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The effect of non-homogeneity on the stability of laminated orthotropic conical shells subjected to hydrostatic pressure

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Abstract. In this study, the stability of laminated homogeneous and non-homogeneous orthotropic truncated conical shells with freely supported edges under a uniform hydrostatic pressure is investigated. It is assumed that the composite material is orthotropic and the material properties depend only on the thickness coordinate. The basic relations, the modified Donnell type stability and compatibility equations have been obtained for laminated non-homogeneous orthotropic truncated conical shells. Applying Galerkin method to the foregoing equations, the expression for the critical hydrostatic pressure is obtained. The appropriate formulas for the single-layer and laminated, cylindrical and complete conical shells made of homogeneous and non-homogeneous, orthotropic and isotropic materials are found as a special case. Finally, effects of non-homogeneity, number and ordering of layers and variations of shell characteristics on the critical hydrostatic pressure are investigated.

Keywords: laminated conical shells; non-homogeneous orthotropic materials; freely supported edges; stability; critical hydrostatic pressure

1. Introduction

Non-homogeneous composites have considerable technical and engineering importance. Therefore analysis of the overall mechanical properties of the non-homogeneous composites must be studied comprehensively. The materials and structural components are often non-homogeneous, because of design, manufacturing process, production techniques, surface and thermal polishing processes or physical composition and imperfections in the underlying material. Thus, the physical properties of materials change from point to point as random, piecewise continuous or continuous functions of coordinates. In an up-to-date survey of literature, authors have come across various models to account for non-homogeneity of the shell material proposed by researchers dealing with the stability and vibration (Babich and Khoroshun 2001, Shen and Noda 2007, Ootao and Tanigawa 2007). By using these models, aspects of the buckling and vibration of single-layer non-homogeneous shells have been examined (Massalas *et al.* 1981, Zhang and Hasebe 1999, Elishakoff 2001, Ding *et al.* 2003, Gupta *et al.* 2010).

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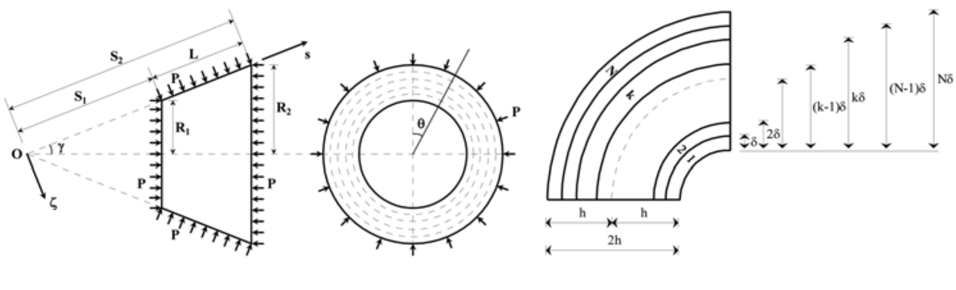
The laminated composite conical shells are being widely used in construction of engineering structures is an important field of current area of research. The conical shells made up of composite materials are one of the important structural elements in a variety of high performance engineering systems including aircraft, submarine, and space structures. Considerable efforts have been made in the past by researchers on the predictions of the dynamic and buckling response of laminated homogeneous composite shells (Wang and Wang 1991, Tong 1999, Wu and Chen 2001, Li *et al.* 2005, Civalek 2005, 2007, Yas and Garmsiri 2010, Patel *et al.* 2011). However, a limited literature is available on the analysis of laminated composite shells with non-homogeneous material properties (Mecitoglu 1996, Zenkour and Fares 2001, Sofiyev 2002, Sofiyev and Schnack 2003, Goldfeld and Arbocz 2004, Li and Batra 2005, Sofiyev *et al.* 2010). Furthermore, studies of the stability and vibration of freely supported (held circular but unrestrained axially) single-layer conical shells are fewer in the literature. Aganesov and Sachenkov (1964) presented the stability and free vibration of freely supported truncated and complete conical shells, by using energy method. Yakushev (1991) studied the stability of freely supported truncated and complete conical shells under dynamic external pressure. Sofiyev and Aksogan (2002) presented the dynamic stability of the freely supported non-homogeneous orthotropic elastic truncated conical shell under the time dependent external pressure. Sofiyev (2009) investigated the stability and vibration of freely supported FGM conical shells under the uniform external pressure.

It is evident from the available literature that the studies on the buckling response of freely supported laminated composite conical shells with non-homogeneous material properties are not dealt by the researchers due to complexity associated with the conical shells. In the present work, an attempt is made to address this problem.

2. Basic relations and equations

Consider a circular conical shell as shown in Fig. 1(a), which is assumed to be thin, laminated and composed of N layers of non-homogeneous orthotropic composite materials. A thickness of the layers is equal, i.e., is $\delta = h_1/N$. Here $h_1 = 2h$ is the total thickness, L is the length of truncated conical shell and γ is the semi-vertex angle of the conical shell. R_1 and R_2 indicate the radii of the cone at its small and large ends, respectively. S_1 and R_2 are the distances from the vertex to the small and large bases, respectively. A set of curvilinear coordinates (ζ, θ, S) is located on the



(a) (b)

Fig. 1 (a) Geometry of a laminated conical shell, (b) cross sectional view of thickness of the laminated composite truncated conical shell reference surface. The ζ-axis is always normal to the moving S-axis, lies in the plane generated by the S-axis and the axis of the cone, and points inwards. The θ-axis is in the direction perpendicular to the $S-\zeta$ plane. Fig. 1(b) shows a cross-sectional configuration of the laminated composite conical shell in thickness direction. The reference surface $\zeta = 0$ is located at a layer interface for even values of N, whereas for odd values of N the reference surface is located at the center of the middle layer.

The Young's moduli and shear modulus of the layers are defined as continuous function of the thickness coordinate ζ as

$$[E_S^{(k+1)}(\bar{\zeta}), E_\theta^{(k+1)}(\bar{\zeta}), G^{(k+1)}(\bar{\zeta})] = \bar{\varphi}_1^{(k+1)}(\bar{\zeta})[E_{0S}^{(k+1)}, E_{0\theta}^{(k+1)}, G_0^{(k+1)}]$$
$$\bar{\zeta} = \zeta/h,\ -1+k\bar{\delta} \le \bar{\zeta} \le -1+(k+1)\bar{\delta};\quad \bar{\delta} = \delta/h;\quad k = 0,1,2,\ldots(N-1) \tag{1}$$

where $E_{0S}^{(k+1)}$ and $E_{0\theta}^{(k+1)}$ are Young's moduli of the layer k+1 along the S and θ directions, respectively, $G_0^{(k+1)}$ is the shear modulus on the plane of the layer k+1. Additionally

$$\bar{\varphi}_1^{(k+1)}(\bar{\zeta}) = 1 + \mu\varphi^{(k+1)}(\bar{\zeta}) \tag{2}$$

where $\varphi^{(k+1)}(\bar{\zeta}), k = 0,1,2,\ldots(N-1)$ is continuous function giving the variation of the Young's moduli in the layers, respectively, satisfying the condition $|\varphi^{(k+1)}(\bar{\zeta})| \le 1$; μ is variation coefficient of the Young's moduli and shear modulus, and satisfying the following inequality $0 \le \mu < 1$.

The stress-strain relations for non-homogeneous orthotropic layers are given as follows (Sofiyev 2009, Volmir 1967, Reddy 2004)

$$\begin{bmatrix} \sigma_S^{(k+1)} \\ \sigma_\theta^{(k+1)} \\ \sigma_{S\theta}^{(k+1)} \end{bmatrix} = \begin{bmatrix} Q_{11}^{(k+1)} & Q_{12}^{(k+1)} & 0 \\ Q_{12}^{(k+1)} & Q_{22}^{(k+1)} & 0 \\ 0 & 0 & Q_{66}^{(k+1)} \end{bmatrix} \begin{bmatrix} \varepsilon_S^0 - \zeta\dfrac{\partial^2 w}{\partial S^2} \\ \varepsilon_\theta^0 - \zeta\left(\dfrac{1}{S^2}\dfrac{\partial^2 w}{\partial\theta_1^2} + \dfrac{1}{S}\dfrac{\partial w}{\partial S}\right) \\ \varepsilon_{S\theta}^0 - \zeta\left(\dfrac{1}{S}\dfrac{\partial^2 w}{\partial S\partial\theta_1} - \dfrac{1}{S^2}\dfrac{\partial w}{\partial\theta_1}\right) \end{bmatrix} \tag{3}$$

where $\sigma_S^{(k+1)}, \sigma_\theta^{(k+1)}$ and $\sigma_{S\theta}^{(k+1)}$ are stresses in the layers; $\varepsilon_S^0, \varepsilon_\theta^0, \varepsilon_{S\theta}^0$ are strain components on the reference surface; $\theta_1 = \theta\sin\gamma$; w is the displacement of the reference surface in the normal direction, positive towards the axis of the cone and assumed to be much smaller than the thickness. The quantities $Q_{ij}^{(k+1)}$, $i,j = 1,2,6$ for orthotropic lamina are

$$Q_{11}^{(k+1)} = \frac{E_{0S}^{(k+1)}\bar{\varphi}_1^{(k+1)}(\bar{\zeta})}{1-\nu_{S\theta}^{(k+1)}\nu_{\theta S}^{(k+1)}},\quad Q_{22}^{(k+1)} = \frac{E_{0\theta}^{(k+1)}\bar{\varphi}_1^{(k+1)}(\bar{\zeta})}{1-\nu_{S\theta}^{(k+1)}\nu_{\theta S}^{(k+1)}},\quad Q_{12}^{(k+1)} = \nu_{\theta S}^{(k+1)}Q_{11}^{(k+1)} = \nu_{S\theta}^{(k+1)}Q_{22}^{(k+1)}$$
$$Q_{66}^{(k+1)} = 2G_0^{(k+1)}\bar{\varphi}_1^{(k+1)}(\bar{\zeta}),\quad k = 0,1,2,\ldots,(N-1) \tag{4}$$

in which $\nu_{S\theta}^{(k+1)}$ and $\nu_{\theta S}^{(k+1)}$ are Poisson's ratios of the layer k+1, assumed to be constant.

The force and moment resultants are expressed by (Volmir 1967)

$$[(N_S, N_\theta, N_{S\theta}), (M_S, M_\theta, M_{S\theta})] = \sum_{k=0}^{N-1} \int_{-h+k\delta}^{-h+(k+1)\delta} (1, \zeta)(\sigma_S^{(k+1)}, \sigma_\theta^{(k+1)}, \sigma_{S\theta}^{(k+1)}) d\zeta \tag{5}$$

The relations between the forces N_S, N_θ and $N_{S\theta}$ and the stress function Ψ are given by

$$(N_S, N_\theta, N_{S\theta}) = \left(\frac{1}{S^2}\frac{\partial^2 \Psi}{\partial \theta_1^2} + \frac{1}{S}\frac{\partial \Psi}{\partial S}, \ \frac{\partial^2 \Psi}{\partial S^2}, \ -\frac{1}{S}\frac{\partial^2 \Psi}{\partial S \partial \theta_1} + \frac{1}{S^2}\frac{\partial \Psi}{\partial \theta_1} \right) \tag{6}$$

The laminated orthotropic truncated conical shell is freely supported and subjected to a uniform hydrostatic pressure, P

$$N_S^0 = -0.5 P S \tan\gamma : \quad N_\theta^0 = -P S \tan\gamma : \quad N_{S\theta}^0 = 0 \tag{7}$$

where N_S^0, N_θ^0 and $N_{S\theta}^0$ are the membrane forces for the condition with zero initial moments.

The modified Donnell type stability and compatibility equations of a laminated truncated conical shell are, respectively, as follows (Volmir 1967)

$$\begin{gathered} \frac{\partial^2 M_S}{\partial S^2} + \frac{2}{S}\frac{\partial M_S}{\partial S} + \frac{2}{S}\frac{\partial^2 M_{S\theta}}{\partial S \partial \theta_1} - \frac{1}{S}\frac{\partial M_\theta}{\partial S} + \frac{2}{S^2}\frac{\partial M_{S\theta}}{\partial \theta_1} + \frac{1}{S^2}\frac{\partial^2 M_\theta}{\partial \theta_1^2} + \\ \frac{N_\theta}{S}\cot\gamma + N_S^0 \frac{\partial^2 w}{\partial S^2} + \frac{N_\theta^0}{S}\left(\frac{1}{S}\frac{\partial^2 w}{\partial \theta_1^2} + \frac{\partial w}{\partial S} \right) + 2 N_{S\theta}^0 \frac{\partial}{\partial S}\left(\frac{1}{S}\frac{\partial w}{\partial \theta_1} \right) = 0 \end{gathered} \tag{8}$$

$$\frac{\cot\gamma}{S}\frac{\partial^2 w}{\partial S^2} - \frac{2}{S}\frac{\partial^2 \varepsilon_{S\theta}^0}{\partial S \partial \theta_1} - \frac{2}{S^2}\frac{\partial \varepsilon_{S\theta}^0}{\partial \theta_1} + \frac{\partial^2 \varepsilon_\theta^0}{\partial S^2} + \frac{1}{S^2}\frac{\partial^2 \varepsilon_S^0}{\partial \theta_1^2} + \frac{2}{S}\frac{\partial \varepsilon_\theta^0}{\partial S} - \frac{1}{S}\frac{\partial \varepsilon_S^0}{\partial S} = 0 \tag{9}$$

Substituting Eq. (3) in Eq. (5) after some rearrangements, the relations found for moments and strains, being substituted into Eqs. (8) and (9) together with relation Eq. (7), then considering the variable $S = S_2 e^x$, after lengthy computations, the modified Donnell type stability and strain compatibility equations of the laminated non-homogeneous orthotropic truncated conical shells can be written in the following form

$$\begin{aligned} L_1(w, \Psi) \equiv{}& \delta_1 e^{-4x} \frac{\partial^4 \Psi}{\partial x^4} + \delta_2 e^{-4x} \frac{\partial^3 \Psi}{\partial x^3} + \delta_3 e^{-4x} \frac{\partial^2 \Psi}{\partial x^2} + \delta_4 e^{-4x} \frac{\partial \Psi}{\partial x} \\ &+ S_2 e^{-3x} \cot\gamma \left(\frac{\partial^2 \Psi}{\partial x^2} - \frac{\partial \Psi}{\partial x} \right) + \delta_5 e^{-4x} \frac{\partial^4 \Psi}{\partial \theta_1^4} + \delta_6 e^{-4x} \frac{\partial^4 \Psi}{\partial x^2 \partial \theta_1^2} \\ &+ \delta_7 e^{-4x} \frac{\partial^3 \Psi}{\partial x \partial \theta_1^2} + \delta_8 e^{-4x} \frac{\partial^2 \Psi}{\partial \theta_1^2} - \delta_9 e^{-4x} \frac{\partial^4 w}{\partial \theta_1^4} - \delta_{10} e^{-4x} \frac{\partial^4 w}{\partial x^2 \partial \theta_1^2} \\ &+ \delta_{11} e^{-4x} \frac{\partial^3 w}{\partial x \partial \theta_1^2} - \delta_{12} e^{-4x} \frac{\partial^2 w}{\partial \theta_1^2} - \delta_{13} e^{-4x} \frac{\partial^4 w}{\partial x^4} + \delta_{14} e^{-4x} \frac{\partial^3 w}{\partial x^3} \\ &+ \delta_{15} e^{-4x} \frac{\partial^2 w}{\partial x^2} + \delta_{16} e^{-4x} \frac{\partial w}{\partial x} - 0.5 P S_2^3 e^{-x} \tan\gamma \left(\frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2} + 2\frac{\partial^2 w}{\partial \theta_1^2} \right) = 0 \end{aligned} \tag{10}$$

$$L_2(w,\Psi) \equiv \Delta_1 e^{-4x}\frac{\partial^4\Psi}{\partial\theta_1^4} + \Delta_2 e^{-4x}\frac{\partial^4\Psi}{\partial x^2\partial\theta_1^2} - \Delta_3 e^{-4x}\frac{\partial^3\Psi}{\partial x\partial\theta_1^2} + \Delta_4 e^{-4x}\frac{\partial^2\Psi}{\partial\theta_1^2}$$
$$+\Delta_5 e^{-4x}\frac{\partial^4\Psi}{\partial x^4} + \Delta_6 e^{-4x}\frac{\partial^3\Psi}{\partial x^3} + \Delta_7 e^{-4x}\frac{\partial^2\Psi}{\partial x^2} + \Delta_8 e^{-4x}\frac{\partial\Psi}{\partial x} - \Delta_9 e^{-4x}\frac{\partial^4 w}{\partial\theta_1^4}$$
$$+\Delta_{10} e^{-4x}\frac{\partial^4 w}{\partial x^2\partial\theta_1^2} + \Delta_{11} e^{-4x}\frac{\partial^3 w}{\partial x\partial\theta_1^2} + \Delta_{12} e^{-4x}\frac{\partial^2 w}{\partial\theta_1^2} - \Delta_{13} e^{-4x}\frac{\partial^4 w}{\partial x^4}$$
$$+\Delta_{14} e^{-4x}\frac{\partial^3 w}{\partial x^3} + \Delta_{15} e^{-4x}\frac{\partial^2 w}{\partial x^2} + \Delta_{16} e^{-4x}\frac{\partial w}{\partial x} + S_2 e^{-3x}\cot\gamma\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial w}{\partial x}\right) = 0 \tag{11}$$

where the expressions δ_j, Δ_j $(j = 1, 2, \ldots, 16)$ are given in Appendix A.

3. Solution of basic equations

For the laminated non-homogeneous orthotropic conical shell, freely supported boundary conditions at both ends are considered and expressed as (Aganesov and Sachenkov 1964, Yakushev 1991, Sofiyev 2009)

$$v = w = M_S = T_S = 0 \quad \text{at} \quad S = S_1(x = -x_0) \quad \text{and} \quad S = S_2(x = 0) \tag{12}$$

where v is displacement in the circumferential direction.

The solution of Eqs. (10) and (11) is sought in the following form (Sofiyev 2009)

$$w = \xi e^{\lambda x}\sin\beta_1 x\cos\beta_2\theta_1, \quad \Psi = \zeta S_2 e^{(\lambda+1)x}\sin\beta_1 x\cos\beta_2\theta_1 \tag{13}$$

where λ is a parameter which is found from the minimum condition of the critical hydrostatic pressure, ξ and ζ are amplitudes and the following definitions apply

$$\beta_1 = \frac{\pi}{x_0}, \quad \beta_2 = \frac{n}{\sin\gamma}, \quad x_0 = \ln\frac{S_2}{S_1}, \quad x = \ln\frac{S}{S_2} \tag{14}$$

Multiplying Eq. (10) by $wS_2^2e^{2x}dxd\theta_1$ and Eq. (11) by $\Psi S_2^2e^{2x}dxd\theta_1$, for $0 \le \theta_1 \le 2\pi\sin\gamma$ and $-x_0 \le x \le 0$, applying the Galerkin method to Eqs. (10) and (11), thus obtained, one gets

$$\int_0^{2\pi\sin\gamma}\int_{-x_0}^{0} L_1(w,\Psi)wS_2^2e^{2x}dxd\theta_1 = 0 \tag{15}$$

$$\int_0^{2\pi\sin\gamma}\int_{-x_0}^{0} L_2(w,\Psi)\Psi S_2^2e^{2x}dxd\theta_1 = 0 \tag{16}$$

Substituting Eq. (13) into the Eqs. (15) and (16) after some rearrangements, for the critical hydrostatic pressure of the laminated non-homogeneous orthotropic truncated conical shells with freely supported edges, the following expression is obtained

$$P_{Hcr}^{TC} = \frac{Q_1Q_5 + Q_2Q_4}{Q_4Q_3} \tag{17}$$

where the expressions $Q_i (i = 1, 2, \ldots, 5)$ are defined as follows

$$Q_1 = [\,\delta_1 A_1 + \delta_2 A_2 + (\delta_3 - \delta_6 \beta_2^2) A_3 + 3\delta_4 + 2\delta_5 \beta_2^4 - 3\delta_7 \beta_2^2 - 2\delta_8 \beta_2^2\,]\theta_{-1} - 2(\lambda^2 + \beta_1^2)\theta_0 S_2 \cot\gamma$$
$$Q_2 = [\,2\delta_9 \beta_2^4 - (\delta_{10}\beta_2^2 + \delta_{15}) A_4 + 2\beta_2^2(\delta_{11} - \delta_{12}) + \delta_{13} A_5 - \delta_{14} A_6 - 2\delta_{16}\,]\theta_{-2}$$
$$Q_3 = [\,(\lambda + 1)\lambda + \beta_1^2 + 0.5 + 2\beta_2^2\,]\theta_1 S_2^3 \tan\gamma$$
$$Q_4 = [\,2\Delta_1 \beta_2^4 - \Delta_2 \beta_2^2 B_1 + 2\Delta_3 \beta_2^2 - 2\Delta_4 \beta_2^2 + \Delta_5 B_2 + \Delta_6 B_3 + \Delta_7 B_1 + 2\Delta_8\,]\theta_0$$
$$Q_5 = \begin{bmatrix} -2\Delta_9 \beta_2^4 + (\Delta_{15} - \Delta_{10}\beta_2^2) B_4 - \Delta_{11}\beta_2^2 - \\ -2\Delta_{12}\beta_2^2 - \Delta_{13} B_5 + \Delta_{14} B_6 + \Delta_{16} \end{bmatrix} \theta_{-1} - 2(\lambda^2 + \beta_1^2)\theta_0 S_2 \cot\gamma \tag{18}$$

in which expressions $A_i, B_i (i = 1, 2, \ldots, 6)$ are defined as follows

$$A_1 = -2[3(\lambda+1)^3(\lambda-1) + 2\beta_1^2(\lambda+1)(\lambda+4) - \beta_1^4];\;\; A_2 = -[\beta_1^2(4\lambda+7) + (\lambda+1)^2(4\lambda-5)]$$
$$A_3 = -2[(\lambda-2)(\lambda+1) + \beta_1^2];\;\; A_4 = -2[(\lambda-2)\lambda + \beta_1^2]$$
$$A_5 = -2[\lambda^3(3\lambda-4) + 2\beta_1^2\lambda(\lambda+2) - \beta_1^4];\;\; A_6 = -[\beta_1^2(4\lambda+2) + \lambda^2(4\lambda-6)]$$
$$B_1 = -2[(\lambda-1)(\lambda+1) + \beta_1^2];\;\; B_2 = -2[(\lambda+1)^3(3\lambda-1) + 2\beta_1^2(\lambda+1)(\lambda+3) - \beta_1^4]$$
$$B_3 = -[\beta_1^2(4\lambda+6) + (\lambda+1)^2(4\lambda-2)];\; B_4 = -2[(\lambda-1)\lambda + \beta_1^2]$$
$$B_5 = -2[\lambda^3(3\lambda-2) + 2\beta_1^2\lambda(3\lambda-2) - \beta_1^4];\;\; B_6 = -[\beta_1^2(4\lambda+1) + \lambda^2(4\lambda-3)]$$
$$\theta_i = \frac{\beta_1^2[1 - e^{-(2\lambda+i)x_0}]}{[(2\lambda+i)^2 + 4\beta_1^2](2\lambda+i)};\; i = -2;\, -1;\, 0;\, 1 \tag{19}$$

Eq. (17) can be used for the study of stability of laminated non-homogeneous orthotropic complete conical and cylindrical shells with freely supported edges;

a) The laminated truncated conical shell is transformed into the laminated complete conical shell when $R_1 \to 0, x_0 \to \infty, \beta_1 \to 0, e^{-ax_0} \to 0, a > 0$ (Aganesov and Sachenkov 1964). In this case, P_{Hcr}^{TC} in Eq. (17) is transformed into P_{Hcr}^{CC}, respectively.

b) If $\gamma = \pi/180000 \to 0, S_1 \to \infty, S_1 \sin\gamma = R, \beta_1 \sin\gamma = \frac{m\pi R}{L} = m_2$, $S_2 = S_1 + L$, $x_0 = \ln\frac{S_2}{S_1} = \ln\left(1 + \frac{L}{S_1}\right) \approx \frac{L}{S_1}$, $e^{-ax_0} \approx 1 - a\frac{L}{S_1}$, $a > 0$ are substituted in Eq. (17) corresponding formula for the laminated cylindrical shell is obtained. In this case, P_{Hcr}^{TC} in Eq. (17) is transformed into P_{Hcr}^{Cyl}, respectively.

c) When $\mu = 0$, from Eq. (17), the expressions for critical hydrostatic pressures are obtained for the homogeneous conical and cylindrical shells.

The minimum values of the critical hydrostatic pressure of the laminated cylindrical, truncated and complete conical shells are obtained by minimizing Eq. (17) with respect to *m*, *n* and *λ*. The values of the parameter, *λ*, changed depending on the geometry of the shell, loading and boundary conditions. After the various numerical computations and analyses for the critical hydrostatic pressure of the freely supported cylindrical, truncated and complete conical shells, the following generalized values are obtained for parameter, λ. The minimum values of the critical hydrostatic pressure of laminated and single-layer cylindrical, truncated and complete conical shells are obtained, at $\lambda = 0$, $\lambda = 2.4$ and $\lambda = 4$, respectively.

In numerical computations part, by taking into account these values for λ and for the longitudinal wave number $m = 1$, the critical hydrostatic pressure are minimized only according to n.

4. Numerical computations and results

4.1 Comparative studies

To check the accuracy of the present work, two comparisons with the results in open literature are made. In the first comparison, the values of the critical uniform hydrostatic pressure of the freely supported orthotropic truncated conical shell are compared with the results of Yakushev (1991), Sofiyev (2009) is given in Table 1. By taking $\mu = 0$; $k = 0$; $E_{0S}^{(k+1)} = E_{0S}$; $E_{0\theta}^{(k+1)} = E_{0\theta}$; $\nu_{S\theta}^{(k+1)} = \nu_{S\theta}$; $\nu_{\theta S}^{(k+1)} = \nu_{\theta S}$ in Eq. (17), the formula for a single-layer homogeneous orthotropic truncated conical shell is obtained. The compared values of the critical hydrostatic pressure, which are shown in Table 1, are taken from Sofiyev (2009). Computations have been carried out for the following data: orthotropic material properties and conical shell characteristics are $E_{0S} = 2.225 \times 10^{10}$ Pa; $E_{0\theta} = 1.085 \times 10^{10}$ Pa; $\nu_{0S} = 0.117$; $\nu_{0\theta} = 0.057$; $h_1 = 1.3 \times 10^{-4}$ m; $R_1 = 0.0225$ m; $R_2 = 0.08$ m (Yakushev 1991, Sofiyev 2009). The values of the critical hydrostatic pressure of orthotropic truncated conical shell of the present study are in good agreement with results of Yakushev (1991), Sofiyev (2009).

Table 1 Comparison of P_{Hcr}^{TC}(MPa) for orthotropic truncated conical shells with results of Yakushev (1991), Sofiyev (2009)

P_{Hcr}^{TC}(MPa), (n_{cr}, $\lambda = 2.4$)			
γ	Yakushev (1991)	Sofiyev (2009)	Present study
10°	0.01110	0.01129(5)	0.01125(5)
30°	0.02636	0.02649(9)	0.02646(9)
50°	0.02582	0.02561(10)	0.02555(10)
70°	0.01230	0.01207(9)	0.01202(9)

Table 2 Comparison of P_{Hcr}^{CC}(MPa) for isotropic complete conical shells with results of Aganesov and Sachenkov (1964), Sofiyev (2009)

P_{Hcr}^{CC}(MPa), (n_{cr}, $\lambda = 4$)			
γ	Aganesov and Sachenkov (1964)	Sofiyev (2009)	Present study
10°	0.28712	0.28832(4)	0.28830(4)
30°	0.66708	0.68102(7)	0.68100(7)
50°	0.63743	0.65217(7)	0.65215(7)
70°	0.29207	0.29411(7)	0.29409(7)

Table 2 shows, the comparison of the critical hydrostatic pressure for homogeneous complete conical shells, with results presented in studies of Aganesov and Sachenkov (1964), Sofiyev (2009). By taking $\mu = 0$; $k = 0$; $E_{0S}^{(k+1)} = E_{0\theta}^{(k+1)}$; $\nu_{S\theta}^{(k+1)} = \nu_{\theta S}^{(k+1)} = \nu_0$, $R_1 \to 0$, $x_0 \to \infty$, $\beta_1 \to 0$, $e^{-ax_0} \to 0$, $a > 0$ in Eq. (17), the formula for a single-layer homogeneous isotropic conical shell is obtained. The compared values of the critical hydrostatic pressure, which are shown in Table 2, are taken from Sofiyev (2009). Computations have been carried out for the following data: material properties are $E_0 = 1.93 \times 10^{11}$ Pa; $\nu_0 = 0.3$ and shell characteristics are $h_1 = 0.00105$ m; $h_1/R_2 = 0.006$, $R_1 = 10^{-50} \approx 0$. It is seen that the values of the critical hydrostatic pressure for the single-layer isotropic complete conical shell of the present study are in good agreement with results of Aganesov and Sachenkov (1964), Sofiyev (2009).

4.2 Selection of non-homogeneous functions and buckling analysis

Numerical computations, for the symmetric cross-ply laminated homogeneous and non-homogeneous truncated conical shells with freely supported edges have been carried out using Eq. (17). The Young's moduli variation function of materials of the layers are assumed to be linear and quadratic functions which, $\varphi^{(k+1)}(\bar{\zeta}) = \bar{\zeta}$ and $\varphi^{(k+1)}(\bar{\zeta}) = \bar{\zeta}^2$. In all tables and figures the variation coefficient is taken into account as $\mu = 0.9$ for the non-homogeneous case. In tables and figures, H and NH are corresponding to the homogeneous and non-homogeneous cases, respectively. The material of lamina is assumed to be graphite/epoxy, with the following orthotropic properties (Han and Simitses 1991): $E_{0S}^{(k+1)} = 21.7 \times 10^6$ psi; $E_{0\theta}^{(k+1)} = 1.44 \times 10^6$ psi; $G_0^{(k+1)} = 0.65 \times 10^6$ psi; $\nu_{S\theta}^{(k+1)} = 0.28$; $k = 0, 1, \ldots, 7$. The following expression is used for percents: $[(NH-H)/H] \times 100\%$. The negative sign in front of the percents show that the values of the critical hydrostatic pressure in non-homogeneity case are smaller than in the homogeneity case.

Table 3 Variations of P_{Hcr}^{TC}(psi), P_{Hcr}^{CC}(psi) and n_{cr} for H and NH orthotropic laminated conical shells with different ordering of layers ($R_1/h_1 = 100$ or $R_1 = 10^{-50} \approx 0$; $R_1/h_1 = 200$; $\gamma = 30^{\circ}$; $\mu = 0.9$)

	P_{Hcr}^{TC}(psi), (n_{cr}, $\lambda = 2.4$)			P_{Hcr}^{CC}(psi), (n_{cr}, $\lambda = 4$)		
Stacking sequence	Homog.	$\bar{\zeta}$	$\bar{\zeta}^2$	Homog.	$\bar{\zeta}$	$\bar{\zeta}^2$
$(0_2/90)_S$	5.788(9)	5.387(9)	7.983(9)	5.868(8)	5.531(8)	8.040(8)
$(0/90/0)_S$	11.928(8)	9.932(8)	15.870(8)	12.690(7)	10.587(7)	16.772(7)
$(0/90_2)_S$	12.045(7)	10.624(7)	16.180(7)	12.733(6)	11.428(7)	17.058(6)
$(90/0_2)_S$	20.716(7)	12.757(7)	31.413(6)	22.453(6)	13.966(7)	34.567(6)
$(90/0/90)_S$	20.537(7)	15.828(7)	30.312(6)	21.975(6)	17.028(6)	33.712(5)
$(90_2/0)_S$	23.555(6)	17.358(7)	33.712(6)	26.387(5)	18.708(6)	37.024(5)
$(0_2/90_2)_S$	8.266(8)	7.531(8)	10.956(8)	8.525(7)	7.898(7)	11.202(7)
$(0/90)_{2S}$	12.889(7)	11.075(8)	17.701(7)	13.798(6)	11.858(7)	18.849(6)
$(0/90_2/0)_S$	14.680(7)	12.438(7)	19.729(7)	15.747(6)	13.482(7)	21.032(6)
$(90/0_2/90)_S$	18.263(7)	14.137(7)	28.609(7)	19.646(6)	15.348(6)	30.650(6)
$(90/0)_{2S}$	20.054(7)	14.742(7)	30.030(6)	21.595(6)	16.023(6)	32.820(6)
$(90_2/0_2)_S$	23.213(6)	15.463(7)	33.587(6)	25.494(6)	16.863(6)	37.500(5)

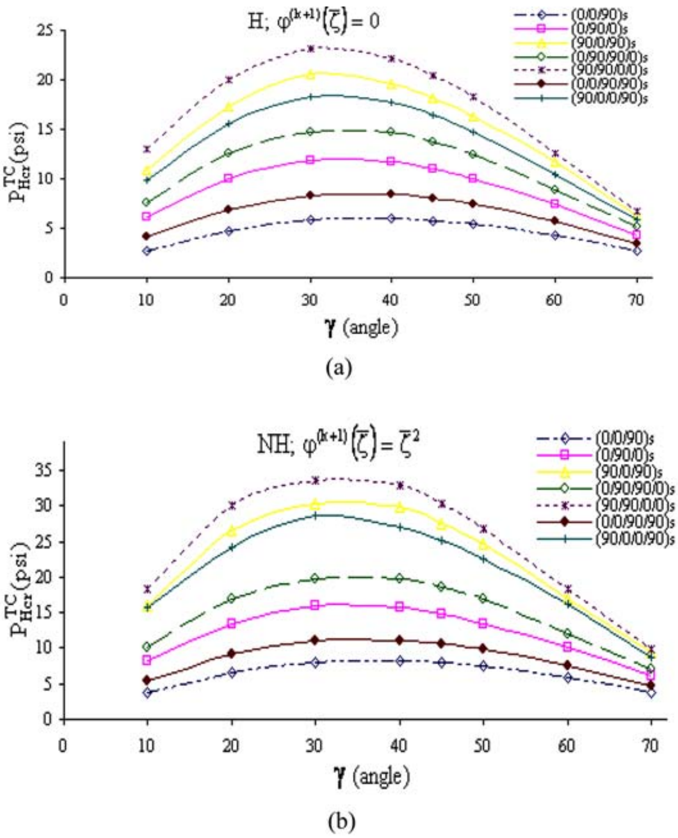


Fig. 2 Variations of P_{Hcr}^{TC}(psi) for (a) H and (b) NH orthotropic laminated truncated conical shells, respectively, with the semi-vertex angle γ for seven ordering of layers ($R_1/h_1 = 100$; $R_2/h_1 = 200$; $\varphi^{(k+1)}(\bar{\zeta}) = \bar{\zeta}^2$; $k = 5$ and 7; $\mu = 0.9$)

In Table 3, variations of the critical hydrostatic pressure and corresponding circumferential wave numbers for the symmetric cross-ply laminated homogeneous and non-homogeneous orthotropic truncated and complete conical shells, in which the Young's moduli of the materials of the layers, graded through the thickness, are described by linear and quadratic functions with different numbers and ordering of layers are presented. The non-homogeneous symmetric laminated orthotropic truncated and complete conical shells are compared with the symmetric laminated fully homogeneous orthotropic truncated and complete conical shells, respectively; for example; the Young's moduli being linear function, the highest effect of non-homogeneity on P_{Hcr}^{TC} and P_{Hcr}^{CC} is (−38%) in a conical shell $(90/0_2)_S$, whereas, the lowest effect are (−6.93%) and (−5.74%), respectively, in a conical shell $(0_2/90)_S$. The Young's moduli being quadratic function, the highest effect of non-homogeneity on P_{Hcr}^{TC} and P_{Hcr}^{CC} is 56% in a conical shell $(90/0_2/90)_S$, whereas, the lowest effect is 33% in conical shells with lamination $(0/90/0)_S$ and $(0_2/90_2)_S$. It is observed that compared to the homogeneous case, the foregoing effect of the variation of material properties in the thickness direction on the critical hydrostatic pressure is more pronounced for the quadratic variation than that for the linear one.

In Fig. 2, variations of P_{Hcr}^{TC}(psi) for the symmetric cross-ply laminated orthotropic truncated

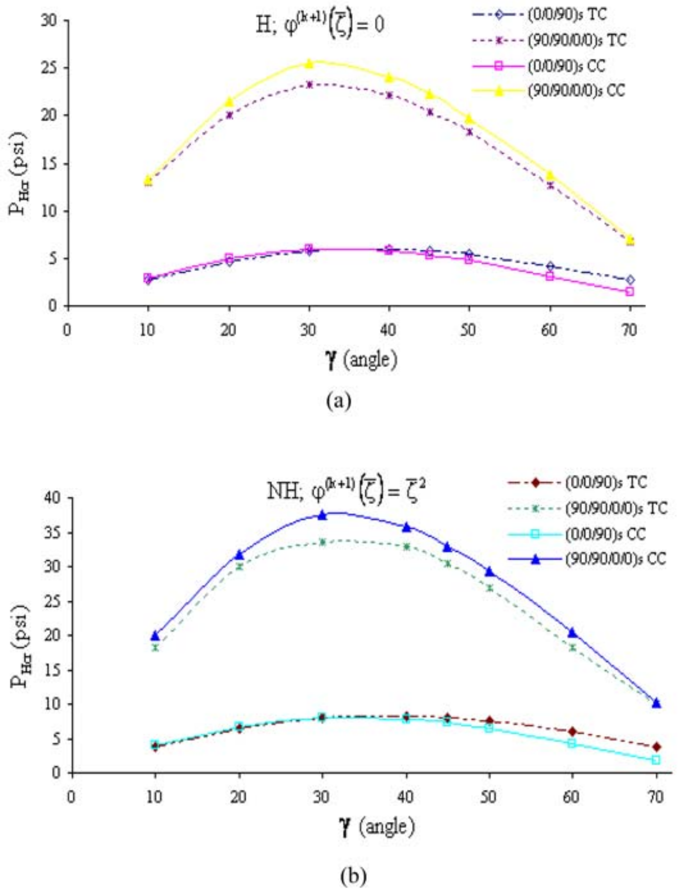


Fig. 3 Variations of P_{Hcr}^{TC}(psi) and P_{Hcr}^{CC}(psi) for (a) *H* and (b) *NH* orthotropic laminated conical shells, respectively, with the semi vertex-angle, γ, for two ordering of layers ($R_1/h_1 = 100$ and $R_1 = 10^{-50} \approx 0$; $R_2/h_1 = 200$; $\varphi^{(k+1)}(\bar{\zeta}) = \bar{\zeta}^2$; $k = 5$ and 7; $\mu = 0.9$)

conical shells made of a) homogeneous and b) non-homogeneous materials with the semi-vertex angle, γ, for seven ordering of layers and the quadratic non-homogeneity profile are presented. As the semi-vertex angle, γ, varies from 10° to 30° the values of P_{Hcr}^{TC}(psi) increase, whereas, $\gamma > 30°$ the values of P_{Hcr}^{TC}(psi) decrease for the homogeneous and non-homogeneous cases. The effect of non-homogeneity is independent from the variation of values of the semi-vertex angle, γ. It is observed that in the homogeneous and non-homogeneous cases, the number and ordering of the layers affect the values of P_{Hcr}^{TC}(psi) appreciably. Variations of the values of P_{Hcr}^{TC}(psi) for *H* and *NH* conical shells with the semi-vertex angle, γ, are similar.

In Fig. 3, variations of the values of critical hydrostatic pressure for the symmetric cross-ply laminated a) homogeneous and b) non-homogeneous orthotropic, truncated and complete conical shells with the semi-vertex angle, γ, for two ordering of layers and the quadratic non-homogeneity profile are presented. The effect of non-homogeneity on the values of P_{Hcr}^{TC}(psi) and P_{Hcr}^{CC}(psi) is considerable. It is seen that the values of P_{Hcr}^{TC}(psi) is lower than the values of P_{Hcr}^{CC}(psi) as $\gamma \le 45^{\circ}$, whereas, this rule is changing with the variation of the ordering of layers as $\gamma > 45^{\circ}$.

Table 4 Variations of $P_{Hcr}^{TC}(\text{psi})$ and n_{cr} for H and NH orthotropic laminated truncated conical shells with R_1/h_1 with different ordering of layers ($L/R_1 = 1$; $\gamma = 30^{\circ}$; $\mu = 0.9$)

	$P_{Hcr}^{TC}(\text{psi})$, (n_{cr}, $\lambda = 2.4$)					
	Hom.			NH, $\varphi^{(k+1)}(\bar{\zeta}) = \bar{\zeta}^2$		
Stacking sequence \ R_1/h_1	100	200	300	100	200	300
$(0_2/90)_S$	14.718(10)	2.602(13)	0.961(14)	20.600(10)	3.612(13)	1.328(14)
$(0/90/0)_S$	27.407(9)	5.155(11)	1.943(12)	36.998(9)	6.891(11)	2.588(12)
$(0/90_2)_S$	27.864(8)	5.227(10)	1.970(12)	37.696(8)	7.025(10)	2.634(12)
$(90/0_2)_S$	43.712(7)	8.688(9)	3.384(11)	65.438(7)	13.130(9)	5.072(10)
$(90/0/90)_S$	43.406(7)	8.549(9)	3.288(10)	64.895(7)	12.935(9)	4.944(10)
$(90_2/0)_S$	49.653(7)	10.011(9)	3.839(10)	72.441(7)	14.429(8)	5.591(10)
$(0_2/90_2)_S$	20.053(9)	3.675(12)	1.359(13)	27.007(9)	4.873(12)	1.802(13)
$(0/90)_{2S}$	29.149(8)	5.539(10)	2.097(12)	40.451(8)	7.643(10)	2.900(12)
$(0/90_2/0)_S$	32.868(8)	6.325(10)	2.414(11)	44.724(8)	8.539(10)	3.249(11)
$(90/0_2/90)_S$	40.147(7)	7.820(9)	2.990(11)	60.301(7)	11.901(9)	4.591(10)
$(90/0)_{2S}$	42.782(7)	8.437(9)	3.273(10)	63.261(7)	12.588(9)	4.846(10)
$(90_2/0_2)_S$	48.053(7)	9.670(9)	3.739(10)	70.929(7)	14.372(9)	5.510(10)

In Table 4 are given variations of the critical hydrostatic pressure and corresponding circumferential wave numbers for symmetric cross-ply laminated homogeneous and non-homogeneous orthotropic conical shells, in which the Young's moduli of materials of the layers vary quadratic in the thickness direction with the ratio, R_1/h_1, for different ordering of layers. As the ratio, R_1/h_1, increases, the values of the critical hydrostatic pressure decrease, whereas, corresponding circumferential wave numbers increase for the homogeneous and non-homogeneous cases. When the ratio, R_1/h_1, increases, the effect of non-homogeneity on the values of critical hydrostatic pressure does not change. It is observed that, in the homogeneous and non-homogeneous cases, the number and ordering of the layers affect the values of the critical hydrostatic pressure appreciably. The effect of non-homogeneity on the critical hydrostatic pressure also changes with the number and ordering of layers.

In Table 5, variations of $P_{Hcr}^{TC}(\text{psi})$ and corresponding circumferential wave numbers for the symmetric cross-ply laminated homogeneous and non-homogeneous orthotropic conical shells, in which the Young's moduli of the materials of the layers vary parabolic in the thickness direction, with the ratio, L/R_1, for different ordering of layers are presented. It can be seen that as the ratio, L/R_1, increases, the values of $P_{Hcr}^{TC}(\text{psi})$ and corresponding circumferential wave numbers degrease for the homogeneous ($\mu = 0$) and non-homogeneous cases. As $1 \leq L/R_1 \leq 5$ and Young's moduli of the materials of the layers is varied parabolic, the effect of non-homogeneity on the critical hydrostatic pressure chances same, for the certain lamination of the conical shell. It is easily observed that, as the number and ordering of the layers change, $P_{Hcr}^{TC}(\text{psi})$ takes the minimum values at different wave numbers for homogeneous and non-homogeneous cases. Thus, it looks as if the minimum values of $P_{Hcr}^{TC}(\text{psi})$ is affected by number and ordering of layers.

Table 5 Variations of P_{Hcr}^{TC}(psi) and n_{cr} for H and NH laminated truncated conical shells with the ratio L/R_1 with different ordering of layers (R_1/h_1 =100; γ = 30°; μ = 0.9)

	P_{Hcr}^{TC}(psi), (n_{cr}, λ = 2.4)					
	Homog.			NH, $\varphi^{(k+1)}(\bar{\zeta}) = \bar{\zeta}^2$		
Stacking sequence \ L/R_1	1	3	5	1	3	5
$(0_2/90)_S$	14.718(10)	3.252(9)	1.458(9)	20.600(10)	4.481(9)	2.008(9)
$(0/90/0)_S$	27.407(9)	6.961(7)	3.208(7)	36.998(9)	9.219(7)	4.243(7)
$(0/90_2)_S$	27.864(8)	6.832(7)	3.097(7)	37.696(8)	9.166(7)	4.166(7)
$(90/0_2)_S$	43.712(7)	12.395(7)	5.758(7)	65.438(7)	18.416(6)	8.629(6)
$(90/0/90)_S$	43.406(7)	11.812(6)	5.446(6)	64.895(7)	17.635(6)	8.136(6)
$(90_2/0)_S$	49.653(7)	13.762(6)	6.367(6)	72.441(7)	19.838(6)	9.152(6)
$(0_2/90_2)_S$	20.053(9)	4.679(8)	2.112(8)	27.007(9)	6.175(8)	2.788(8)
$(0/90)_{2S}$	29.149(8)	7.373(7)	3.376(7)	40.451(8)	10.147(7)	4.652(7)
$(0/90_2/0)_S$	32.868(8)	8.519(7)	3.914(7)	44.724(8)	11.443(7)	5.255(7)
$(90/0_2/90)_S$	40.147(7)	10.811(7)	4.990(7)	60.301(7)	16.604(6)	7.728(6)
$(90/0)_{2S}$	42.782(7)	11.926(6)	5.528(7)	63.261(7)	17.481(6)	8.132(6)
$(90_2/0_2)_S$	48.053(7)	13.571(6)	6.338(6)	70.929(7)	19.774(6)	9.195(6)

5. Conclusions

In this study, the stability of cross-ply laminated non-homogeneous orthotropic shells with freely supported edges under uniform hydrostatic pressure is investigated. At first, the basic equations have been obtained for cross-ply laminated orthotropic conical shells, the Young's moduli of which vary piecewise continuously in the thickness direction. Applying Galerkin method to basic equations, the expression for the critical hydrostatic pressure is obtained.

The numerical results support the following conclusions:

a) The number and ordering of the layers affect the values of the critical hydrostatic pressures appreciably, in homogeneous and non-homogeneous cases.

b) The effect of non-homogeneity on the critical hydrostatic pressure is more pronounced for the quadratic variation than that for the linear one.

c) The effect of non-homogeneity on the critical hydrostatic pressure changes with the number and ordering of the layers.

d) As the number and ordering of the layers change, the critical hydrostatic pressure takes the minimum values at different wave numbers.

e) As the semi-vertex angle increases, the values of the critical hydrostatic pressure increase firstly and then decrease.

f) As the radius to thickness ratio increases, the values of the critical hydrostatic pressure decrease, whereas, corresponding circumferential wave numbers increase for the homogeneous and non-homogeneous cases.

g) As the length to radius ratio increases, the critical hydrostatic pressure and corresponding circumferential wave numbers decrease, for homogeneous and non-homogeneous cases.

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Appendix. A

The expressions δ_j, Δ_j $(j=1,2,\ldots,16)$ are listed below

$$\begin{gathered}\delta_1=c_{12};\ \delta_2=c_{11}-4c_{12}-c_{22};\ \delta_3=5c_{12}+3c_{22}-3c_{11}-c_{21};\ \delta_4=2(c_{11}-c_{22}-c_{12}+c_{21})\\ \delta_5=c_{21};\ \delta_6=c_{11}-2c_{31}+c_{22};\ \delta_7=4c_{31}-3c_{11}-c_{22};\ \delta_8=2(c_{11}-c_{31}+c_{21});\ \delta_9=c_{24}\\ \delta_{10}=c_{14}+c_{23}+2c_{32};\ \delta_{11}=3c_{14}+c_{23}+4c_{32};\ \delta_{12}=2(c_{14}+c_{32}+c_{24});\ \delta_{13}=c_{13}\\ \delta_{14}=c_{23}-c_{14}+4c_{13};\ \delta_{15}=c_{24}-3c_{23}+3c_{14}-5c_{13};\ \delta_{16}=2(c_{23}-c_{14}-c_{24}+c_{13})\end{gathered}\tag{A1}$$

$$\begin{gathered}\Delta_1=b_{11};\Delta_2=2b_{31}+b_{21}+b_{12};\Delta_3=4b_{31}+3b_{21}+b_{12};\Delta_4=2(b_{31}+b_{21}+b_{11});\Delta_5=b_{22}\\ \Delta_6=b_{21}-4b_{22}-b_{12};\Delta_7=5b_{22}+3b_{12}-b_{11}-3b_{21};\Delta_8=2(b_{21}-b_{22}-b_{12}+b_{11})\\ \Delta_9=b_{14};\Delta_{10}=2b_{32}-b_{13}-b_{24};\Delta_{11}=b_{13}+3b_{24}-4b_{32};\Delta_{12}=2(b_{32}-b_{24}-b_{14});\Delta_{13}=b_{23}\\ \Delta_{14}=b_{13}-b_{24}+4b_{23};\Delta_{15}=b_{14}-3b_{13}+3b_{24}-5b_{23};\ \Delta_{16}=2(b_{13}-b_{24}+b_{23}-b_{14})\end{gathered}\tag{A2}$$

where

$$\begin{gathered}c_{11}=A_{11}^1b_{11}+A_{12}^1b_{21},\ c_{12}=A_{11}^1b_{12}+A_{12}^1b_{22},\ c_{13}=A_{11}^1b_{13}+A_{12}^1b_{23}+A_{11}^2\\ c_{14}=A_{11}^1b_{14}+A_{12}^1b_{24}+A_{12}^2,\ c_{21}=A_{21}^1b_{11}+A_{22}^1b_{21},\quad c_{22}=A_{21}^1b_{12}+A_{22}^1b_{22}\\ c_{23}=A_{21}^1b_{13}+A_{22}^1b_{14}+A_{21}^2,\ c_{24}=A_{21}^1b_{14}+A_{22}^1b_{13}+A_{22}^2,\ \ c_{31}=A_{66}^1b_{31}\\ c_{32}=A_{66}^1b_{32}+A_{66}^2,\ b_{11}=A_{22}^0L_0^{-1},\ b_{12}=-A_{12}^0L_0^{-1},b_{13}=(A_{12}^0A_{21}^1-A_{11}^1A_{22}^0)L_0^{-1}\\ b_{14}=(A_{12}^0A_{22}^1-A_{12}^1A_{22}^0)L_0^{-1},\ b_{21}=-A_{21}^0L_0^{-1},\ b_{22}=A_{11}^0L_0^{-1},\ b_{23}=(A_{21}^0A_{11}^1-A_{21}^1A_{11}^0)L_0^{-1}\\ b_{24}=(A_{21}^0A_{12}^1-A_{22}^1A_{11}^0)L_0^{-1},\ b_{31}=1/A_{66}^0,\ b_{32}=-A_{66}^1/A_{66}^0,\ L_0=A_{11}^0A_{22}^0-A_{12}^0A_{21}^0\end{gathered}\tag{A3}$$

in which

$$\begin{gathered}A_{11}^{k_1}=h^{k_1+1}\sum_{k=0}^{N-1}\frac{E_{0S}^{(k+1)}\hbar^{(k+1)}}{1-\nu_{S\theta}^{(k+1)}\nu_{\theta S}^{(k+1)}},\quad A_{12}^{k_1}=h^{k_1+1}\sum_{k=0}^{N-1}\frac{\nu_{\theta S}^{(k+1)}E_{0S}^{(k+1)}\hbar^{(k+1)}}{1-\nu_{S\theta}^{(k+1)}\nu_{\theta S}^{(k+1)}}\\ A_{22}^{k_1}=h^{k_1+1}\sum_{k=0}^{N-1}\frac{E_{0\theta}^{(k+1)}\hbar^{(k+1)}}{1-\nu_{S\theta}^{(k+1)}\nu_{\theta S}^{(k+1)}},\quad A_{21}^{k_1}=h^{k_1+1}\sum_{k=0}^{N-1}\frac{\nu_{S\theta}^{(k+1)}E_{0\theta}^{(k+1)}\hbar^{(k+1)}}{1-\nu_{S\theta}^{(k+1)}\nu_{\theta S}^{(k+1)}}\\ A_{66}^{k_1}=2h^{k_1+1}\sum_{k=0}^{N-1}G_0^{(k+1)}\hbar^{(k+1)},\ \hbar^{(k+1)}=\int_{-1+2k/N}^{-1+2(k+1)/N}\bar{\zeta}^{k_1}\bar{\phi}_1^{(k+1)}(\bar{\zeta})d\bar{\zeta}\end{gathered}\tag{A4}$$