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Accurate analytical solution for nonlinear free vibration of beams

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Abstract. In this study, Hamiltonian Approach (HA) is applied to analysis the nonlinear free vibration of beams. Two well-known examples are illustrated to show the efficiency of this method. One of them deals with the Nonlinear vibration of an electrostatically actuated microbeam and the other is the nonlinear vibrations of tapered beams. This new approach prepares us to achieve the beam's natural frequencies and mode shapes easily and a rapidly convergent sequence is obtained during the solution. The effects of the small parameters on the frequency of the beams are discussed. Some comparisons are conducted between the results obtained by the Hamiltonian Approach (HA) and numerical solutions using to illustrate the effectiveness and convenience of the proposed methods.

Keywords: non-linear vibration; hamiltonian approach; electrostatically actuated microbeam; tapered beams

1. Introduction

One of the most interesting areas in civil engineering and mechanical engineering is beam vibrations. It is very important to obtain the dynamic response of beams in the design process. Generally, the nonlinear partial differential equations in space and time are presented the governing equations of the continuous systems like beams. Burgreen (1950) applied the classical continuum approach for the large amplitude vibration problems of hinged beams.

Lou and Sikarskie (1975) applied form-function approximations to obtain the nonlinear response of buckled beams.

Prathap and Varadan (1978) used the actual nonlinear equilibrium equations to study the nonlinear vibrations of simply supported beams. Sathyamoorthy (1982) tried to complete the work on the classical methods for the vibrations of the beams with the material, geometric and other types of nonlinearities. Dumir and Bhaskar (1988) brought out the errors of the nonlinear finite element formulations of beam and plate vibrations to the presence of a linearizing function in the strain energy evaluation.

Singh *et al.* (1990) presented a complete report for the formulations of the nonlinear free vibrations of beams.

In fact, it is very difficult to find an exact or close-form solution for the nonlinear response of the

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beam vibrations. Homotopy analysis method was used to study of the nonlinear vibration of tapered beams by Hoseini *et al.* (2009).

Bayat *et al.* (2011) applied the max-min approach and homotopy perturbation method (HPM) to obtain an approximate solution for nonlinear vibrations of tapered beams.

Another new analytical approach was used by Shahidi *et al.* (2011) to prepare an accurate solution in the beams vibrations.

In the recent decades many new analytical and numerical approaches have been investigated. The most useful methods for solving nonlinear equations are perturbation methods. They are not valid for strongly nonlinear equations and they have many shortcomings. Many new techniques have appeared in the open literature to overcome the shortcomings of traditional analytical methods such as Energy Balance (Bayat 2011a, b, c, Jamshidi 2010, Mehdipour *et al.* 2010),

Variational Approach (Liu 2009, Bayat 2011d, Pakar 2011a), Variational iteration Method (Pakar 2012), Iteration Perturbation (Bayat 2011e), Homotopy analysis method (Ganji 2009), Max-min approach (Bayat 2011f), Homotopy perturbation method (Baki 2011) and Hamiltonian approach (He 2010, Xu and He 2010, He *et al.* 2010) other numerical and analytical methods (He 2002, Ghasemi *et al.* 2011, Pakar 2011b, Bayat 2011g, h, Ganji and Kachapi 2011, Fu and Wang 2011).

The paper has been collocated as follows:

First, we describe the basic concept of Hamiltonian approach. The mathematical formulations of the problems are considered in second section. Then for the third section, applications of Hamiltonian approach have been studied, to demonstrate the applicability and preciseness of the method. In the fourth section, some comparisons between analytical and numerical solutions are presented. Eventually we show that HA can converge to a precise cyclic solution for nonlinear systems.

2. Basic idea of Hamiltonian approach

Previously, He (2002) had introduced the Energy Balance method based on collocation and the Hamiltonian. This approach is very simple but strongly depends upon the chosen location point. Recently, He (2010) has proposed the Hamiltonian approach to overcome the shortcomings of the energy balance method. This approach is a kind of energy method with a vast application in conservative oscillatory systems. In order to clarify this approach, consider the following general oscillator

$$\ddot{u}+f(u,\dot{u},\ddot{u}) = 0 \tag{1}$$

With initial conditions

$$u(0) = A, \quad \dot{u}(0)=0 \tag{2}$$

Oscillatory systems contain two important physical parameters, i.e., the frequency ω and the amplitude of oscillation A. It is easy to establish a variational principle for Eq. (1), which reads

$$J(u) = \int_0^{T/4}\left\{-\frac{1}{2}\dot{u}^2+F(u)\right\}dt \tag{3}$$

Where T is period of the nonlinear oscillator, $\partial F/\partial u = f$.

In the Eq. (3), $\frac{1}{2}\dot{u}^2$ is kinetic energy and $F(u)$ potential energy, so the Eq. (3) is the least

Lagrangian action, from which we can immediately obtain its Hamiltonian, which reads

$$H(u) = \frac{1}{2}\dot{u}^2 + F(u) = \text{constant} \tag{4}$$

From Eq. (4), we have

$$\frac{\partial H}{\partial A} = 0 \tag{5}$$

Introducing a new function, $\overline{H}(u)$, defined as

$$\overline{H}(u) = \int_0^{T/4} \left\{\frac{1}{2}\dot{u}^2 + F(u)\right\}dt = \frac{1}{4}TH \tag{6}$$

Eq. (5) is, then, equivalent to the following one

$$\frac{\partial}{\partial A}\left(\frac{\partial \overline{H}}{\partial T}\right) = 0 \tag{7}$$

or

$$\frac{\partial}{\partial A}\left(\frac{\partial \overline{H}}{\partial(1/\omega)}\right) = 0 \tag{8}$$

From Eq. (8) we can obtain approximate frequency-amplitude relationship of a nonlinear oscillator.

3. Applications

In order to assess the advantages and the accuracy of the Hamiltonian Approach, we will consider the following examples:

3.1 Example 1

We consider the nonlinear vibration of an electrostatically actuated microbeam. Fig. 1 represent a fully clamped micro beam with uniform thickness h, length l, width b ($b >> 5h$), effective modulus $\overline{E} = E/(1-\nu^2)$, Young's modulus E, Poisson's ratio υ and density ρ. By applying the Galerkin Method and employing the classical beam theory and taking into account of the mid-plane stretching effect as well as the distributed electrostatic force, the dimensionless equation of motion for the micro beam is as follow (Fu 2011)

$$\ddot{u}(a_1u^4 + a_2u^2 + a_3) + a_4u + a_5u^3 + a_6u^5 + a_7u^7 = 0,\ u(0) = A,\ \dot{u}(0) = 0 \tag{9}$$

Where u is the dimensionless deflection of the micro beam, a dot denotes the derivative with respect to the dimensionless time variable $\tau = t\sqrt{\overline{E}I/(\rho bhl^4)}$ with I and t being the second moment of area of the beam cross-section and time, respectively.

In Eq. (9), the physical parameters $a_i(i = 1-7)$ are given by Fu (2011)

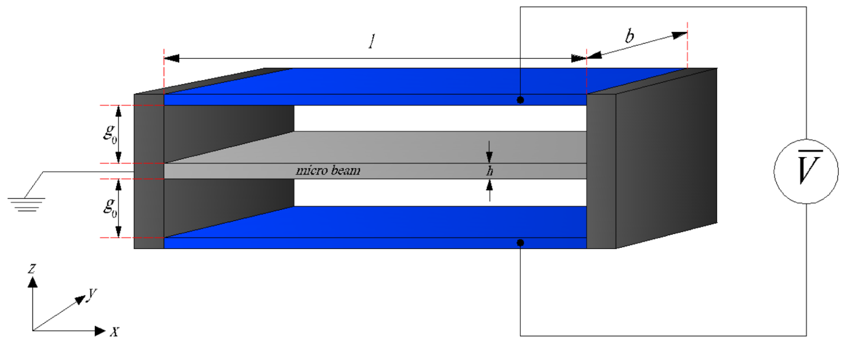

Fig. 1 Schematics of a double-sided driven clamped-clamped microbeam-based electromechanical resonator

$$a_1 = \int_0^1 \phi^6 d\xi \tag{10.a}$$

$$a_2 = -2\int_0^1 \phi^4 d\xi \tag{10.b}$$

$$a_3 = \int_0^1 \phi^2 d\xi \tag{10.c}$$

$$a_4 = \int_0^1 (\phi'''' \phi - N\phi'' \phi - V^2 \phi) d\xi \tag{10.d}$$

$$a_5 = -\int_0^1 (2\phi'''' \phi^3 - 2N\phi'' \phi^3 + \alpha\phi'' \phi \int_0^1 (\phi')^2 d\xi) d\xi \tag{10.e}$$

$$a_6 = \int_0^1 (\phi'''' \phi^5 - N\phi'' \phi^5 + 2\alpha\phi'' \phi^3 \int_0^1 (\phi')^2 d\xi) d\xi \tag{10.f}$$

$$a_7 = -\int_0^1 (\alpha\phi'' \phi^5 \int_0^1 (\phi')^2 d\xi) d\xi \tag{10.g}$$

In which, the following nondimensional variables and parameters are introduced

$$\alpha = \frac{6g_0^2}{h^2}, \quad \xi = \frac{x}{l}, \quad N = \frac{\bar{N}l^2}{\bar{E}I}, \quad V^2 = \frac{24\varepsilon_0 l^4 \bar{V}^2}{\bar{E}h^3 g_0^3} \tag{11}$$

While a prime (') indicates the partial differentiation with respect to the coordinate variable ξ.

The trial function is $\phi(\xi) = 16\xi^2(1-\xi)^2$. The parameter $\bar{N}$ denotes the tensile or compressive axial load, g_0 is initial gap between the microbeam and the electrode, $\bar{V}$ the electrostatic load and ε_0 vacuum permittivity.

The complete formulation of Eq. (9) can be referred to (Fu 2011) for details.

The Hamiltonian of Eq. (9) is constructed as

$$H = \frac{1}{2}(a_1u^4 + a_2u^2 + a_3)\dot{u}^2 + \frac{1}{2}a_4u^2 + \frac{1}{4}a_5u^4 + \frac{1}{6}a_5u^6 + \frac{1}{8}a_7u^8 \tag{12}$$

Integrating Eq. (12) with respect to τ from 0 to $T/4$, we have

$$\overline{H} = \int_0^{T/4}\left(\frac{1}{2}(a_1u^4+a_2u^2+a_3)\dot{u}^2+\frac{1}{2}a_4u^2+\frac{1}{4}a_5u^4+\frac{1}{6}a_5u^6+\frac{1}{8}a_7u^8\right)d\tau \tag{13}$$

Assume that the solution can be expressed as

$$u(\tau) = A\cos(\omega\tau) \tag{14}$$

Substituting Eq. (14) into Eq. (13), we obtain

$$\overline{H} = \int_0^{T/4}\begin{pmatrix}\frac{1}{2}\alpha_1\omega^2A^6\cos^4(\omega\tau)\sin^2(\omega\tau)+\frac{1}{2}\alpha_2\omega^2A^4\cos^2(\omega\tau)\sin^2(\omega\tau)+\frac{1}{2}\alpha_3\omega^2A^2\sin^2(\omega\tau)\\ +\frac{1}{2}\alpha_4A^2\cos^2(\omega\tau)+\frac{1}{4}\alpha_5A^4\cos^4(\omega\tau)+\frac{1}{6}\alpha_6A^6\cos^6(\omega\tau)+\frac{1}{8}\alpha_7A^8\cos^8(\omega\tau)\end{pmatrix}d\tau$$

$$= \int_0^{\pi/2}\begin{pmatrix}\frac{1}{2}\alpha_1\omega^2A^6\cos^4\tau\sin^2\tau+\frac{1}{2}\alpha_2\omega^2A^4\cos^2\tau\sin^2\tau+\frac{1}{2}\alpha_3\omega^2A^2\sin^2\tau\\ +\frac{1}{2}\alpha_4A^2\cos^2\tau+\frac{1}{4}\alpha_5A^4\cos^4\tau+\frac{1}{6}\alpha_6A^6\cos^6\tau+\frac{1}{8}\alpha_7A^8\cos^8\tau\end{pmatrix}d\tau$$

$$= \frac{1}{64}A^6\omega\alpha_1\pi+\frac{1}{32}A^4\omega\alpha_2\pi+\frac{1}{8}A^2\omega\alpha_3\pi+\frac{1}{8\omega}A^2\alpha_4\pi+\frac{3}{64\omega}A^4\alpha_5\pi+\frac{5}{192\omega}A^6\alpha_6\pi+\frac{34}{2048\omega}A^8\alpha_7\pi \tag{15}$$

Setting

$$\frac{\partial}{\partial A}\left(\frac{\partial\overline{H}}{\partial(1/\omega)}\right) = -\frac{3}{32}A^5\pi\omega^2\alpha_1-\frac{1}{8}A^3\pi\omega^2\alpha_2-\frac{1}{4}A\pi\omega^2\alpha_3+\frac{1}{4}A\pi\alpha_4+\frac{3}{16}A^3\pi\alpha_5+\frac{5}{32}A^5\pi\alpha_6+\frac{35}{256}A^7\pi\alpha_7 = 0 \tag{16}$$

Solving the above equation, an approximate frequency as a function of amplitude equals

$$\omega_{HA} = \frac{\sqrt{2}}{4}\frac{\sqrt{(48a_5A^2+40a_6A^4+35a_7A^6+64a^4)}}{\sqrt{4A^2a_2+3A^4a_1+8a_3}} \tag{17}$$

According to Eqs. (14) and (17), we can obtain the following approximate solution

$$u(\tau) = A\cos\left(\frac{\sqrt{2}}{4}\frac{\sqrt{(48a_5A^2+40a_6A^4+35a_7A^6+64a^4)}}{\sqrt{4A^2a_2+3A^4a_1+8a_3}}\tau\right) \tag{18}$$

3.2 Example 2

The nonlinear vibration of tapered beams is considered for the second example. In dimensionless form, Goorman is given the governing differential equation corresponding to fundamental vibration mode of a tapered beam (Gorman 1975)

$$\ddot{u}+\varepsilon_1(u^2\ddot{u}+u\dot{u}^2)+u+\varepsilon_2u^3 = 0 \tag{19}$$

Where u is displacement and ε_1 and ε_2 are arbitrary constants. Subject to the following initial conditions

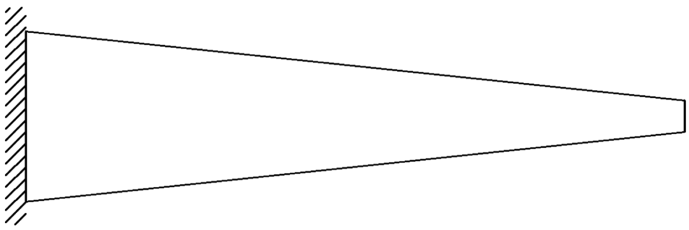

Fig. 2 Schematic representation of a tapered beam

$$u(0) = A, \quad \dot{u}(0) = 0 \tag{20}$$

The Hamiltonian of Eq. (19) is constructed as

$$H = \frac{1}{2}\dot{u}^2 + \frac{1}{2}\varepsilon_1\dot{u}^2u^2 + \frac{1}{2}u^2 + \frac{1}{4}\varepsilon_2u^4 \tag{21}$$

Integrating Eq. (21) with respect to t from 0 to $T/4$, we have

$$\overline{H} = \int_0^{T/4}\left(\frac{1}{2}\dot{u}^2 + \frac{1}{2}\varepsilon_1\dot{u}^2u^2 + \frac{1}{2}u^2 + \frac{1}{4}\varepsilon_2u^4\right)dt \tag{22}$$

Assume that the solution can be expressed as

$$u(t) = A\cos(\omega t) \tag{23}$$

Substituting Eq. (23) into Eq. (22), we obtain

$$\overline{H} = \int_0^{T/4}\left(\frac{1}{2}A^2\omega^2\sin^2(\omega t) + \frac{1}{2}\varepsilon_1A^4\omega^2\sin^2(\omega t)\cos^2(\omega t) + \frac{1}{2}A^2\cos^2(\omega t) + \frac{1}{4}\varepsilon_2A^4\cos^4(\omega t)\right)dt$$

$$= \int_0^{\pi/2}\left(\frac{1}{2}A^2\omega\sin^2t + \frac{1}{2}\varepsilon_1A^4\omega\sin^2t\cos^2t + \frac{1}{2\omega}A^2\cos^2t + \frac{1}{4\omega}\varepsilon_2A^4\cos^4t\right)dt$$

$$= +\frac{1}{8}\omega A^2\pi + \frac{1}{32}\omega A^4\varepsilon_1\pi\frac{1}{8\omega}A^2\pi + \frac{3}{64\omega}A^4\varepsilon_2\pi \tag{24}$$

Setting

$$\frac{\partial}{\partial A}\left(\frac{\partial\overline{H}}{\partial(1/\omega)}\right) = -\frac{1}{4}A\pi\omega^2 - \frac{1}{8}\varepsilon_1A^3\pi\omega^2 + \frac{1}{4}A\pi + \frac{3}{16}\varepsilon_2A^3\pi \tag{25}$$

Solving the above equation, an approximate frequency as a function of amplitude equals

$$\omega_{HA} = \frac{\sqrt{2}}{2}\frac{\sqrt{(\varepsilon_1A^2+2)(3\varepsilon_2A^2+4)}}{(\varepsilon_1A^2+2)} \tag{26}$$

Hence, the approximate solution can be readily obtained

$$u(t) = A\cos\left(\frac{\sqrt{2}}{2}\frac{\sqrt{(2+\varepsilon_1A^2)(4+3\varepsilon_2A^2)}}{(2+\varepsilon_1A^2)}\right) \tag{27}$$

4. Results and discussions

To illustrate and verify the accuracy of this new approximate analytical approach, some comparisons of the time history oscillatory displacement responses with the Energy Balance Method are presented in Table 1 and Figs. 3 to 4 for example 1 and Table 2 and Figs. 5 to 8 for example 2.

Table 1 gives the comparison of obtained results with those obtained by Fu (2011) with EBM solution for different N, α, V and initial conditions. It can be observed from Table 1 that there is high level of agreement between the results obtained from the Hamiltonian approach and the results of Fu (2011). Figs. 3 and 4 represent comparisons of the analytical solution of $u(\tau)$ based on τ and $\dot{u}$ versus u with the EBM solution. The motion of the system is a periodic motion and the amplitude of vibration is a function of the initial conditions. The best accuracy can be seen at extreme points.

For the second example the exact frequency ω_{ex} for a dynamic system governed by Eq. (19) can be derived, as shown in Eq. (28), as follows

$$\omega_{Exact} = \frac{2\pi}{4\sqrt{2}A}\int_0^{\pi/2} \frac{\sqrt{1+\varepsilon_1 A^2\cos^2 t}\sin t}{\sqrt{A^2(1-\cos^2 t)(\varepsilon_2 A^2\cos^2 t+\varepsilon_2 A^2+2)}}dt \tag{28}$$

Table 1 Comparison of frequency corresponding to various parameters of system

Constant parameters				Energy balance solution	Hamiltonian Approach solution
A	N	α	V	ω_{EBM} (Fu 2011)	ω_{HA}
0.3	10	24	0	26.3867	26.3644
0.3	10	24	10	24.2753	24.2526
0.3	10	24	20	16.3829	16.3556
0.6	10	24	0	28.9227	28.5579
0.6	10	24	10	26.5324	26.1671
0.6	10	24	20	17.5017	17.0940

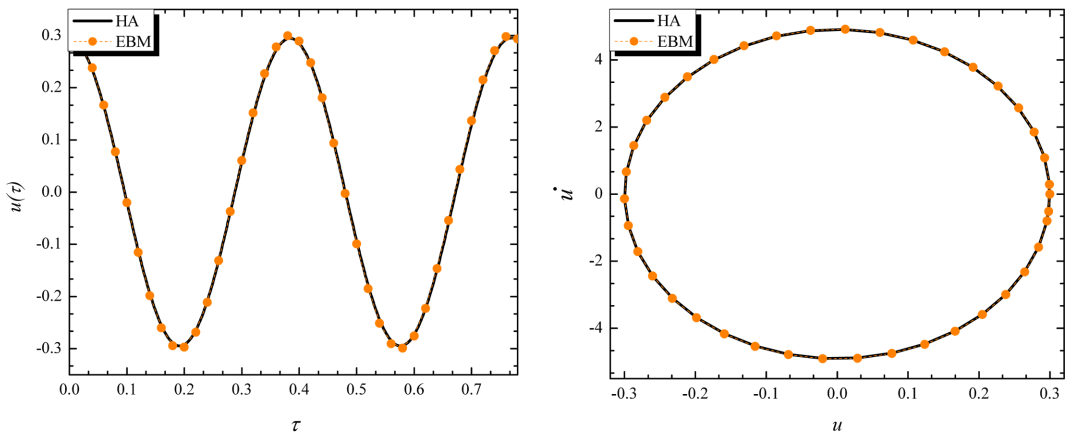

Fig. 3 Comparison of analytical solution of $u(\tau)$ based on time and $\dot{u}$ versus u with the EBM solution for $N = 10$, α =24, $V = 20$, $A = 0.3$

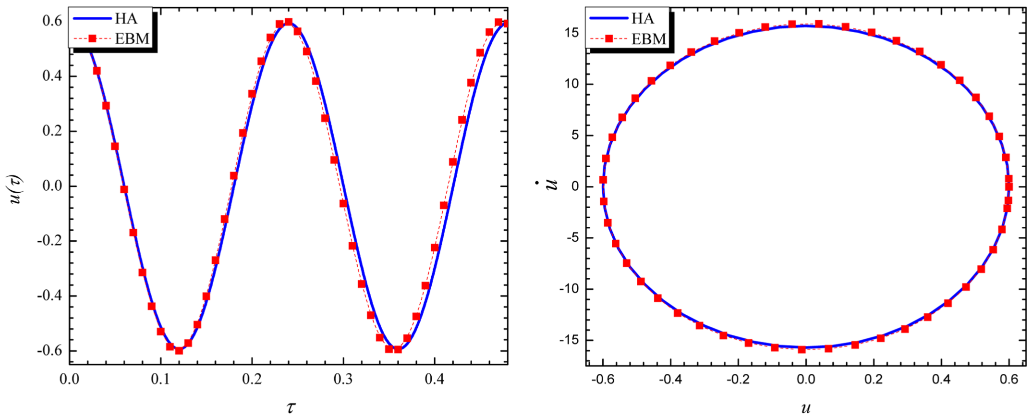

Fig. 4 Comparison of analytical solution of $u(\tau)$ based on time and, $\dot{u}$ versus u with the EBM solution for $N = 10$, $\alpha = 24$, $V = 10$, $A = 0.6$

Table 2 Comparison of frequency corresponding to various parameters of system

Constant parameters			Approximate solution	Exact solution	Relative error %
A	ε_1	ε_2	ω_{HA}	ω_{Exact}	$\left\lvert \frac{\omega_{EX} - \omega_{HA}}{\omega_{EX}} \right\rvert$
0.1	0.1	0.1	1.0001	1.0005	0.0374
0.1	1	0.2	0.9983	0.9983	0.0002
0.5	0.5	1	1.0572	1.0573	0.0084
0.5	1	0.5	0.9860	0.9870	0.1018
1	1	1	1.0801	1.0904	0.9382
1	0.5	0.2	0.9592	0.9623	0.3262
1.5	0.3	1	1.4175	1.4210	0.2458
1.5	0.8	0.2	0.8390	0.8550	1.8654
2	0.4	0.2	0.9428	0.9593	1.7212
2	1	0.8	1.0646	1.0917	2.4853

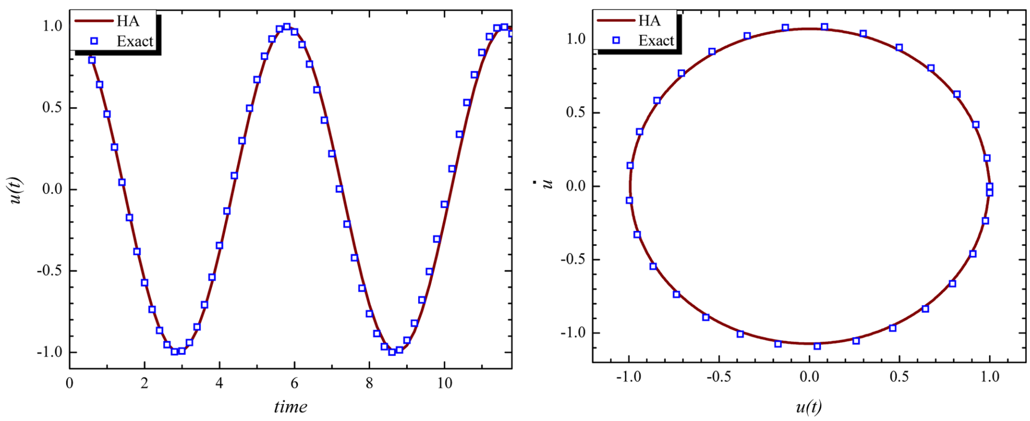

Fig. 5 Comparison of analytical solution of $u(t)$ based on time and, $\dot{u}$ versus u with the exact solution for $\varepsilon_1 = 1$, $\varepsilon_2 = 1$, $A = 1$

Table 2 represents the comparsion of frequencies with the Hamiltonian Approach (HA) and the exact ones for different value of A, ε_1 and ε_2. The maximum relative error between the Hamiltonian Approach (HA) results and exact results is 2.4853%.

Figs. 5 and 6 represent comparisons of the analytical solution of $u(t)$ based on time and $\dot{u}$ versus u, with the exact solution. From Figs. 5, 6, the motion of the system is a periodic motion and the amplitude of vibration is a function of the initial conditions. Comparison of frequency corresponding to various parameters of amplitude (A) and ε_2 for $\varepsilon_1 = 0.5$ has been studied in the figure shows in Fig. 7. The effect of small parameters ε_1 on the frequency corresponding to various parameters of amplitude (A) has been studied in Fig. 8 for $\varepsilon_2 = 1$. It can be observed that Hamiltonian Approach results are accurate and require smaller computational effort. It is evident that Hamiltonian Approach (HA) shows an excellent agreement with the exact solution and quickly

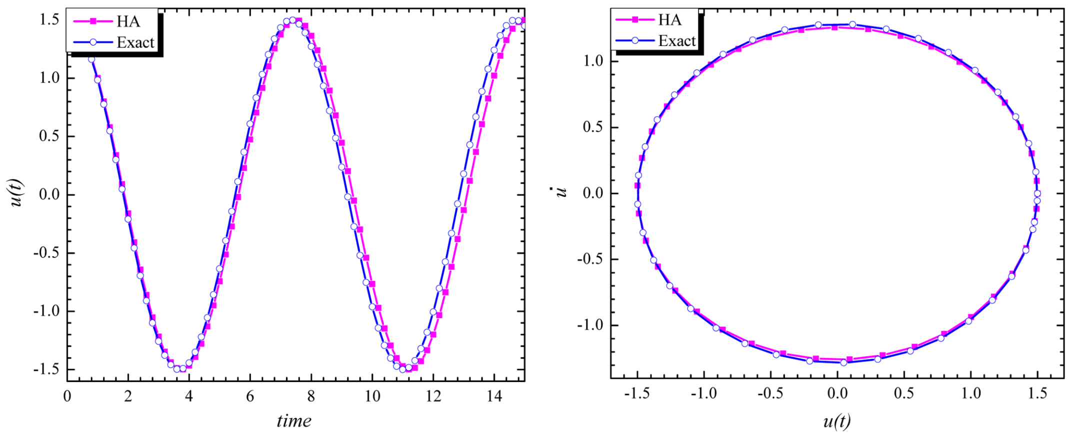

Fig. 6 Comparison of analytical solution of $u(t)$ based on time and, $\dot{u}$ versus u with the exact solution for $\varepsilon_1 = 0.8$, $\varepsilon_2 = 0.2$, $A = 1.5$

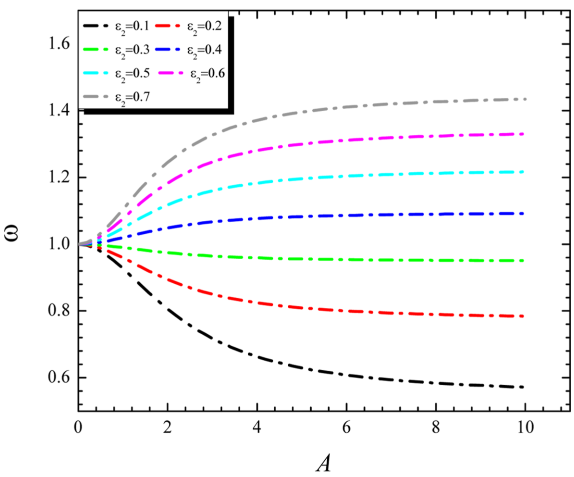

Fig. 7 Comparison of frequency corresponding to various parameters of amplitude (A) for $\varepsilon_1 = 0.5$

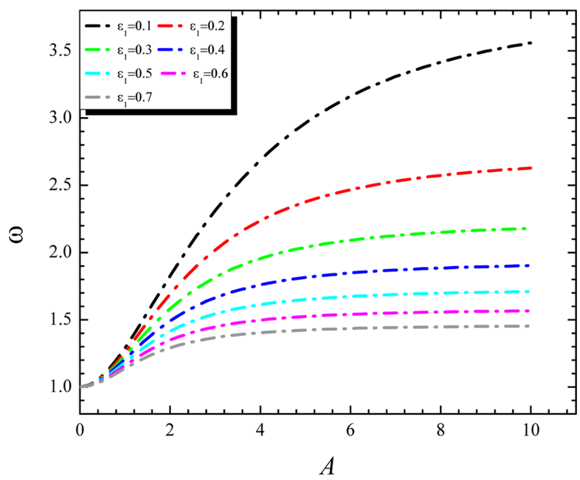

Fig. 8 Comparison of frequency corresponding to various parameters of amplitude (A) for $\varepsilon_2 = 1$

convergent and valid for a wide range of vibration amplitudes and initial conditions.

5. Conclusions

In this paper, a new novel method has been used to obtain analytical solutions for nonlinear vibration of an electrostatically actuated microbeam and tapered beams. The analytical solutions yield a thoughtful and insightful understanding of the effect of system parameters and initial conditions. An excellent accuracy of the Hamiltonian Approach (HA) results indicates that those methods can be used for problems in which the strong nonlinearities are taken into account. Hamiltonian approach (HA) can be powerful mathematical tools for studying of strong nonlinear problems. We can suggest Hamiltonian approach as novel and simple method for oscillation systems which provide easy and direct procedures for determining approximations to the periodic solutions.

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