Structural Engineering and Mechanics, Vol. 41, No. 5 (2012) 605-615 605
DOI: http://dx.doi.org/10.12989/sem.2012.41.5.605

Guided viscoelastic wave in circumferential direction of orthotropic cylindrical curved plates

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(Received January 20, 2011, Revised January 25, 2012, Accepted February 1, 2012)

Abstract. In this paper, guided circumferential wave propagating in an orthotropic viscoelastic cylindrical curved plate subjected to traction-free conditions is investigated in the frame of the Kelvin-Voight viscoelastic theory. The obtained three wave equations are decoupled into two groups, Lamb-like wave and SH wave. They are separately solved by the Legendre polynomial series approach. The availability of the method is confirmed through the comparison with the published data of the SH wave for a viscoelastic flat plate. The dispersion curves and attenuation curves for the carbon fiber and prepreg cylindrical plates are illustrated and the viscous effect on dispersion curves is shown. The influences of the ratio of radius to thickness are analyzed.

Keywords: viscoelastic materials; guided circumferential wave; Legendre polynomial series; dispersion curves; Attenuation curves

1. Introduction

Circumferential wave, because its sensitivity to the longitudinal stress-corrosion cracks in cylindrical curved plates (or large-diameter hollow cylinders), has received many researchers' attention. As early as in 1966, Grace and Goodman (1996) investigated the circumferential wave traveling around a solid cylinder immersed in water. Liu and Qu (1998) studied guided circumferential waves in a circular annulus Valle *et al.* (1999) concerned the circumferential waves in layered cylinders. Zhao and Rose (2004) studied the guided circumferential SH wave characteristics in isotropic hollow cylinders. Towfighi *et al.* (2002) and Yu *et al.* (2007) studied the anisotropic cylindrical curved plates. Sharma and Pathania (2005) and Yu *et al.* (2010) investigated the circumferential generalized thermoelastic wave in cylindrical plates. Circumferential wave in coupled electro-elastic (Yu and Ma 2008) and magneto-electro-elastic (Yu and Wu 2009) structures were also discussed.

Over the past decades, the attenuation caused by viscoelasticity had puzzled researchers who embarked on ultrasonic nondestructive testing (NDT). Because the viscoelastic material constants are complex and frequency dependent, the governing differential equations for guided wave propagation in elastic and viscoelastic structures are similar. But solving the frequency-dependent differential equations with complex coefficients is difficult, especially for anisotropic materials and

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curved structures. By the spectral finite-element method, Shorter (2004) investigated wave propagation and damping in linear viscoelastic laminates. Simonetti (2004) studied Lamb wave characteristics in an elastic plate coated with viscoelastic materials layer. Barshinger and Rose (2004), Luo and Rose (2004) investigated the guided wave in elastic hollow cylinders coated with viscoelastic layers. Luo *et al.* (2006) used the cirrunferential waves for defect detection in viscoelastic coated pipes. The structural materials considered in the above work were all assumed isotropic. Castaings and Hosten (2003) investigated the guided wave propagating in anisotropic and viscoelastic sandwich structures. But the viscous coefficients considered in this paper are frequency independent. In this article, the propagation of guided cirrunferential waves in viscoelastic cylindrical curved plates is investigated in the frame of the Kelvin-Voight viscoelastic theory. The dispersion curves, attenuation curves for both the cirrunferential Lamb-like wave and SH wave are illustrated. The influences of the ratio of radius to thickness are discussed.

2. Theoretical formulation

Consider an orthotropic, viscoelastic cylindrical curved plate which is infinite axially with a thickness h, as shown in Fig. 1. In the cylindrical coordinate system (θ, z, r), a, b are the inner and outer radii respectively.

The orthotropic viscoelastic constitutive equation can be written in the following form

$$\begin{aligned}
T_{\theta\theta} &= C_{11}^*\varepsilon_{\theta\theta} + C_{12}^*\varepsilon_{zz} + C_{13}^*\varepsilon_{rr} \\
T_{zz} &= C_{12}^*\varepsilon_{\theta\theta} + C_{22}^*\varepsilon_{zz} + C_{23}^*\varepsilon_{rr} \\
T_{rr} &= C_{13}^*\varepsilon_{\theta\theta} + C_{23}^*\varepsilon_{zz} + C_{33}^*\varepsilon_{rr} \qquad \text{with } C_{ij}^* = C_{ij} + i\omega\mu_{ij} \\
T_{rz} &= 2C_{44}^*\varepsilon_{rz} \\
T_{r\theta} &= 2C_{55}^*\varepsilon_{r\theta} \\
T_{\theta z} &= 2C_{66}^*\varepsilon_{\theta z}
\end{aligned} \tag{1}$$

where T_{ij}, ε_{ij} are the stress and strain; C_{ij}, μ_{ij} are the elastic and viscos coefficients, and ω is the angular frequency.

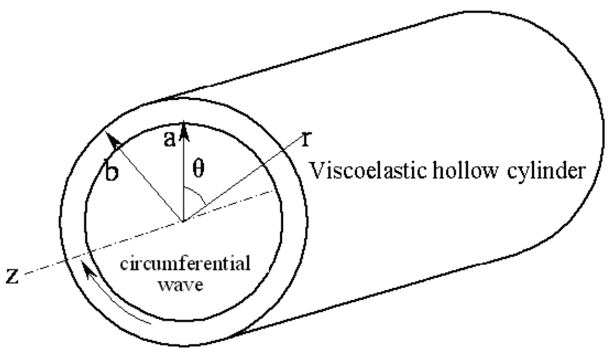


Fig. 1 The scheme of circumferential wave in a viscoelastic cylindrical curved plate

The relationship between the strain and displacement can be expressed as

$$\varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r},\quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z},\quad \varepsilon_{rr} = \frac{\partial u_r}{\partial r},$$

$$\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right),\quad \varepsilon_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right),\quad \varepsilon_{\theta z} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{\partial u_z}{r\partial \theta}\right) \tag{2}$$

where u_i is the elastic displacements.

The dynamic equation for the viscoelastic cylindrical curved plate in the absence of body forces is governed by

$$\begin{aligned} \frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta} + \frac{\partial T_{rz}}{\partial z} + \frac{T_{rr} - T_{\theta\theta}}{r} &= \rho\frac{\partial^2 u_r}{\partial t^2} \\ \frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{\partial T_{\theta z}}{\partial z} + \frac{2T_{r\theta}}{r} &= \rho\frac{\partial^2 u_\theta}{\partial t^2} \\ \frac{\partial T_{rz}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta z}}{\partial \theta} + \frac{\partial T_{zz}}{\partial z} + \frac{T_{rz}}{r} &= \rho\frac{\partial^2 u_z}{\partial t^2} \end{aligned} \tag{3}$$

By introducing the rectangular window function $\pi(r)$

$$\pi(r) = \begin{cases} 1, & a \le r \le b \\ 0, & elsewhere \end{cases}$$

the stress-free boundary conditions ($T_{rr} = T_{r\theta} = T_{rz} = 0$ at $r = a, r = b$) are automatically incorporated in the constitutive relations (Elmaimouni *et al.* 2005)

$$\begin{aligned} T_{\theta\theta} &= C_{11}^*\varepsilon_{\theta\theta} + C_{12}^*\varepsilon_{zz} + C_{13}^*\varepsilon_{rr} \\ T_{zz} &= C_{12}^*\varepsilon_{\theta\theta} + C_{22}^*\varepsilon_{zz} + C_{23}^*\varepsilon_{rr} \\ T_{rr} &= (C_{13}^*\varepsilon_{\theta\theta} + C_{23}^*\varepsilon_{zz} + C_{33}^*\varepsilon_{rr})\pi(r) \\ T_{rz} &= 2C_{44}^*\varepsilon_{rz}\pi(r) \\ T_{r\theta} &= 2C_{55}^*\varepsilon_{r\theta}\pi(r) \\ T_{\theta z} &= 2C_{66}^*\varepsilon_{\theta z} \end{aligned} \tag{4}$$

For a free harmonic wave propagating in the circumferential direction in a circular cylinder of infinite length, we assume the displacement components, temperature change to be of the form

$$u_r(r,\theta,z,t) = \exp(ikb\theta - i\omega t)U(r) \tag{5a}$$

$$u_\theta(r,\theta,z,t) = \exp(ikb\theta - i\omega t)V(r) \tag{5b}$$

$$u_z(r,\theta,z,t) = \exp(ikb\theta - i\omega t)W(r) \tag{5c}$$

$$T(r,\theta,z,t) = \exp(ikb\theta - i\omega t)X(r) \tag{5d}$$

$U(r)$, $V(r)$ and $W(r)$ represent the amplitude of vibration in the r, θ, z directions respectively. k is the magnitude of the wave vector in the propagation direction, and ω is the angular frequency.

Substituting Eqs. (2), (4), (5) into Eq. (3), we obtain the governing differential equations in terms of displacement components

$$[r^2C_{33}^*U'' + rC_{33}^*U' + rikb(C_{13}^* + C_{55}^*)V' - (k^2b^2C_{55}^* + C_{11}^*)U - ikb(C_{55}^* + C_{11}^*)V]\pi(r) \\ +(\delta(r-a) - \delta(r-b))[r^2C_{33}^*U' + r(C_{13}^*U + ikbC_{13}^*V)] = -\rho r^2\omega^2 U \quad (6a)$$

$$[r^2C_{55}^*V'' + rikb(C_{13}^* + C_{55}^*)U' + rC_{55}^*V' + ikb(C_{55}^* + C_{11}^*)U - (C_{55}^* + k^2b^2C_{11}^*)V]\pi(r) \\ +(\delta(r-a) - \delta(r-b))(r^2C_{55}^*V' + rikbC_{55}^*U - rC_{55}^*V) = -\rho r^2\omega^2 V \quad (6b)$$

$$(r^2C_{44}^*W'' + rC_{44}^*W' - k^2b^2C_{66}^*W)\pi(r) + (\delta(r-a) - \delta(r-b))r^2C_{44}^*W' = -\rho r^2\omega^2 W \quad (6c)$$

Here, Eq. (6c) is independent of the other two equations. In fact, Eq. (6c) represents the propagating circumferential SH waves while Eqs. (6a) and (6b) the propagating circumferential Lamb-like waves.

To obtain the solutions of the viscoelastic circumferential Lamb-like waves and SH waves, we expand $U(z)$, $W(z)$ and $V(z)$ to Legendre orthogonal polynomial series

$$U(r) = \sum_{m=0}^{\infty} p_m^{(1)} Q_m(r), \; V(r) = \sum_{m=0}^{\infty} p_m^{(2)} Q_m(r), \; W(r) = \sum_{m=0}^{\infty} p_m^{(3)} Q_m(r) \quad (7)$$

Here $Q_m(r) = \sqrt{\frac{2m+1}{(b-a)}} P_m\left(\frac{2r-(b+a)}{(b-a)}\right)$ is the expansion coefficients and with P_m being the mth Legendre polynomial. Theoretically, m runs from 0 to ∞. In practice, the summation over the polynomials in Eq. (7) can be truncated at some finite value $m = M$, when higher order terms become essentially negligible.

Eqs. (6a), (6b) and (6c) are multiplied by $Q_j(r)$ with j running from 0 to M, and then integrated over r from a to b, which gives the following $3(M+1)$ equations

$$A_{11}^{j,m} p_m^{(1)} + A_{12}^{j,m} p_m^{(2)} = 0 \quad (8a)$$

$$A_{21}^{j,m} p_m^{(1)} + A_{22}^{j,m} p_m^{(2)} = 0 \quad (8b)$$

$$A_{33}^{j,m} p_m^{(3)} = 0 \quad (8c)$$

where summation over repeated index m is implied with m ranging from 0 to M. $A_{\alpha\beta}^{j,m}$ $(\alpha, \beta = 1, 2, 3)$ are the elements of a non-symmetric matrix. They can be obtained according to Eq. (6).

The non-zero solutions of Eqs. (8a), (8b) and Eq. (8c) can only exist when the determinant of the coefficient of $p_m^{(i)}$ $(i = 1, 2, 3)$ equals zero, which yields

$$f_1(k, \omega) = \begin{vmatrix} {}^{l}A_{11}^{j,m} & {}^{l}A_{12}^{j,m} \\ {}^{l}A_{21}^{j,m} & {}^{l}A_{22}^{j,m} \end{vmatrix} = 0 \quad (9a)$$

$$f_2(k, \omega) = \left| {}^{l}A_{33}^{j,m} \right| = 0 \quad (9b)$$

So, the Lamb-like wave dispersion curves and attenuation curves can be obtained by numerically solving Eq. (9a) and SH wave by Eq. (9b). Here, the Newton downhill method is used to solve the nonlinear equation with complex coefficients. Because Eqs. (9) are two nonlinear complex coefficient equations about frequency ω and wave number k, we keep ω real and let k to be complex. Then, the phase velocity is defined as $c = \omega/\mathrm{Re}(k)$, and the imaginary part of the k is a measure of the attenuation. The obtained solutions are asymptotic. So, not all solutions can be accepted. We determine that the solutions obtained are converged solutions when a further increase in the matrix dimension or M does not result in a significant change. The computer program was written using Mathmatica.

3. Numerical results

3.1 Comparison with the available data

Because there is not solution for the viscoelastic wave in cylindrical curved plates so far, we calculate Lamb-like wave dispersion curves for a pure elastic cylindrical plate with $\eta = 10$ to make a comparison with the available data (Fig. 15(b) in the paper of Towfighi *et al.* 2002), as shown in Fig. 2. Here, we define the ratio of outer radius to thickness is η. We also calculate SH wave dispersion and attenuation curves for a viscoelastic orthotropic cylindrical plate with $\eta = 100$ to make a comparison with the available data on a viscoelastic orthotropic flat plate (Figs. 18-5 and 18-6(b) in the book of Rose 1999), as shown in Fig. 3 (As is well known, the wave characteristics for a flat plate are almost the same to those for a cylindrical plate with very large ratio of radius to thickness.). The used material properties can be seen in the two references. It can be seen that our solutions agree well with the published data for both dispersion and attenuation curves.

3.2 Wave characteristics in viscoelastic cylindrical curved plates

This section analyzes two cylindrical curved plates with $\eta = 10$. The materials are prepreg and carbon fiber, respectively. Their material properties are listed in Table 1. Obtained dispersion curves

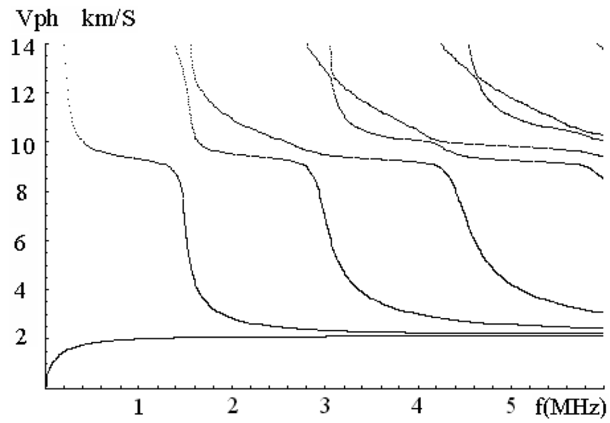
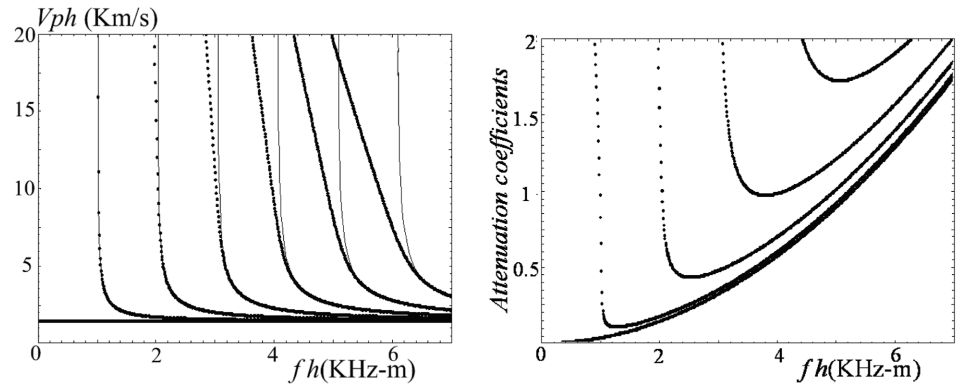


Fig. 2 Circumferential Lamb- like wave dispersion curves for a unidirectional composite cylindrical curved plate

Table 1 Material properties of the two homogeneous materials

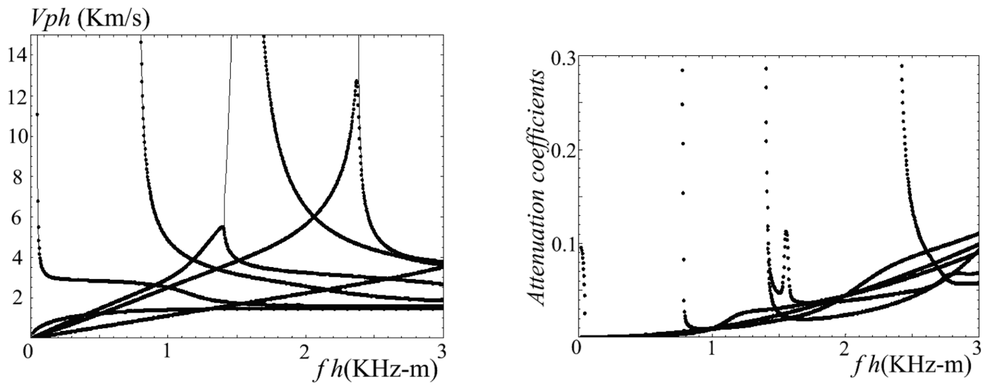
Property	C_{11}	C_{13}	C_{33}	C_{44}	C_{55}	C_{66}	ρ
prepreg	15	7.7	16	7.8	7.8	3.9	1.595
carbon fiber	12.1	5.5	12.3	6.15	6.15	3.32	1.5
	μ_{11}	μ_{13}	μ_{33}	μ_{44}	μ_{55}	μ_{66}	
prepreg	0.014	0.0064	0.011	0.0042	0.0042	0.0034	
carbon fiber	0.043	0.021	0.037	0.02	0.02	0.009	

Units: C_{ij} (GPa), μ_{ij} (GPa·ms), ρ (g/cm^3).



(a) (b)

Fig. 3 Circumferential SH wave characteristic curves for a cylindrical curved plate of η = 100; (a) phase velocity dispersion curves, (b) attenuation curves; the solid line in (a) is the dispersion curves for the pure elastic cylindrical curved plate



(a) (b)

Fig. 4 Circumferential Lamb- like wave characteristic curves for the prepreg cylindrical curved plate of η = 10; (a) phase velocity dispersion curves, (b) attenuation curves; the solid line in (a) is the dispersion curves for the pure elastic cylindrical curved plate

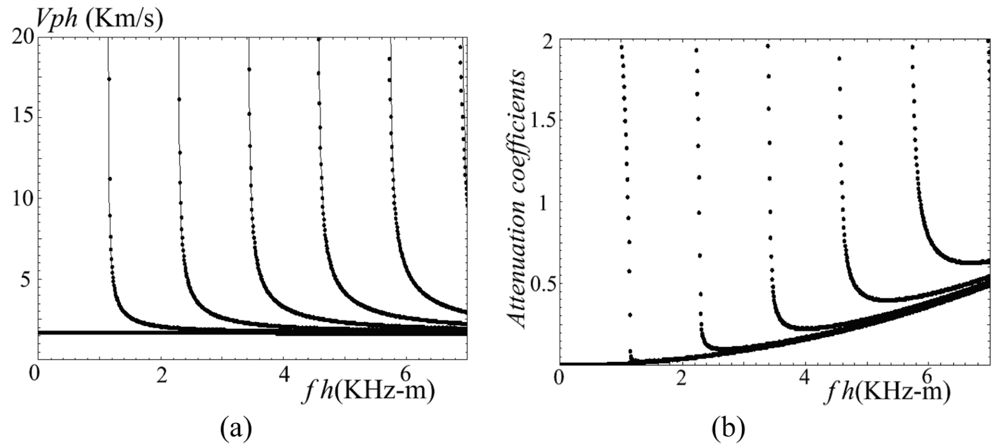


Fig. 5 Circumferential SH wave characteristic curves for the prepreg cylindrical curved plate of η = 10; (a) phase velocity dispersion curves, (b) attenuation curves; the solid line in (a) is the dispersion curves for the pure elastic cylindrical curved plate

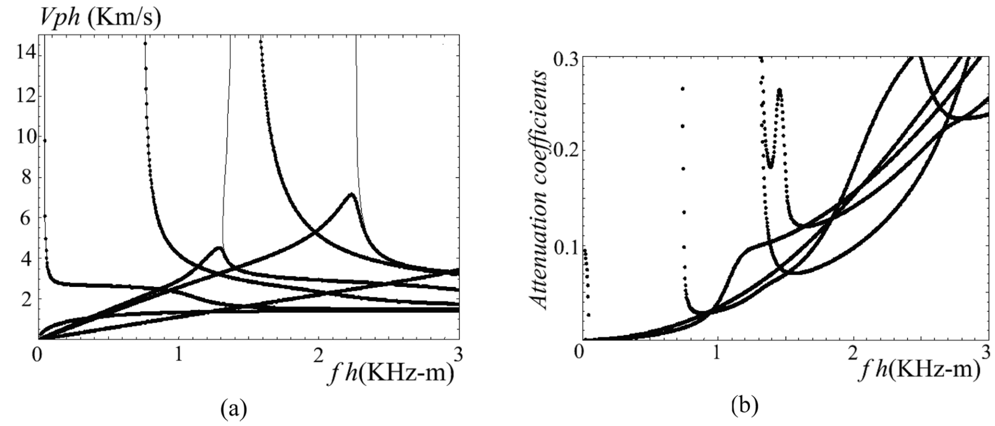


Fig. 6 Circumferential Lamb- like wave characteristic curves for the carbon fiber cylindrical curved plate of η = 10; (a) phase velocity dispersion curves, (b) attenuation curves; the solid line in (a) is the dispersion curves for the pure elastic cylindrical curved plate

and attenuation curves are shown in Figs. 4-7.

Above all, the common characteristics for the two cylindrical curved plates are discussed. For circumferential Lamb-like waves, the viscous effect on the dispersion curves mainly occurs on quasi-longitudinal wave modes at the lower frequencies. For circumferential SH waves, the viscous effect on the dispersion curves also mainly occurs at the lower frequencies, and becomes stronger with the increase of the modal order. For the attenuation curves, there are common points for Lamb-like waves and SH waves. The attenuation of lower order modes is smaller than that of higher order modes. The attenuation of each mode is higher at low frequencies. As the frequency increases, the attenuation of each mode becomes small until it reaches a minimum at a certain frequency. Exceeding the minimum attenuation frequency, the attenuation becomes higher. The minimum attenuation frequency should be chiefly considered for the guided ultrasonic NDT for viscoelastic

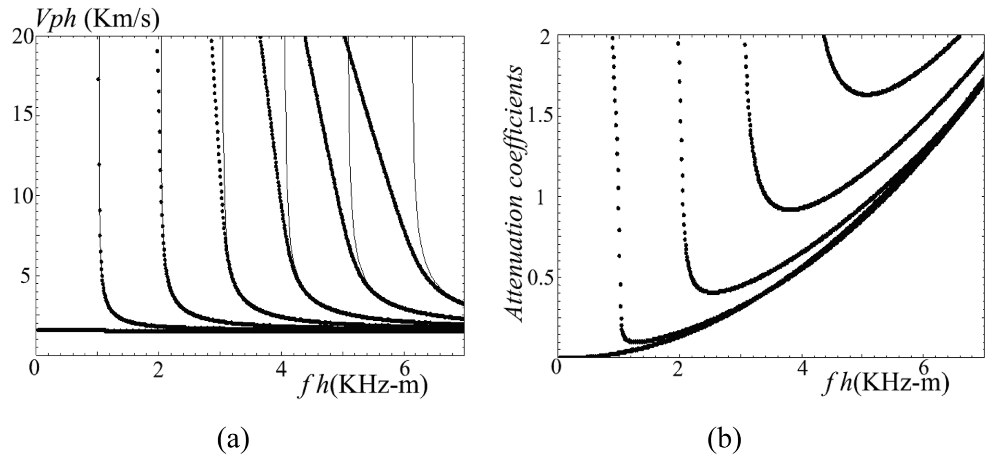


Fig. 7 Circumferential SH wave characteristic curves for the carbon fiber cylindrical curved plate of η = 10; (a) phase velocity dispersion curves, (b) attenuation curves; the solid line in (a) is the dispersion curves for the pure elastic cylindrical curved plate

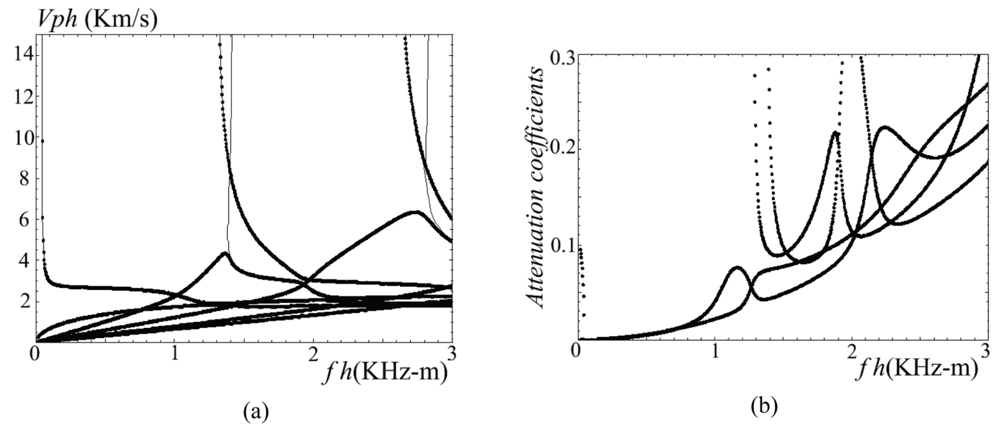


Fig. 8 Circumferential Lamb- like wave characteristic curves for the carbon fiber cylindrical curved plate of changed parameter; (a) phase velocity dispersion curves, (b) attenuation curves; the solid line in (a) is the dispersion curves for the pure elastic cylindrical curved plate

waveguides. The differences between the two waves lie in that the attenuation curves of Lamb-like waves are more offbeat than those of SH waves.

The differences of the two cylindrical curved plates lie in that the viscous effect on the dispersion curves for the prepreg plate is weaker than that for the carbon fiber plate and that the attenuation for the prepreg plate is smaller than that for the carbon fiber plate. The reason is that the viscous coefficients of carbon fiber are about three times of prepreg.

From Figs. 4(a) and 6(a), the viscous effect on the Lamb-like wave dispersion curves mainly occurs on quasi-longitudinal wave modes. According to Eq. (1), quasi-longitudinal wave are mainly determined by C_{11}^{*} and C_{33}^{*}; quasi-transverse wave are mainly determined by C_{55}^{*}. The viscous coefficient of C_{55}^{*} is about half of C_{11}^{*} and C_{33}^{*}. We enlarge C_{55}^{*} to its triple, and than observe its Lamb-like dispersion and attenuation curves, as shown in Fig. 8. Obviously, the viscous effect on

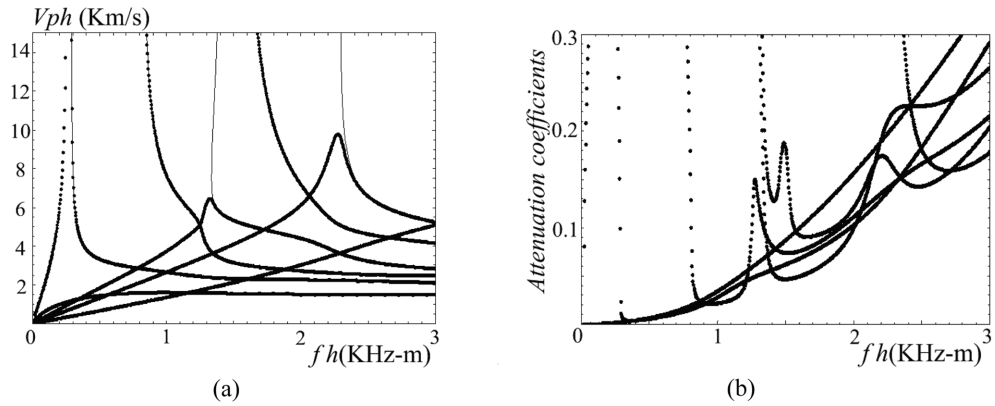


Fig. 9 Circumferential Lamb- like wave characteristic curves for the carbon fiber cylindrical curved plate of $\eta = 2$; (a) phase velocity dispersion curves, (b) attenuation curves; the solid line in (a) is the dispersion curves for the pure elastic cylindrical curved plate

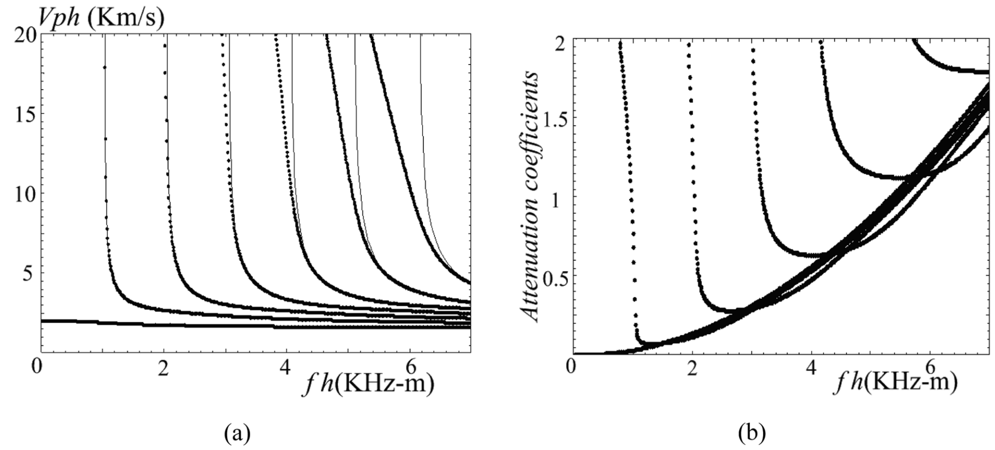


Fig. 10 Circumferential SH wave characteristic curves for the carbon fiber cylindrical curved plate of $\eta = 2$; (a) phase velocity dispersion curves, (b) attenuation curves; the solid line in (a) is the dispersion curves for the pure elastic cylindrical curved plate

the dispersion curves mainly occurs on quasi-transverse wave modes.

3.3 The case of a small ratio of radius to thickness

In this section, another carbon fiber cylindrical curved plate which $\eta = 2$ is investigated to show the influences of the ratio of radius to thickness, as shown in Figs. 9 and 10. Comparing them with the figures of large ratio (Figs. 6 and 7), we can see that the ratio has prominent influences on both the dispersion and the attenuation. For the small η, the dispersion is more serious; the attenuation becomes smaller and the attenuation curves are more contorted; the viscous effect on the dispersion curves of SH wave becomes a little weaker but that on the dispersion curves of the first order quasi-longitudinal wave becomes stronger.

4. Conclusions

In the frame of the Kelvin-Voight viscoelastic theory, circumferential wave characteristics in orthotropic cylindrical curved plates are investigated. Based on the calculated results, the following conclusions can be drawn:

(a) For both circumferential Lamb-like waves and circumferential SH waves in viscoelastic cylindrical curved plates, the viscous effect on the dispersion curves mainly occurs at the lower frequencies.

(b) For the viscoelastic circumferential Lamb-like waves, the viscous effect on the dispersion curves mainly occurs on quasi-longitudinal wave modes or mainly occurs on quasi-transverse wave modes, which is determined by the material parameters.

(c) The ratio of radius to thickness has prominent influences on both dispersion and attenuation. As the ratio decrease, the dispersion becomes more serious and the attenuation becomes smaller.

Acknowledgments

The work is supported by the National Natural Science Foundation of China (No. 10802027) and by the high-performance grid computing platform of Henan Polytechnic University. The authors also wish to express their sincere thanks to the reviewers for their detailed suggestions.

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