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# Analytical approximate solutions for large post-buckling response of a hygrothermal beam

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**Abstract.** This paper deals with large deformation post-buckling of a linear-elastic and hygrothermal beam with axially nonmovable pinned-pinned ends and subjected to a significant increase in swelling by an alternative method. Analytical approximate solutions for the geometrically nonlinear problem are presented. The solution for the limiting case of a string is also obtained. By coupling of the well-known Maclaurin series expansion and orthogonal Chebyshev polynomials, the governing differential equation with sinusoidal nonlinearity can be reduced to form a cubic-nonlinear equation, and supplementary condition with cosinoidal nonlinearity can also be simplified to be a polynomial integral equation. Analytical approximations to the resulting boundary condition problem are established by combining the Newton's method with the method of harmonic balance. Two approximate formulae for load along axis, potential strain for free hygrothermal expansion and periodic solution are established for small as well as large angle of rotation at the end of the beam. Illustrative examples are selected and compared to "reference" solution obtained by the shooting method to substantiate the accuracy and correctness of the approximate analytical approach.

**Keywords:** buckling; approximation method; large deformation; Chebyshev polynomials

## 1. Introduction

Mechanical buckling has been identified to be a primary mode of failure for beams subjected to in-plane compressive loads. Beams may also experience thermal buckling due to change in temperature, and hygroscopic buckling due to change in moisture concentrations. The hygrothermal buckling of a constrained beam is due to axial hygrothermal expansion (Li *et al.* 2009, Anandrao *et al.* 2010, Kocaturk and Akbas 2011, 2012). Understanding the buckling and post-buckling behavior of an elastic and hygrothermal beam is important for the designers of railroad tracks, optical fibers, satellite tethers, subsea and buried pipelines. Such a buckling differs substantially from that of the common buckling of beams subjected to mechanical compressive loads. For this problem, the extensibility of the beam must be considered and the compressive normal stress is due to an increase in either temperature or moisture content.

Many post-buckling studies, based on classical theory, of elastic beams subjected to mechanical or thermal loading are available in the literature, see, for example, based on only the highest-order

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nonlinearity in the expression for axial strain, the first-order approximation to the post-buckling response was derived by Nowinski (1978), Ziegler *et al.* (1989), Boley *et al.* (1997). Their derivations indicate that the axial force after initial buckling remains the same as the compressive load at critical buckling temperature change. By using the framework of the general branching theory of discrete systems, El Naschie (1976) also investigated the thermal buckling problem and treated the initial post-buckling response. He predicted that the axial load would increase in the initial post-buckling response. It has been pointed out later by Coffin *et al.* (1999) that the conclusion of El Naschie (1976) was not valid because of his incorrect independent assumption of the axial strain and the angle of rotation. Coffin and Bloom (1999) developed an elliptic integral solution for the post-buckling response of a linearly elastic and hygrothermal beam with axially nonmovable pinned-pinned ends, by considering a relation between these two quantities, and further assuming that the strain of free expansion was linearly related to changes in temperature and moisture. A set of differential equations for the undeformed configuration was derived which resulted in the requirement to simultaneously solve two coupled integral elliptic equations.

The thermal post-buckling of a beam made of physically nonlinear thermoelastic material was examined by Jekot (1996), but he did not consider the geometric nonlinearity of the central axis curvature and only used simplified form of nonlinear axial strain. His solution also predicted a constant axial load after buckling. Li *et al.* (2000) proposed a computational analysis for the thermal post-buckling behavior of beams with axially nonmovable pinned-pinned ends as well as fixed-fixed ends, using a shooting method. Focusing on the bifurcations of the resulting equilibrium equations under both traction and displacement boundary conditions, Cisternas *et al.* (2002) considered thermal expansion effects in the extensible rod theory. They determined the subcritical and supercritical pitchfork bifurcations. Vaz *et al.* (2003) investigated the post-buckling response of an initially straight slender beam made of linear elastic material with a nonlinear strain-temperature relationship. Uniform temperature gradient along the beam was assumed and expansion is prevented by double-hinged nonmovable ends. By using uncoupled elliptic integrals, they obtained the solution of this problem, which are derived from the governing equations in the deformed configuration, hence completely defining the shape of the beam (elastica).

In recent years, with fiber-reinforced composite laminated shell structures being widely used in the aerospace, marine, automobile and other engineering industries, many post-buckling studies, based on classical shell theory, of composite laminated thin cylindrical panels subjected to thermal loading are available in the literature. Among these studies, Dafedar *et al.* (2002) presented a novel, analytical mixed theory based on the potential energy principle to investigate buckling response of laminated composite plates subjected to mechanical and hygrothermal loads. It was shown that solutions from the models were in excellent agreement with the available three-dimensional elasticity solutions. Taking into account a non-uniform temperature distribution through the thickness, Liu *et al.* (2006) investigated the response of composite columns under axial compressive loading. Due to effect of the non-uniform temperature, the structure behaves like an imperfect column and responds by bending, the neutral axis moves away from the centroid of the cross-section. The results are obtained by solving linearized equation. By applying energy method, Wang *et al.* (2007) studied hygrothermal effects on local buckling for different delaminated shapes near the surface of cylindrical laminated shells, and then obtained the relationships between critical strain value and the geometrical and physical parameters of cylindrical laminated shells and sub-laminated shells. Aoki *et al.* (2008) investigated the combined effects of water absorption and thermal environment on compression after impact (CAI) characteristics of CFRP laminates. Kundu *et al.*

(2009) used finite element method Nonlinear buckling analysis of hygrothermoelastic composite shell panels, and the result shows that large deformation due to the bending of shell panels under hygrothermal environmental condition may cause structural instability. It is different from above that Lal *et al.* (2011) investigated the effect of random system properties on the post buckling load of geometrically nonlinear laminated composite cylindrical shell panel subjected to hygrothermomechanical loading, and by comparing the present results with those available in the literature and independent Monte Carlo simulation they showed that the performance of outlined stochastic approach had been validated.

In this paper, an alternative approach is presented and discussed to solve large deformation post-buckling of an elastic and hygrothermal beam with axially nonmovable pinned–pinned ends and subjected to a significant increase in swelling. Geometrically large nonlinear strain is considered in the analysis, but any form of material nonlinearity is not included. The solution is based on the governing equations derived by Coffin *et al.* (1999). The proposed approach forms a significant extension of constructing analytical approximate solutions to (i) non-linear oscillations (Wu *et al.* 2006a, b); and (ii) large deformation post-buckling of elastic rings under uniform hydrostatic pressure (Wu *et al*. 2007). We establish analytical approximate solutions to large deformation post-buckling of a linear-elastic and hygrothermal pinned beam in terms of the angle of rotation at the end of the beam, by using linearization of the governing equation and the method of harmonic balance. Unlike the classical method of harmonic balance, the linearization is performed prior to harmonic balancing and thus a set of linear algebraic equations instead of one of non-linear algebraic equations is derived. We are hence able to establish analytical approximate solutions. These analytical approximate solutions show excellent agreement with the "reference" solution obtained by the shooting method for small as well as large angle of rotation at the end of the beam.

## 2. Formulation

As shown in Fig. 1, the beam is assumed to exhibit pure elastic response, dimensional changes in the cross-section are regarded as negligible, and there is no transverse shear deformation. The axial stress is assumed to be proportional to the difference between the stretch and the hygrothermal extension of the beam. For the case of extensible elastica, the total strain is not a linear superposition of the stain along the neutral axis and the curvature of the neutral axis, which brings the extra nonlinear. The presented analysis of large post-buckling behavior of the beam needs to consider the nonlinear due to extensibility.

The governing equation of an elastic and hygrothermal beam which is fully restrained against axial expansion and is subjected to an increase in either temperature or moisture content can be written in the form (Coffin *et al.* 1999)

$$\frac{d^2\theta}{dS^2} = \left(\frac{N}{EI}\right)\left(\frac{N}{EA}\cos\theta + 1 + \varepsilon_{ht}\right)\sin\theta \tag{1}$$

where $N$ is the force component along the horizontal $x$-axis, $A$ is the cross-sectional area of beam, $E$ is Young's modulus, $\theta$ is the angle formed by the $x$-axis and the tangent to neutral axis, $S \in [-L, L]$ is the original lengthwise coordinate system of the beam, $\varepsilon_{ht}$ is the potential strain for free hygrothermal expansion and is defined as

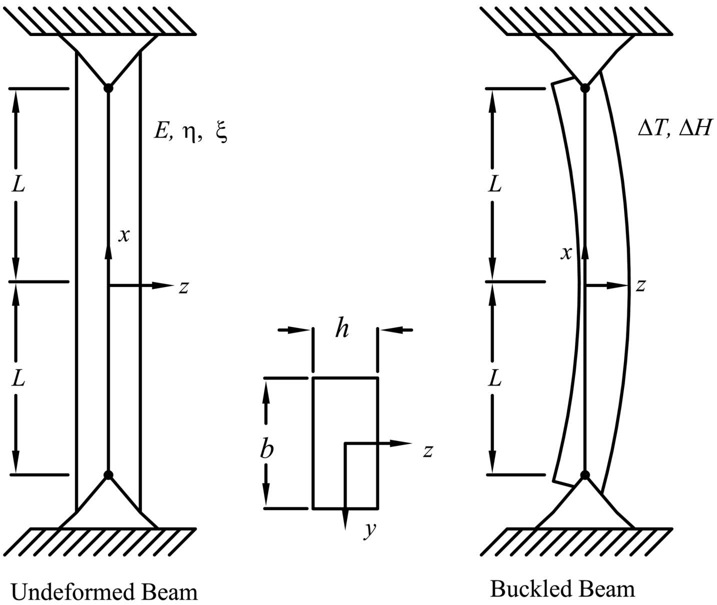


Fig. 1 Sketch of a simply supported beam subjected to hygrothermal loads

$$\varepsilon_{ht} = \eta \cdot \Delta T + \xi \cdot \Delta H \tag{2}$$

Here, the terms $\Delta T$ and $\Delta H$ are the change in temperature and moisture, respectively, and $\eta$ and $\xi$ are the coefficients of thermal expansion and hygroexpansion, respectively.

Let the angle of rotation at $S = -L$ be $a$, we seek a deformed state that is symmetric about $S = 0$, such that the angle at $S = L$ is $-a$. At the pinned ends of the beam, the bending moments must vanish, we have the boundary conditions

$$\frac{d\theta}{dS}(-L) = \frac{d\theta}{dS}(L) = 0 \tag{3}$$

In addition, the beam is fully restrained against end displacement, thus, we have the supplementary condition (Coffin *et al.* 1999)

$$2L = \int_{-L}^{L}\left(\frac{N}{EA}\cos\theta + 1 + \varepsilon_{ht}\right)\cos\theta dS \tag{4}$$

Once $\theta(S), N$ and $\varepsilon_{ht}$ are obtained from differential and integration formulation in Eqs. (1), (3) and (4), the axial deflection $x(S)$ and lateral deflection $w(S)$ of the beam, at any point $S$ along the beam can then be calculated from the following relations

$$x(S) = \int_{-L}^{S}\left(\frac{N}{EA}\cos\theta(\zeta) + 1 + \varepsilon_{ht}\right)\cos\theta(\zeta)d\zeta \tag{5}$$

$$w(S) = \int_{-L}^{S}\left(\frac{N}{EA}\cos\theta(\zeta) + 1 + \varepsilon_{ht}\right)\sin\theta(\zeta)d\zeta \tag{6}$$

For details of derivation in this section, we refer the readers to Coffin *et al.* (1999).

## 3. Solution methodology

A new independent variable $\tau = \pi S/(2L) + \pi/2$ is introduced. Then, Eqs. (1), (3) and (4) can be rewritten in the following dimensionless forms

$$\frac{d^2\theta}{d\tau^2} = \frac{2\Lambda^2\rho^2}{\pi^2}\sin 2\theta - \frac{4\Lambda\mu}{\pi^2}\sin\theta \tag{7}$$

$$\frac{d\theta}{d\tau}(0) = \frac{d\theta}{d\tau}(\pi) = 0 \tag{8}$$

$$\pi = \int_0^{\pi}(\mu - \Lambda\rho^2\cos\theta)\cos\theta d\tau \tag{9}$$

where

$$\Lambda = \lambda^2,\ \lambda^2 = -NL^2/(EI),\quad \rho^2 = I/AL^2 \quad \text{and} \quad \mu = \varepsilon_{ht} + 1 \tag{10}$$

are dimensionless parameters and

$$\theta(0) = a,\quad \theta(\pi) = -a \tag{11}$$

are the angles of rotation at the ends of the beam.

### *3.1 Maclaurin series expansion and Chebyshev polynomials*

Along with the Maclaurin series expansion and the Chebyshev polynomials (Denman 1969, Jonckheere 1971, Li *et al*. 2008, Beléndez 2009), we arrive at a new nonlinear equation with no circular functions. Introducing a variable $u = \theta/a$ (Denman 1969, Jonckheere 1971, Li *et al*. 2008, Beléndez 2009) to Eqs. (7)-(9) and applying the Maclaurin series representation (Abramowitz 1965) for the functions $\sin(au)/a$, $\sin(2au)/(2a)$, $\cos(au)$, and $\cos(2au)$ by taking the first five terms yield a series of equations. Expressing the powers of $u$ in the resulting equations in the form of Chebyshev polynomials as $T_k(k = 1, 2, \ldots)$, and then neglecting all terms associated with those Chebyshev polynomials for $T_i(i > 3)$ yield

$$\frac{d^2u}{d\tau^2} - \frac{4\Lambda^2\rho^2}{\pi^2}[B_1u + B_2u^3] + \frac{4\Lambda\mu}{\pi^2}[C_1u + C_2u^3] = 0 \tag{12}$$

$$u(0) = 1,\quad u(\pi) = -1,\quad \frac{du}{d\tau}(0) = \frac{du}{d\tau}(\pi) = 0 \tag{13}$$

$$\int_0^{\pi}\{2\mu[D_0 + D_1u^2 + D_2u^4] - \Lambda\rho^2[1 + F_0 + F_1u^2 + F_2u^4] - 2\}d\tau = 0 \tag{14}$$

The expressions for $B_1, B_2, C_1, C_2, D_0, D_1, D_2, F_0, F_1, F_2$ are presented in the Appendix.

### *3.2 Solution procedure*

We will establish the analytical approximate solution to Eqs. (12)-(14) in terms of the initial value $u(0) = 1$. A reasonable and simple initial approximation satisfying conditions in Eq. (12) can be taken as

$$u_0(\tau) = \cos\tau, \quad \tau \in [0, \pi] \tag{15}$$

Here, $u_0(\tau)$ is a periodic function of $\tau$, of period $2\pi$.

Substituting Eq. (15) into Eq. (12) and setting the resulting coefficient of term $\cos\tau$ equal zero, give

$$(4B_1+3B_2)\rho^2\Lambda_0^2-(4C_1+3C_2)\mu_0\Lambda_0+\pi^2 = 0 \tag{16}$$

Substituting Eq. (15) into Eq. (14), and simplifying yield

$$16+(8+8F_0+4F_1+3F_2)\rho^2\Lambda_0-2(8D_0+4D_1+3D_2)\mu_0 = 0 \tag{17}$$

From Eqs. (16) and (17), the first analytical approximations for $\Lambda$ and $\mu$ can be solved and expressed as functions of $a$, as

$$\Lambda_0(a) = \frac{8(4C_1+3C_2)+\sqrt{[8(4C_1+3C_2)]^2-2M(8D_0+4D_1+3D_2)\pi^2}}{M} \tag{18}$$

$$\mu_0(a) = \frac{16+(8+8F_0+4F_1+3F_2)\rho^2\Lambda_0}{2(8D_0+4D_1+3D_2)} \tag{19}$$

where

$$M = [2(4B_1+3B_2)(8D_0+4D_1+3D_2)-(4C_1+3C_2)(8+8F_0+4F_1+3F_2)]\rho^2$$

and the corresponding analytical approximate solution is given by Eq. (15). Applying Eq. (10), we can obtain the first analytical approximations for $\lambda$, and $\varepsilon_{ht}$ as

$$\lambda_0(a) = \sqrt{\Lambda_0(a)}, \quad \varepsilon_{ht0}(a) = \mu_0(a)-1 \tag{20}$$

And the first analytical approximate periodic solution can be expressed as

$$\theta_0(\tau) = a\cos\tau, \quad \tau \in [0, \pi] \tag{21}$$

Next, we express the solution $(u(\tau), \Lambda, \mu)$ of Eqs. (12)-(14) as

$$u(\tau) = u_0(\tau)+\Delta u_0(\tau), \quad \mu = \mu_0+\Delta\mu_0, \quad \Lambda = \Lambda_0+\Delta\Lambda_0 \tag{22}$$

Here, $(u_0(\tau), \Lambda_0, \mu_0)$ is the principal part and $(\Delta u_0(\tau), \Delta\Lambda_0, \Delta\mu_0)$ is the correction part. Substituting Eq. (22) into Eqs. (12)-(14) and linearizing with respect to $(\Delta u_0(\tau), \Delta\Lambda_0, \Delta\mu_0)$ lead to

$$\begin{aligned}&\frac{d^2u_0}{d\tau^2}+\frac{d^2\Delta u_0}{d\tau^2}-\frac{4\rho^2}{\pi^2}[(\Lambda_0^2+2\Lambda_0\cdot\Delta\Lambda_0)(B_1u_0+B_2u_0^3)+\Lambda_0^2(B_1+3B_2u_0^2)\Delta u_0]\\&+\frac{4}{\pi^2}[\Lambda_0\mu_0(C_1+3C_2u_0^2)\Delta u_0+(\Lambda_0\mu_0+\Delta\Lambda_0\mu_0+\Lambda_0\Delta\mu_0)(C_1u_0+C_2u_0^3)] = 0\end{aligned} \tag{23}$$

$$\Delta u_0(0) = \Delta u_0(\pi) = \frac{d\Delta u_0}{d\tau}(0) = \frac{d\Delta u_0}{d\tau}(\pi) = 0 \tag{24}$$

$$\begin{aligned}\int_0^\pi\{&2(\mu_0+\Delta\mu_0)(D_0+D_1u_0^2+D_2u_0^4)+4\mu_0(D_1u_0+2D_2u_0^3)\Delta u_0-\rho^2[(\Lambda_0+\Delta\Lambda_0)(1+F_0+F_1u_0^2+F_2u_0^4)\\&+2\Lambda_0(F_1u_0+2F_2u_0^3)\Delta u_0-2\}d\tau = 0\end{aligned} \tag{25}$$

where $\Delta u_0(\tau)$, a periodic function of period $2\pi, \Delta\mu_0$ and $\Delta\Lambda_0$ are unknown quantities. The second approximate solution can be obtained by solving via the method of harmonic balance the resulting linear Eqs. (23)-(25) in $\Delta u_0(\tau)$, $\Delta\mu_0$ and $\Delta\Lambda_0$.

In view of the expression in Eq. (15), $\Delta u_0(\tau)$ in Eqs. (23)-(25) is taken of the form

$$\Delta u_0(\tau) = z_0(\cos\tau - \cos 3\tau) \tag{26}$$

which satisfies the initial condition in Eq. (24) at the outset. Substituting Eqs. (15) and (26) into Eq. (23), expanding the expression into a trigonometric series and setting the resulting coefficients of the items $\cos\tau$ and $\cos 3\tau$ to zeros, respectively, and similarly, substituting Eqs. (15) and (26) into Eq. (25), and simplifying it yield

$$\begin{gathered} \alpha_1 \times \Delta\mu_0 + \alpha_2 \times \Delta\Lambda_0 + \alpha_3 \times z_0 + \alpha_4 = 0 \\ \beta_1 \times \Delta\mu_0 + \beta_2 \times \Delta\Lambda_0 + \beta_3 \times z_0 + \beta_4 = 0 \\ \gamma_1 \times \Delta\mu_0 + \gamma_2 \times \Delta\Lambda_0 + \gamma_3 \times z_0 + \gamma_4 = 0 \end{gathered} \tag{27}$$

Solving Eq. (27) gives $z_0$, $\Delta\mu_0$ and $\Delta\Lambda_0$

$$\begin{aligned} \Delta\Lambda_0 &= \frac{\alpha_4\beta_3\gamma_1 - \alpha_3\beta_4\gamma_1 - \alpha_4\beta_1\gamma_3 + \alpha_1\beta_4\gamma_3 + \alpha_3\beta_1\gamma_4 - \alpha_1\beta_3\gamma_4}{\alpha_3\beta_2\gamma_1 - \alpha_2\beta_3\gamma_1 - \alpha_3\beta_1\gamma_2 + \alpha_1\beta_3\gamma_2 + \alpha_2\beta_1\gamma_3 - \alpha_1\beta_2\gamma_3} \\ \Delta\mu_0 &= \frac{-\alpha_4\beta_3\gamma_2 + \alpha_3\beta_4\gamma_2 + \alpha_4\beta_3\gamma_3 - \alpha_2\beta_4\gamma_3 - \alpha_3\beta_2\gamma_4 + \alpha_2\beta_3\gamma_4}{\alpha_3\beta_2\gamma_1 - \alpha_2\beta_3\gamma_1 - \alpha_3\beta_1\gamma_2 + \alpha_1\beta_3\gamma_2 + \alpha_2\beta_1\gamma_3 - \alpha_1\beta_2\gamma_3} \\ z_0 &= \frac{-\alpha_4\beta_2\gamma_1 + \alpha_2\beta_4\gamma_1 + \alpha_4\beta_1\gamma_2 - \alpha_1\beta_4\gamma_2 - \alpha_2\beta_1\gamma_4 + \alpha_1\beta_2\gamma_4}{\alpha_3\beta_2\gamma_1 - \alpha_2\beta_3\gamma_1 - \alpha_3\beta_1\gamma_2 + \alpha_1\beta_3\gamma_2 + \alpha_2\beta_1\gamma_3 - \alpha_1\beta_2\gamma_3} \end{aligned} \tag{28}$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4, \gamma_1, \gamma_2, \gamma_3, \gamma_4$ are given in Appendix.

Then we get the second analytical approximation to the post-buckling deformation as

$$\mu_1(a) = \mu_0(a) + \Delta\mu_0(a), \quad \Lambda_1(a) = \Lambda_0(a) + \Delta\Lambda_0(a) \tag{29a}$$

$$u_1(\tau) = \cos\tau + z_0(\cos\tau - \cos 3\tau), \quad \tau \in [0, \pi] \tag{29b}$$

Applying Eq. (10), we obtain the second analytical approximate $\lambda$, and $\varepsilon_{ht}$ as

$$\lambda_1(a) = \sqrt{\Lambda_0(a) + \Delta\Lambda_0(a)}, \quad \varepsilon_{ht1}(a) = \mu_1(a) - 1 \tag{30}$$

and get the second analytical approximate periodic solution $\theta(\tau)$ as

$$\theta_1(\tau) = a[\cos\tau + z_0(\cos\tau - \cos 3\tau)] \tag{31}$$

It should be clear how the procedure works for constructing further analytical approximate solutions. It will be shown in the next section that Eqs. (30)-(31) provide excellent analytical approximations with respect to the "reference" solution obtained by the shooting method for small as well as large $a$.

## 4. Results and discussion

In this section, the accuracy of the proposed analytical approximations will be illustrated by comparing with the “reference” solutions obtained by the shooting method (Ascher *et al*. 1988).

The “reference” solutions $\theta_r(\tau, a), \lambda_r(a)$ and $\varepsilon_{htr}(a)$ can be obtained by first transforming Eqs. (7)-(9) into the following first-order system

$$\frac{d\theta}{d\tau} = \varphi; \quad \frac{d\varphi}{d\tau} = \frac{2\lambda^4\rho^2}{\pi^2}\sin 2\theta - \frac{4\lambda^2(1+\varepsilon_{ht})}{\pi^2}\sin\theta$$

$$\frac{d\chi}{d\tau} = (1 + \varepsilon_{ht} - \lambda^2\rho^2\cos\theta)\cos\theta - 1 \tag{32a}$$

with boundary conditions

$$\theta(0) = a; \quad \varphi(0) = 0; \quad \varphi(\pi) = 0; \quad \chi(0) = 0; \quad \chi(\pi) = 0 \tag{32b}$$

and then applying the shooting method. It should be noted that the “reference” solutions are the same as the elliptic integral solutions derived by Coffin *et al.* (1999) where $\lambda$, and $\varepsilon_{ht}$ are solved for a given value of $K = \sin(a/2)$. Therefore $\theta_r(\tau, a), \lambda_r(a)$ and $\varepsilon_{htr}(a)$ also represent the solutions given by Coffin *et al.* (1999).

For an illustration of the results from the preceding analysis, consider a hygrothermal beam with width $b = 40$ mm and total length $2L = 200$ mm, the corresponding elastic modulus of the beam is $E = 70$ GPa. The dimensionless parameter $\rho$ changes with respect to the thickness $h$, such as $h = 0$ corresponds to the situation of $\rho = 0$, and $h = 35$ mm corresponds to $\rho = 0.1$.

For comparison, the variations of the “reference” and approximate values of $\lambda$, and $\varepsilon_{ht}$ with the angle $a$ of rotation are shown in Figs. 2 and 3, (a) for $\rho = 0$ and (b) for $\rho = 0.1$.

Figs. 2 and 3 indicate that Eq. (30) is able to provide excellent approximate values of $\lambda$ and $\varepsilon_{ht}$ for both small and large $a$, while Eq. (20) may give acceptable results for small $a$. By the way, the dimensionless buckling parameter, $\lambda$, is fairly insensitive to the slenderness ratio of the beam, and

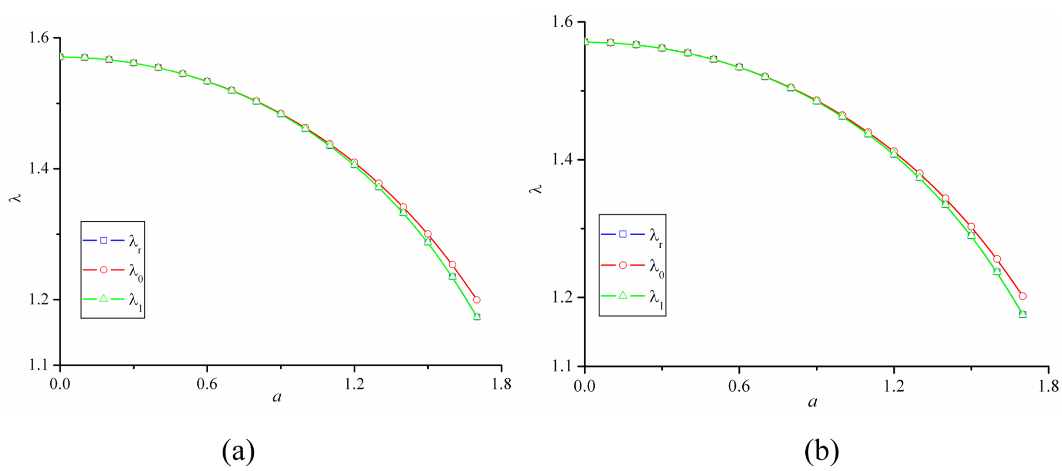


Fig. 2 The variations of the “reference” and approximate values of $\lambda$ with the angle of rotation $a$, (a) for $\rho = 0$ and (b) for $\rho = 0.1$

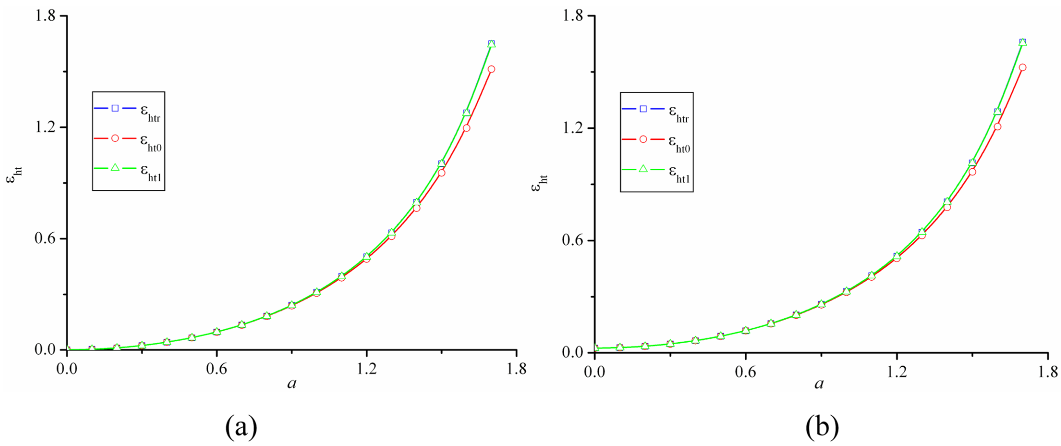


Fig. 3 The variations of the “reference” and approximate values of $\varepsilon_{ht}$ with the angle of rotation $a$, (a) for $\rho = 0$ and (b) for $\rho = 0.1$

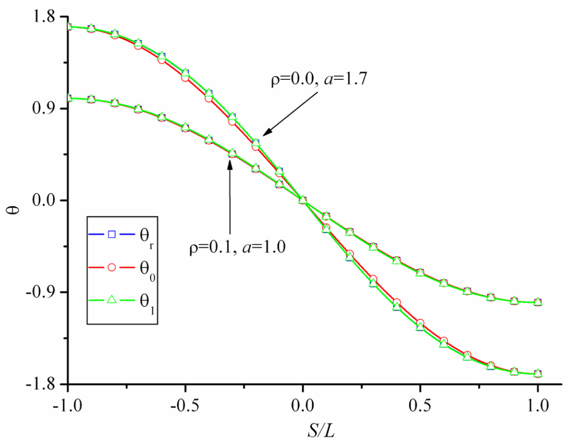


Fig. 4 Comparison of the “reference” and approximate solutions

the prediction for this value obtained for the limiting string case could give a excellent estimate for slender beams.

The “reference” solution $\theta_r$, approximate solutions $\theta_0$ and $\theta_1$ given in Eqs. (21), (31), respectively, are plotted in Fig. 4. These graphs correspond to $\rho = 0.1$, $a = 1.0$ and $\rho = 0$, $a = 1.7$, respectively. Fig. 4 demonstrates that Eq. (31) provides the best approximations with respect to the “reference” solution for both small and large $a$ despite significant increases in nonlinear effects. Note that the angle of rotation at the end of the beam is $a$. For $a = 1.7$, the angle of rotation exceeds $\pi/2$ and it is quite a large post-buckling deformation.

By using Eqs. (5), (6) and the “reference”solution $\theta_r$, approximate solutions $\theta_0$ and $\theta_1$ given in Eqs. (21) and (31), respectively, the “reference” and approximate dimensionless axial and lateral deflection components $x(S)/L$ and $w(S)/L$ of the beam can be determined. Fig. 5 presents the “reference” and approximate post-buckling geometric configurations for $\rho = 0.1$, $a$ = 1.0 and $\rho = 0$,

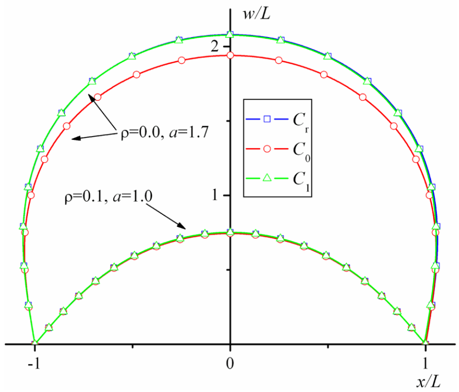


Fig. 5 Comparison of the “reference” and approximate post-buckling geometric configurations

$a$ = 1.7, respectively. The post-buckling geometric configurations obtained by Eq. (31) provide the best approximations to the “reference” configurations for both small and large $a$.

As noticed, since it is assumed that there is no transverse shear deformation, the mechanical model of the beam is valid for slender beam, but not very thick beam.

## 5. Conclusions

In this paper, we present an alternative, accurate approach to solve large post-buckling deformation of an elastic and hygrothermal beam fully restrained against end displacement and subjected to a significant increase in swelling. The solution for the limiting case of a string has also been obtained.

The new approach combines linearization of governing equation and the method of harmonic balance to establish excellent analytical approximate solutions to large post-buckling deformation of the beam in terms of angle of rotation at the end of the beam. At the same time, it is an alternative approach for solving the large post-buckling response problem of a hygrothermal beam without using the Bessel functions. Furthermore, the present analysis demonstrates excellent results as compared to the “reference” solutions for small as well as large angle of rotation at the end of the beam.

The “reference” solutions are less advantageous to the understanding of the physics of nonlinear response because only numerical solutions can be obtained. The alternative method presented here offers analytical approximate solutions which help in understanding of the roles of specific physical parameters in the nonlinear system.

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## Notations

The following symbols are used in the paper:

| | |
|---|---|
| $2L$ | : length of the beam |
| $A$ | : cross-sectional area of beam |
| $I$ | : Moment of inertia |
| $\theta$ | : angle formed by the $x$-axis and the tangent to neutral axis |
| $u$ | : $\theta/a$ |
| $N$ | : force component along the $x$-axis |
| $E$ | : Young's modulus |
| $S$ | : original lengthwise coordinate system of the beam |
| $\varepsilon_{ht}$ | : potential strain for free hygrothermal expansion |
| $\eta$ | : coefficient of thermal expansion |
| $\xi$ | : coefficient of hygroexpansion |
| $\Delta T$ | : change in temperature |
| $\Delta H$ | : change in moisture |
| $\lambda$ | : dimensionless loading parameter |
| $\Lambda$ | : $\lambda^2$ |
| $a$ | : angle of rotation at the end $S = -L$ of the beam |
| $\mu$ | : $\varepsilon_{ht} + 1$ |
| $\rho$ | : dimensionless density parameter of the beam |
| $\tau$ | : $\pi S/(2L) + \pi/2$ a new independent variable |
| $x$, $w$ | : the axial and lateral deflection components of a point on the beam |
| $h_n$ | : coefficient of the Fourier series expansion of $\theta$ |
| $u_i, \Lambda_i, \mu_i$ | : the $(i+1)$th analytical approximation to $u, \Lambda, \mu$ respectively |
| $\Delta u_i, \Delta\Lambda_i, \Delta\mu_i$ | : correction to $u_i, \Lambda_i, \mu_i$, respectively |
| $z_0$ | : coefficient to be determined in the method of harmonic balance |
| $u_r, \Lambda_r, \mu_r$ | : "reference" value of $u, \Lambda, \mu$, respectively |

**Appendix. Expressions of coefficients in Eqs. (12)-(14) and (27)**

$$B_1 = 1 - \frac{a^4}{24} + \frac{a^6}{180} - \frac{a^8}{2880}$$

$$B_2 = -\frac{2a^2}{3} + \frac{a^4}{6} - \frac{a^6}{60} + \frac{a^8}{1080}$$

$$C_1 = 1 - \frac{a^4}{384} + \frac{a^6}{11520} - \frac{a^8}{737280}$$

$$C_2 = -\frac{a^2}{6} + \frac{a^4}{96} - \frac{a^6}{3840} + \frac{a^8}{276480}$$

$$D_0 = 1 - \frac{a^6}{23040} + \frac{a^8}{737280} - \frac{a^{10}}{51609600}$$

$$D_1 = -\frac{a^2}{2} + \frac{a^6}{1280} - \frac{a^8}{46080} + \frac{a^{10}}{3440640}$$

$$D_2 = \frac{a^4}{24} - \frac{a^6}{480} + \frac{a^8}{23040} - \frac{a^{10}}{1935360}$$

$$F_0 = 1 - \frac{a^6}{360} + \frac{a^8}{2880} - \frac{a^{10}}{50400}$$

$$F_1 = -2a^2 + \frac{a^6}{20} - \frac{a^8}{180} + \frac{a^{10}}{3360}$$

$$F_2 = \frac{2a^4}{3} - \frac{2a^6}{15} + \frac{a^8}{90} - \frac{a^{10}}{1890}$$

$$\alpha_1 = \left(4C_1 + 3C_2\right)\Lambda_0$$

$$\alpha_2 = -\left(8B_1 + 6B_2\right)\rho^2\Lambda_0 + \left(4C_1 + 3C_2\right)\mu_0$$

$$\alpha_3 = 2\left(2C_1 + 3C_2\right)\mu_0\Lambda_0 - 2\left(2B_1 + 3B_2\right)\rho^2{\Lambda^2}_0 - \pi^2$$

$$\alpha_4 = \left(4C_1 + 3C_2\right)\mu_0\Lambda_0 - \left(4B_1 + 3B_2\right)\rho^2{\Lambda^2}_0 - \pi^2$$

$$\beta_1 = C_2\Lambda_0$$

$$\beta_2 = -2B_2\rho^2\Lambda_0 + C_2\mu_0$$

$$\beta_3 = -\left(4C_1 + 3C_2\right)\mu_0\Lambda_0 + \left(4B_1 + 3B_2\right)\rho^2{\Lambda^2}_0 + 9\pi^2$$

$$\beta_4 = C_2\mu_0\Lambda_0 - B_2\rho^2{\Lambda^2}_0$$

$$\gamma_1 = 16D_0 + 8D_1 + 6D_2$$

$$\gamma_2 = -\left(8 + 8F_0 + 4F_1 + 3F_2\right)\rho^2$$

$$\gamma_3 = 16\left(D_1 + D_2\right)\mu_0 - 8\left(F_1 + F_2\right)\rho^2\Lambda_0$$

$$\gamma_4 = -\left(8 + 8F_0 + 4F_1 + 3F_2\right)\rho^2\Lambda_0 + \left(16D_0 + 8D_1 + 6D_2\right)\mu_0 - 16$$