjected to dynamic load", *Struct. Eng. Mech.*, **37**(1), 61-77.

Challamel, N. (2011), "On the post-buckling of elastic beams on gradient foundation", *C. R. Mecanique*, **339**, 396-405.

Golmakani, M.E., Kadkhodayan, M. (2011), "Nonlinear bending analysis of annular FGM plates using higher-order shear deformation plate theories", *Compos. Struct*., **93**, 9739-982.

Hosseini-Hashemi, Sh., Akhavan, H., Rokni, D.T., Daemi, N. and Alibeigloo, A. (2010), "Differential quadrature analysis of functionally graded circular and annular sector plates on elastic foundation", *Mater. Des.*, **31**(4), 1871-1880.

Huang, Z.Y., Lu, C.F. and Chen, W.Q. (2008), "Benchmark solution for functionally graded thick plates resting on Winkler- Pasternak elastic foundations", *Compos. Struct*., **85**, 95-104.

Kacar, A., Tugba Tan, H. and Kaya, M.O. (2011), "Free vibration analysis of beams on variable Winkler elastic foundation by using the differential transform method", *Math. Comput. Appl*., **16**(3), 773-783.

Li, X.Y., Ding, H.J. and Chen, W.Q. (2008), "Elasticity solutions for a transversely isotropic functionally graded circular plate subject to an axisymmetric transverse load qr^k", *Int. J. Solids Struct*., **45**(1), 191–210.

Lu, C.F., Lim, C.W. and Chen, W.Q. (2009), "Exact solutions for free vibrations of functionally graded thick plates on elastic foundations", *Mech. Adv. Mater. Struct.,* **16**, 576-584.

Malekzadeh, P. (2009), "Three-dimensional free vibration analysis of thick functionally graded plates on elastic foundations", *Compos. Struct*., **89**, 367-373.

Malekzadeh, P., Shahpari, SA. and Ziaee, HR. (2010), "Three-dimensional free vibration of thick functionally graded annular plates in thermal environment", *J. Sound Vib*., **329**, 425-442.

Mirtalaie, S.H. and Hajabasi, M.A. (2011), "Free vibration analysis of functionally graded thin annular sector plates using the differential quadrature method", *Proc. IMechE Part C: J. Mech. Eng. Sci.*, **225**, 568-583.

Naderi, A. and Saidi, A.R. (2011),"Exact solution for stability analysis of moderately thick functionally graded sector plates on elastic foundation", *Compos. Struct*., **93**, 629-638.

Nemat-Alla, M. (2003), "Reduction of thermal stresses by developing two-dimensional functionally graded materials", *Int. J. Solids Struct*., **40**, 7339-7356.

Nie, G.J. and Zhong, Z. (2007), "Axisymmetric bending of two-directional functionally graded circular and annular plates", *Acta Mech. Solid Sin*., **20**(4), 289-295.

Nie, G.J. and Zhong, Z. (2010), "Dynamic analysis of multi-directional functionally graded annular plates", *Appl. Math. Model*., **34**(3), 608-616.

Ponnusamy, P. and Selvamani, R. (2012), "Wave propagation in a generalized thermo elastic plate embedded in elastic medium", *Interac. Multi. Mech*., **5**(1), 13-26.

Pradhan, S.C. and Murmu, T. (2009), "Thermo-mechanical vibration of FGM sandwich beam under variable elastic foundations using differential quadrature method", *J. Sound Vib*., **321**, 342-362.

Praveen, GN. and Reddy, J.N. (1998),"Nonlinear transient thermo elastic analysis of functionally graded ceramic-metal plates", *Int. J. Solids Struct.*, **35** (33), 4457–4476.

Shu, C. (2000), *Differential Quadrature and Its Application in Engineering*, Springer, New York.

Sburlati, R. and Bardella, L. (2011), "Three-dimensional elastic solutions for functionally graded circular plates", *Eur. J. Mech. A-Solid*., **30**, 219-235.

Wang, Y., Xu, R.Q. and Ding, H.J. (2009),"Free axisymmetric vibration of FGM plates", *Appl. Math. Mech*., **30**(9), 1077-1082.

Yang, J. and Shen, H.S. (2001), "Dynamic response of initially stressed functionally graded rectangular thin plates", *Compos. Struct.*, **54**, 497-508.

Yas, M.H. and Tahouneh, V. (2012), "3-D free vibration analysis of thick functionally graded annular plates on Pasternak elastic foundation via differential quadrature method (DQM)", *Acta Mech*., **223**(1), 43-62.

Yu, L.H. and Wang, C.Y. (2010), "Buckling mosaic of a circular plate on a partial elastic foundation", *Struct. Eng. Mech*., **34**(1), 135-138.

Yun, W., Rongqiao, X. and Haojiang, D. (2010), "Three-dimensional solution of axisymmetric bending of functionally graded circular plates", *Compos. Struct*., **92**, 1683-1693.

Zong, Z. and Zhang, Y. (2009), *Advanced Differential Quadrature Methods,* CRC Press, New York.

Appendix 1

The elements of weighting coefficients of the first-order derivative matrix A can be obtained from the following algebraic formulation (Shu 2000)

$$A_{ik} = \frac{\prod_{j=1, j\neq i}^{N} (r_i - r_j)}{(r_i - r_k) \prod_{k=1, j\neq k}^{N} (r_k - r_j)} \qquad i \neq k, \quad i, k = 1, 2, 3, \ldots, N$$

$$A_{ii} = -\sum_{j=1, j=i}^{N} A_{ij} \qquad i = k, \quad i = 1, 2, 3, \ldots, N \tag{A1}$$

The weighting coefficients of the second-order derivative can be obtained from the following recurrence relation (Zong and Zhang 2009).

$$A_{ik}^{(2)} = 2\left[A_{ii}A_{ik} - \frac{A_{ik}}{r_i - r_k}\right] \qquad i \neq k, \quad i, k = 1, 2, 3, \ldots, N$$

$$A_{ii}^{(2)} = -\sum_{j=1, j=i}^{N} A_{ij}^{(2)} \qquad i = k, \quad i = 1, 2, 3, \ldots, N \tag{A2}$$

In this study the following criterions are used for nodes discretization.

1- Richard-Shu criterion for solid circular plate

$$r_i = \frac{r_o}{2}\left[1 - \cos\frac{(i-1)\pi}{N-1}\right] \qquad i = 1, 2, 3, \ldots, N \tag{A3}$$

2- Chebyshev criterion for annular plate

$$r_i = b + \frac{1}{2}\left[1 - \cos\frac{(i-1)\pi}{N-1}\right](a - b) \qquad i = 1, 2, 3, \ldots, N \tag{A4}$$

Appendix 2

The elements of matrix D_i

$$D_i = \begin{bmatrix} [0]_{N\times N} & [0]_{N\times N} & [\delta_{ij}]_{N\times N} & [0]_{N\times N} \\ [0]_{N\times N} & [0]_{N\times N} & [0]_{N\times N} & [\delta_{ij}]_{N\times N} \\ [d_{ij}^{31}]_{N\times N} & [d_{ij}^{32}]_{N\times N} & [d_{ij}^{33}]_{N\times N} & [d_{ij}^{34}]_{N\times N} \\ [d_{ij}^{41}]_{N\times N} & [d_{ij}^{42}]_{N\times N} & [d_{ij}^{43}]_{N\times N} & [d_{ij}^{44}]_{N\times N} \end{bmatrix}_{4N\times 4N}$$

where $\delta_{ij} = 0(i \neq j)$; $\delta_{ii} = 1$

$$d_{ii}^{31} = -\left(\frac{2(1-\nu)}{1-2\nu}\right)(s)^2\beta_{1i}^2\left(A_{ii}^{(2)} + \left(\frac{1}{\eta_i} + \Gamma_2 + 2\beta_{2i}\right)A_{ii} + \left(\left(\frac{1}{\eta_i} + \Gamma_2\right)\beta_{2i} + \beta_{3i} - \frac{1}{\eta_i^2}\right)\right) - \left(\frac{2\nu}{(1-2\nu)}\right)(s)^2\beta_{1i}^2\frac{\Gamma_2}{\eta_i} \qquad (i = j)$$

$$d_{ij}^{31} = -\left(\frac{2(1-\nu)}{1-2\nu}\right)(s)^2\beta_{1i}^2\left(\sum_{j=1}^{N} A_{ij}^{(2)} + \left(\frac{1}{\eta_i} + \Gamma_2 + 2\beta_{2i}\right)\sum_{j=1}^{N} A_{ij}\right) \qquad (i \neq j)$$

$$d_{ii}^{32} = -\Gamma_1 s\beta_{1i}(A_{ii}+\beta_{2i}) \;\; (i=j), \quad d_{ij}^{32} = -\Gamma_1 s\beta_{1i}\sum_{j=1}^{N} A_{ij} \;\; (i\neq j), \quad d_{ii}^{33} = -\Gamma_1, \quad d_{ij}^{33} = 0$$

$$d_{ii}^{34} = -\left(\frac{1}{1-2\nu}\right)s\beta_{1i}A_{ii} - \left(\frac{2\nu}{(1-2\nu)}\right)s\beta_{1i}\Gamma_2 \quad (i=j), \quad d_{ij}^{34} = -\left(\frac{1}{1-2\nu}\right)s\beta_{1i}\sum_{j=1}^{N} A_{ij} \;\; (i\neq j)$$

$$d_{ii}^{41} = -\Gamma_1\left(\frac{\nu}{1-\nu}\right)s\beta_{1i}\left(A_{ii}+\left(\beta_{2i}+\frac{1}{\eta_i}\right)\right) \;\; (i=j), \quad d_{ij}^{41} = -\Gamma_1\left(\frac{\nu}{1-\nu}\right)s\beta_{1i}\sum_{j=1}^{N} A_{ij} \;\; (i\neq j)$$

$$d_{ij}^{42} = -\left(\frac{1-2\nu}{2(1-\nu)}\right)s\beta_{1i}^2\left(A_{ii}^{(2)}+\left(\frac{1}{\eta_i}+2\beta_{2i}+\Gamma_2\right)A_{ii}+\left(\beta_{3i}+\beta_{2i}\left(\frac{1}{\eta_i}+\Gamma_2\right)\right)\right) \;\; (i=j)$$

$$d_{ij}^{42} = -\left(\frac{1-2\nu}{2(1-\nu)}\right)s^2\beta_{1i}^2\left(\sum_{j=1}^{N} A_{ij}^{(2)}+\left(\frac{1}{\eta_i}+2\beta_{2i}+\Gamma_2\right)\sum_{j=1}^{N} A_{ij}\right) \;\; (i\neq j)$$

$$d_{ii}^{43} = -\left(\frac{1}{2(1-\nu)}\right)s\beta_{1i}\left(A_{ii}+\frac{1}{\eta_i}\right) - \left(\frac{1-2\nu}{2(1-\nu)}\right)s\beta_{1i}\Gamma_2 \quad (i=j)$$

$$d_{ij}^{43} = -\left(\frac{1}{2(1-\nu)}\right)s\beta_{1i}\sum_{j=1}^{N} A_{ij} \;\; (i\neq j)$$

$$d_{ii}^{44} = -\Gamma_1, \quad d_{ij}^{44} = 0 \quad i = 1,2,3,\ldots,N \tag{A5}$$

Notations

r_o	outer radii of the FGM circular plate
r_i	inner radii of the FGM annular plate
$E_{ij}(r, z)$	Young's modulii of any arbitrary point of the plate
E_b	Young's modulii at the center point of lower surface of the plate
E_t	Young's modulii at the corner of upper surface of the plate
ν	Poisson's ratio
h_o	thickness of the plate at the center of the plate
k_{wbo}	elastic coefficient of Winkler's foundation at the center of bottom surface of the FGM plate (N/m^3)
k_{pbo}	elastic coefficient of shear layer foundation at the center of bottom surface of the FGM plate (N/m)
k_1, k_2	the dimensionless coefficients of the elastic foundation
N	number of the grid points in the radial direction
P	external load (uniform transverse pressure)
q	external load (uniform radial shear pressure)
r	radius (radial direction)
u, w	displacements in the radial and transverse directions
z	transverse coordinate
f_1, f_2	the elastic foundation variation coefficients
γ_{rz}	transverse shear strain
$\varepsilon_i(i = r, \theta, z)$	normal strains in the r, θ and z directions
$\gamma_i, n_i(i = 1, 2)$	material properties gradient indices
α_1, α_2	the plate geometry variation coefficients
θ	circumferential coordinate
$\sigma_i(i = r, \theta, z)$	normal stresses in the r, θ and z directions
τ_{rz}	transverse shear stress
A_{ij}	weighting coefficients matrix of the first derivative
$A_{ij}^{(2)}$	weighting coefficients matrix of the second derivative