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Local buckling of thin and moderately thick variable thickness viscoelastic composite plates

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Abstract. This paper addresses the finite strip formulations for the stability analysis of viscoelastic composite plates with variable thickness in the transverse direction, which are subjected to in-plane forces. While the finite strip method is fairly well-known in the buckling analysis, hitherto its direct application to the buckling of viscoelastic composite plates with variable thickness has not been investigated. The equations governing the stiffness and the geometry matrices of the composite plate are solved in the time domain using both the higher-order shear deformation theory and the method of effective moduli. These matrices are then assembled so that the global stiffness and geometry matrices of a moderately thick rectangular plate are formed which lead to an eigenvalue problem that is solved to determine the magnitude of critical buckling load for the viscoelastic plate. The accuracy of the proposed model is verified against the results which have been reported elsewhere whilst a comprehensive parametric study is presented to show the effects of viscoelasticity parameters, boundary conditions as well as combined bending and compression loads on the critical buckling load of thin and moderately thick viscoelastic composite plates.

Keywords: buckling; effective moduli; shear deformation; viscoelasticity; variable thickness; finite strip method

1. Introduction

Advanced composite plates are extensively used in industrial structures, such as aircraft, marine vessels etc. Because of viscoelastic properties of such materials, demand for using the linear viscoelastic theory has grown rapidly in recent years. As a result of memory effect of viscoelastic composite materials, buckling response of such materials depends on the time parameter (Christensen 1982).

Przemieniecki (1973) presented a finite element method for the local buckling analysis of thin plates. His proposed formulations led to the standard eigenvalue equations from which, the bucking stresses were determined. The critical buckling load for simply supported viscoelastic plates subjected to the biaxial in-plane forces was determined by Wilsom and Vinson (1984). The quasi-static stability analysis of fiber-reinforced viscoelastic composite plates subjected to in-plane edge load systems was investigated by Zenkour (2004). The study was based on the transverse shear

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deformation effects and the effective moduli while the solution concerned the determination of the critical in-plane edge loads associated with the asymptotic instability of plates. The three dimensional linearised theory of stability of deformable solid body mechanics was developed by Akbarov *et al*. (2001) and Akbarov (2007) in which the three-dimensional stability loss problem of a viscoelastic composite plate was presented using semi-analytical finite element method and Laplace transformation. They assumed the plate was clamped at longitudinal edges and was simply supported at the loaded edges while all procedures were made on the plate from the composite material that was modelled as homogenous anisotropic linear viscoelastic material with normalised mechanical properties. Wang and Wang (2008) presented the differential equations of motion of the viscoelastic plate with variable thickness that was based on the thin-plate theory and the two-dimensional viscoelastic differential constitutive relation. The general eigenvalue equations of the viscoelastic plate with multiple cracks under different boundary conditions were calculated while the effects of various geometric parameters, dimensionless delay time and dimensionless crack parameters on the transverse vibration characteristics of a viscoelastic plate containing multiple all-over part-through cracks were analysed. Reddy and Phan (1985) developed a higher-order shear deformation theory to determine the natural frequencies and buckling loads of elastic plates in which the proposed theory took into account the parabolic distribution of the transverse shear strains through the thickness of the plate and rotary inertia. It was shown that the exact proposed solutions for simply supported plates agree well with the exact solutions of three-dimensional elasticity theory, the first-order shear deformation theory, and the classical plate theory. Buckling and vibration of rectangular composite viscoelastic sandwich plates under thermal loads were investigated by Vangipuram and Ganesan (2007) while the temperature dependence of viscoelastic core properties and effects of pre-stresses were taken into account.

A thick finite strip method was applied by Hinton (1977) to the problem of flexure of composite laminates that was based on Mindlin's plate theory which took account of transverse shear deformation. While a good agreement was reported between the results from the three-dimensional, thick plate and thick finite strip analyses, it was found that in some cases thin plate theory appears to be inadequate. Eisenberger and Alexandrov (2003) presented the biaxial buckling loads of variable thickness thin isotropic elastic plates with combinations of boundary conditions, using the extended Kantorovich method where the thickness varied in the directions parallel to the two sides of the plate. The buckling load was found as the in-plane load that made the determinant of the stiffness matrix equal to zero. Azhari *et al*. (2004) investigated the inelastic local buckling of flat plate structures containing variable thickness plates, using the semi-analytical complex finite strip method that was augmented with transverse bubble functions. Stiffness and stability matrices, which were based on the deformation theory of plasticity, were derived for inclusion in the finite strip method. The inelastic local buckling of tapered plates subjected to compression and shear with different boundary conditions was investigated and the developed methodology was applied to the inelastic local buckling of channel sections with tapered flanges and stiffened plates with variable thickness and different geometries. An analysis of the local buckling of composite laminated plates and folded plate assemblies subjected to arbitrary loading was presented by Azhari *et al*. (2000). The analysis used the spline finite strip method, which utilised β_3- spline functions for the longitudinal variation of buckling displacements, and an interpolation of Hermitian polynomials for the buckling displacements in the transverse direction. The method was programmed to study the local buckling of laminated flat plates and L-sections. Hatami *et al*. (2008) developed an exact finite strip method for the free vibration analysis of axially moving viscoelastic plates. Using the differential equation that governed the vibration of plates travelling at a constant axial speed, and utilising the rheological models to model the viscoelastic behaviour of materials, the exact stiffness matrix of a finite strip of plate was extracted in the frequency domain. By assembling the stiffness matrices of the finite strips, the global stiffness matrix of a plate moving on intermediate rollers was obtained, from which the eigenvalues defining the free vibration of the plate were extracted within the domain of complex numbers, while the effects of axial speed and viscoelastic parameters on the free vibration of moving plates were examined.

Designing structures using the variable thickness plate may result in lighter structures, so it can be used for cases where weight is an important factor such as space structures, aircrafts and so on. While the finite strip method is fairly well-known in buckling analysis and has been developed for different application (Azhari *et al*. 2004, Hatami *et al*. 2008, Azhari and Bradford 1993, Heidarpour and Bradford 2007, 2008), hitherto its direct application to the buckling of viscoelastic composite plates with variable thickness has not been investigated, and therefore in this paper the finite strip formulations are developed to investigate the buckling behaviour of variable thickness viscoelastic composite rectangular plates subjected to in-plane loading. Utilising the higher-order shear deformation theory and method of effective moduli, the stiffness and the geometry matrices are determined. The critical buckling load of a rectangular plate with linearly variable thickness in the transverse direction is determined by solving the eigenvalue problems related to the global matrices, and the effect of combined bending and compression loads on the critical buckling load is investigated.

2. Theoretical model

2.1 General

The basic constitutive relation for a linear viscoelasticity is expressed as (Shinuk 2009)

$$\sigma_{ij}(t) = \int_0^t \lambda(t-\tau)\dot{\varepsilon}_{kk}(\tau)d\tau + 2\int_0^t \mu(t-\tau)\dot{\varepsilon}_{kk}(\tau)d\tau \tag{1}$$

where $\sigma_{ij}(t)$ and $\varepsilon_{ij}(t)$ are the tensors of stress and strain respectively that are functions of time t, λ and μ are the material properties which depend on the time parameter τ and the dot superscript represents the derivation respect to time. It is further assumed that the viscoelastic composite plate is made up of two phases; phase 1 which is an elastic material and is used as the reinforcement, and phase 2 which is either elastic or viscoelastic material and is used as the filler. E_r (modulus of elasticity) and ν_r (Poisson's ratio) represent the properties of phase 1, while E_f and ν_f represent the properties of phase 2.

For a moderately thick rectangular plate, normal stress σ_z in the z direction shown in Fig.1 is small and negligible. So the stress-strain relationship for a viscoelastic composite plate is expressed as

$$\mathbf{s} = \mathbf{Ce} \tag{2}$$

where **s** and **e** are stress and strain vectors respectively that are given by

$$\mathbf{s}^{\mathrm{T}} = \{\sigma_x\ \sigma_y\ \tau_{xy}\ \tau_{yz}\ \tau_{zx}\} \text{ and} \tag{3}$$

$$\mathbf{e}^{\mathrm{T}} = \{\varepsilon_x\ \varepsilon_y\ \gamma_{xy}\ \gamma_{yz}\ \gamma_{zx}\} \tag{4}$$

In Eq. (2), the square matrix **C** is the effective modulus tensor that is represented by

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & 0 & 0 & 0 \\ c_{21} & c_{22} & 0 & 0 & 0 \\ 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} \tag{5}$$

For a linear isotropic viscoelastic material, the components of tensor **C** can be obtained from (Zenkour 2004)

$$c_{11} = c_{22} = \frac{3K\overline{\varpi}(2+\overline{\varpi})}{1+2\overline{\varpi}}; \ c_{12} = c_{21} = \frac{3K\overline{\varpi}(1-\overline{\varpi})}{1+2\overline{\varpi}}; \ c_{44} = c_{55} = c_{66} = \frac{3K\overline{\varpi}}{2} \tag{6}$$

in which K is the bulk modulus and $\overline{\varpi}$ is the relaxation function which is a dimensionless parameter and is determined from

$$\overline{\varpi}(t) = c_1 + c_2 e^{-\tau} \tag{7}$$

where c_1 and c_2 are constants and

$$\tau = \frac{t}{t_s} \tag{8}$$

in which t is the time, t_s is the relaxation time. It is worth noting that for a viscoelastic composite material, the components of tensor **C** can be also indicated by (Zenkour 2004)

$$c_{11} = \frac{E_x}{1-\nu_{xy}\nu_{yx}}; \ c_{12} = c_{21} = \frac{\nu_{xy}E_y}{1-\nu_{xy}\nu_{yx}}; \ c_{22} = \frac{E_y}{1-\nu_{xy}\nu_{yx}}; \ c_{44} = G_{xy}; \ c_{55} = G_{yz}; \ c_{66} = G_{xz} \tag{9}$$

where

$$E_x = \gamma E_r + (1-\gamma)E_f; \quad E_y = \frac{E_r\, E_f}{\gamma E_f + (1-\gamma)E_r} \text{ and} \tag{10}$$

$$G_{xy} = G_{yz} = \frac{G_r\, G_f}{\gamma G_f + (1-\gamma)G_r}; \quad G_{xz} = \gamma G_r + (1-\gamma)G_f \text{ and} \tag{11}$$

$$\nu_{xy} = \gamma\nu_r + (1-\gamma)\nu_f; \quad \nu_{yx} = \frac{\nu_{xy}\, E_y}{E_x\, E_r\, E_f} \tag{12}$$

In Eqs. (10) to (12) γ is the volume fraction, which is defined as the ratio of the reinforcement area to the total area of plate at $y = 0$, and E_f and ν_f are determined in the Laplace domain from (Zenkour 2004)

$$E_f = \frac{9K\overline{\varpi}}{2+\overline{\varpi}}, \quad \nu_f = \frac{1-\overline{\varpi}}{2+\overline{\varpi}} \text{ and} \tag{13}$$

$$G_f = \frac{E_f}{2(1+\nu_f)}, \quad G_r = \frac{E_r}{2(1+\nu_r)} \tag{14}$$

2.2 Higher-order shear deformation

The higher-order shear deformation theory proposed by Reddy (1997) can be expressed by the following equations

$$u = u_0 + z\psi_x - \frac{4z^3}{3h^2}(\psi_x + w_{0,x}) \tag{15}$$

$$v = v_0 + z\psi_y - \frac{4z^3}{3h^2}(\psi_y + w_{0,y}) \text{ and} \tag{16}$$

$$w = w_0(x, y) \tag{17}$$

In Eqs. (15) to (17) u is in-plane displacement in the x direction, v is in-plane displacement in the y direction and u_0, v_0 and w_0 represent the magnitudes of mid-plane displacements. w is the displacement in the z direction and is assumed to be constant over the thickness of the composite plate, while $w_{0,x}$ and $w_{0,y}$ represent the first derivation of w_0 respect to x and y respectively, and ψ_x and ψ_y are rotations of the mid-plane about y and x directions, respectively. In the Eqs. (15) and (16) h is calculated on the section with variable thickness as

$$h = \frac{h_2 - h_1}{B}x + h_1 \tag{18}$$

in which h_1 and h_2 are the thicknesses of plate at $x = 0$ and $x = B$ respectively, where B is the width of the composite plate.

2.3 Finite strip formulation

The finite strip method is a variant of the finite element method that has been put to highly effective use in the study of the stability of thin-walled structures and provides an incredibly powerful simplification to finite element method since in the finite element method the plate must be subdivided into two perpendicular directions, while the finite strip method replaces numerous elements by a single strip with limited degrees of freedom. From the other side, in the present paper, in order to consider the history of local buckling coefficients of viscoelastic composite plates in Laplace domain, the equations governing the stiffness and geometry matrices of the viscoelastic composite plate are solved in the time domain for which using finite strip method leads to the less unknown displacements, and therefore the stability and stiffness matrices given by finite strip method have less rank in comparison to those given by finite element method.

For the purposes of this paper, rectangular viscoelastic composite plates with variable thickness are modelled by several finite strips where each strip is assumed to have three nodal lines as depicted in Fig. 1. Displacement vector of each strip for the mth harmonic can be written as (Akhras *et al.* 1993)

$$\mathbf{d}_m = \{u_1\ v_1\ w_1\ w_{1,x}\ \psi_{x1}\ \psi_{y1}\ u_2\ v_2\ \psi_{x2}\ \psi_{y2}\ u_3\ v_3\ w_3\ w_{3,x}\ \psi_{x3}\ \psi_{y3}\}^{\mathrm{T}} \tag{19}$$

in which u_i and v_i are displacements of the ith nodal line ($i = 1$ to 3) along x and y directions

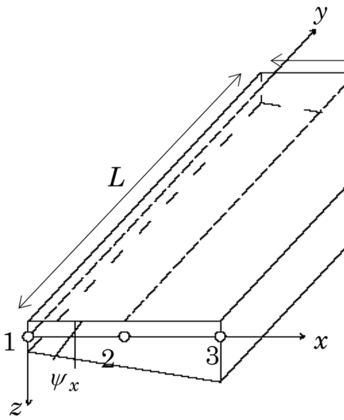

Fig. 1 A typical strip with three nodal lines

respectively, w_i represents the displacement of the ith nodal line ($i = 1, 3$) in the z direction, ψ_{xi} and ψ_{yi} are the rotations of the ith nodal line about y and x axes respectively, and $w_{j,x}$ ($j = 1, 3$) represents the first derivation of w_j respect to x. The magnitude of the mid-plane displacements u_0, v_0 and w_0 can be obtained from interpolation functions so that

$$u_0 = \sum_{m=1}^{M}\sum_{i=1}^{3} N_i(x) u_{im} \sin\frac{m\pi y}{L} \tag{20}$$

$$v_0 = \sum_{m=1}^{M}\sum_{i=1}^{3} N_i(x) u_{im} \cos\frac{m\pi y}{L} \text{ and} \tag{21}$$

$$w_0 = \sum_{m=1}^{M}\sum_{i=1}^{3} [F_i(x) w_{im} + H_i(x) w_{im,x}] \sin\frac{m\pi y}{L} \tag{22}$$

Similarly, the magnitudes of the mid-plane rotations can be determined by

$$\psi(x) = \sum_{m=1}^{M}\sum_{i=1}^{3} N_i(x)\, \psi_m(x) \sin\frac{m\pi y}{L} \text{ and} \tag{23}$$

$$\psi(y) = \sum_{m=1}^{M}\sum_{i=1}^{3} N_i(x)\, \psi_m(y) \cos\frac{m\pi y}{L} \tag{24}$$

where M is the number of harmonics, L is the length of the strip, and $N_i(x)$ is the shape function in the x direction for nodal line i, which can be stated in the form of second order Lagrangian interpolation functions such that

$$N_i(x) = \prod_{j=1}^{3} \frac{x - x_j}{x_i - x_j} \qquad j \neq i \tag{25}$$

In the Eq. (22) $F_i(x)$ and $H_i(x)$ ($i = 1, 3$) are the Hermitian interpolation functions in the x direction for nodal line i, which can be found from

$$F_1(x) = 1-3\left(\frac{x}{b}\right)^2+2\left(\frac{x}{b}\right)^3, \qquad F_3(x) = 3\left(\frac{x}{b}\right)^2-2\left(\frac{x}{b}\right)^3 \tag{26}$$

$$H_1(x) = x\left[1-2\left(\frac{x}{b}\right)+\left(\frac{x}{b}\right)^2\right], \qquad H_3(x) = x\left[\left(\frac{x}{b}\right)^2-\left(\frac{x}{b}\right)\right] \tag{27}$$

where b is the width of each strip.

The strain-displacement relationship is expressed as

$$\mathbf{e} = \mathbf{e}_L + \mathbf{e}_{NL} \tag{28}$$

where $\mathbf{e}_L$ and $\mathbf{e}_{NL}$ are the linear and nonlinear parts of the strain vector, respectively. $\mathbf{e}_L$ can be stated as

$$\mathbf{e}_L = \sum_{m=1}^{M} \mathbf{B}_m \mathbf{d}_m \tag{29}$$

in which $\mathbf{d}_m$ is given by Eqs. (19) and $\mathbf{B}_m$ is the strain matrix for the mth harmonic of the strip which is obtained from

$$\mathbf{B}_m = \begin{bmatrix} N_i'S_m & 0 & -az^3F_i''S_m & -az^3H_i''S_m & (z-az^3)N_i'S_m & 0 \\ 0 & -k_mN_iS_m & k_m^2az^3F_iS_m & k_m^2az^3H_iS_m & 0 & -k_m(z-az^3)N_i'S_m \\ k_mN_iC_m & N_i'C_m & -2k_maz^3F_i'C_m & -2k_maz^3H_i'C_m & k_m(z-az^3)N_i'C_m & (z-az^3)N_i'C_m \\ 0 & 0 & k_m(1-3az^2)F_iC_m & k_m(1-3az^2)H_iC_m & 0 & (1-3az^2)N_iC_m \\ 0 & 0 & (1-3az^2)F_i'S_m & (1-3az^2)H_i'S_m & (1-3az^2)N_iS_m & 0 \end{bmatrix} \tag{30}$$

where $k_m = \dfrac{m\pi}{L}$, $S_m = \sin\dfrac{m\pi}{L}$, $C_m = \cos\dfrac{m\pi}{L}$ and $a = \dfrac{4}{3h^2}$ in which h is the thickness of the plate given by Eq. (18).

The nonlinear part of the strain vector, $\mathbf{e}_{NL}$, in Eq. (28) can be found by

$$\mathbf{e}_{NL} = \begin{bmatrix} \frac{1}{2}\{u_{,x}^2+v_{,x}^2+w_{,x}^2\} \\ \frac{1}{2}\{u_{,y}^2+v_{,y}^2+w_{,y}^2\} \\ u_{,x}u_{,y}+v_{,x}v_{,y}+w_{,x}w_{,y} \\ u_{,y}u_{,z}+v_{,y}v_{,z}+w_{,y}w_{,z} \\ u_{,z}u_{,x}+v_{,z}v_{,x}+w_{,z}w_{,x} \end{bmatrix} \tag{31}$$

and using the equations proposed by Cheung *et al*. (1992) leads to

$$\{u_{,x}\ u_{,y}\}^T = \sum_{m=1}^{M} \mathbf{B}_{Gu,m}\mathbf{d}_m \tag{32}$$

$$\{v_{,x}\ v_{,y}\}^T = \sum_{m=1}^{M} \mathbf{B}_{Gv,m}\mathbf{d}_m \tag{33}$$

$$\{w_{,x}\ w_{,y}\}^T = \sum_{m=1}^{M} \mathbf{B}_{Gw,m}\mathbf{d}_m \tag{34}$$

where m is the number of the series terms, $\mathbf{B}_{Gu,m}$, $\mathbf{B}_{Gv,m}$ and $\mathbf{B}_{Gw,m}$ are the strain matrices used to calculate the geometry matrices for the mth harmonic of the strip.

The stress matrix $\boldsymbol{\sigma}^0$ for a rectangular plate is defined by

$$\boldsymbol{\sigma}^0 = \begin{bmatrix} \sigma_x^0 & \tau_{xy}^0 \\ \tau_{yx}^0 & \sigma_y^0 \end{bmatrix} \tag{35}$$

where σ_x^0 is the compressive stress in the x direction, σ_y^0 is the compressive stress in the y direction and τ_{xy}^0 is the shear stress as shown in Fig. 2. The stiffness and the geometry matrices of each strip can be determined using the principle of virtual work such that (Akhras *et al.* 1994)

$$\mathbf{K}_{mn} = \int_{\Omega}\int_{\Lambda} \mathbf{B}_m^T \mathbf{C} \mathbf{B}_n dz dA \text{ and} \tag{36}$$

$$\mathbf{K}_{G,mn} = \int_{\Omega}\int_{\Lambda} [\mathbf{B}_{Gu,m}^T \boldsymbol{\sigma}^0 \mathbf{B}_{Gu,n} + \mathbf{B}_{Gv,m}^T \boldsymbol{\sigma}^0 \mathbf{B}_{Gv,n} + \mathbf{B}_{Gw,m}^T \boldsymbol{\sigma}^0 \mathbf{B}_{Gw,n}] dz dA \tag{37}$$

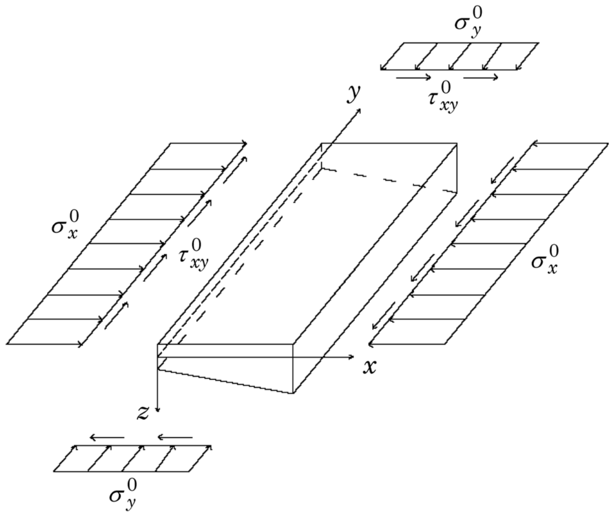

Fig. 2 A typical strip subjected to in-plane stresses

where n is the number of the series terms and $\Omega \in R^2$ and $\Lambda \in [0, h]$ represent the surface of the composite plate and its thickness domains, respectively.

The stiffness and geometric matrices of each strip given by Eqs. (36) and (37) are then assembled so that the critical buckling load can be determined from

$$|\mathbf{K} - \mathbf{K}_G| = 0 \tag{38}$$

where $\mathbf{K}$ is the global stiffness matrix and $\mathbf{K}_G$ is the global geometry matrix of the viscoelastic composite plate. It is noted that once the history of local buckling coefficients of plate is solved in Laplace domain, the time domain is obtained using the inverse Laplace transform, so that the buckling load in the time domain can be represented by

$$\sigma_{crt} = \sum_{j=1}^{5} f_i \phi_j \tag{39}$$

The coefficients f_j in Eq. (39) are determined from the equations given in the Appendix.

3. Illustration

The finite strip methodology proposed in the previous section is used to perform a comprehensive parametric study as described in the following sections. Without loss of generality, the material properties are assumed as follows for all computations unless otherwise stated

$$\nu_r = 0.3, \varsigma = 10, \gamma = 0.1, \overline{\omega} = 0.5, c_1 = 0.1, c_2 = 0.9 \tag{40}$$

where ς is the ratio of bulk modulus to the modulus of elasticity ($\varsigma = \frac{K}{E_r}$). The ratio of width to the average thickness of the composite plate ($2B/h_1 + h_2$) is assumed to be 10 unless otherwise stated. Furthermore, it is assumed that the local buckling coefficient of the plate, k, is formulated by

$$k = \frac{\sigma_{cr} B^2}{D\pi^2} \tag{41}$$

where σ_{cr} is the critical load determined by solving the eigenvalue problem using Eq. (36) and D is the flexural rigidity of the fully elastic plate given by

$$D = \frac{E_r H^3}{12(1 - \nu_r^2)} \tag{42}$$

where H is the average thickness of the composite plate (i.e., $H = (h_1 + h_2)/2$)

3.1 Verification

The accuracy of the presented model is validated against the available results reported elsewhere. Tables 1 and 2 present the magnitudes of the local buckling coefficients, k, of uniaxial and biaxial simply supported square plates ($h_1 = h_2$ and $B/H = 10$) given by Zenkour (2004) in comparison to the finite strip model proposed in this paper for different values of γ and α where. $\alpha = \frac{\sigma_x^0}{\sigma_y^0}$

Table 1. Local buckling coefficient (k) of a viscoelastic square plate ($a = 0$)

$\frac{L}{H}$	$\gamma = 1$		$\gamma = 0$	
	Proposed model	Zenkour's model (2004)	Proposed model	Zenkour's model (2004)
4	2.809	2.9607	49.3605	52.2070
5	3.1344	3.2653	54.6306	57.0209
10	3.7344	3.7866	64.1333	65.0422
20	3.9296	3.9444	67.1639	67.4184
50	3.9887	3.9910	68.0771	68.1155

Table 2. Local buckling coefficient (k) of a viscoelastic square plate ($a = 1$)

$\frac{L}{H}$	$\gamma = 1$		$\gamma = 0$	
	Proposed model	Zenkour's model (2004)	Proposed model	Zenkour's model (2004)
4	1.3999	1.4804	24.5965	26.1035
5	1.5634	1.6327	27.2480	28.5105
10	1.8658	1.8933	32.0426	32.5211
20	1.9643	1.9722	33.5753	33.7092
50	1.9943	1.9955	34.0375	34.0578

As it can be seen, for all values of the ratio of the length to the thickness of the composite plate, the magnitudes of the local buckling coefficients given by the proposed model agree well with the results obtained from Zenkour's model (2004).The results shown in Tables 1 and 2 indicate that the local buckling coefficients increase with the increase of length to thickness ratio, while it is seen that local buckling coefficients for the case of fully viscoelastic plate ($\gamma = 0$) are much greater than the obtained results for the case of fully elastic plate ($\gamma = 1$).

Fig. 3 shows the variation of local buckling coefficients k of simply supported uniaxial rectangular fully elastic ($\gamma = 1$) thin plates for different values of aspect ratio (L/B) given by the proposed model (solid lines) in comparison to those obtained by Eisenberger and Alexandrov (2003) (broken lines). It can be seen that the proposed finite strip method is accurate enough to evaluate the magnitude of local buckling coefficient for rectangular viscoelastic composite plates with variable thickness.

3.2 Parametric study

Fig. 4 shows the local buckling coefficients for a uniaxial viscoelastic square plate with constant thickness ($h = h_1 = h_2$) under combined bending and compression loads. The ratio of width to thickness is assumed to be $B/h = 5$ in this case. The distribution of the applied force can be written as

$$\sigma_y = \sigma_y^0\left(1 - r\frac{x}{B}\right) \tag{43}$$

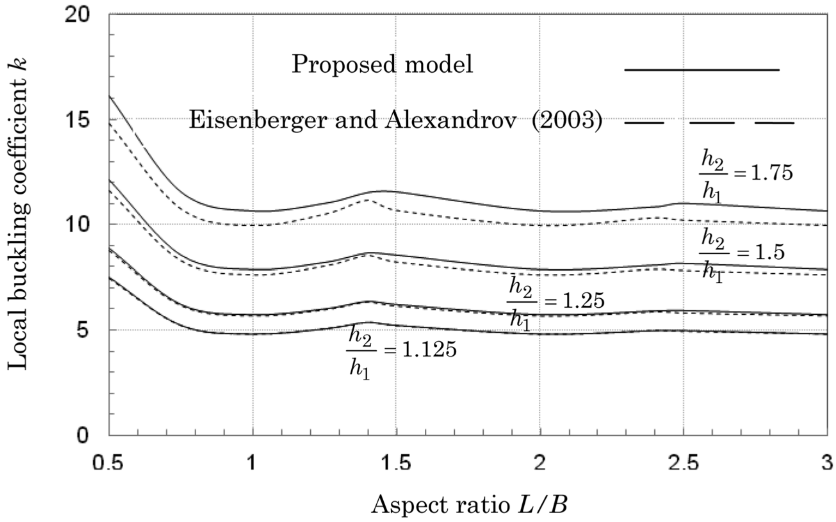

Fig. 3 Local buckling coefficient, k, of uniaxial fully elastic thin plates ($\gamma = 1$) with variable thickness against the variation of aspect ratio given by proposed model in comparison to Eisenberger and Alexandrov's model (2003)

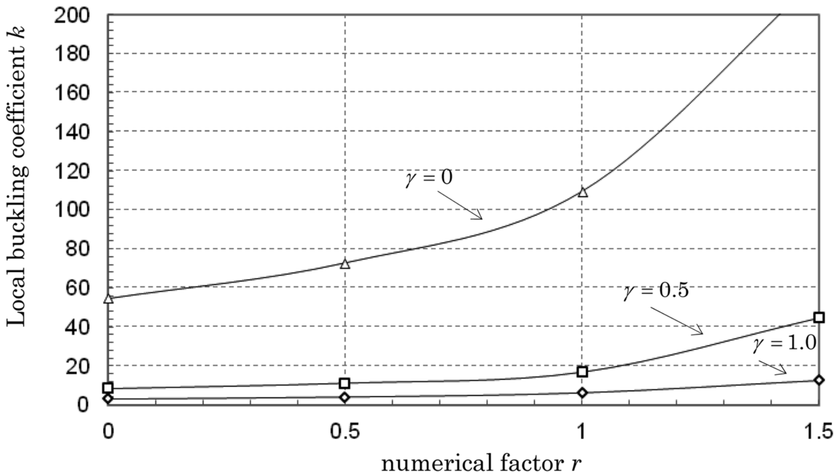

Fig. 4 Local buckling coefficient, k, of uniaxial viscoelastic composite plates with simply supported edges against the numerical factor r for different values of $\gamma = (\alpha = 0)$

where $0 \leq r \leq 2$ is a numerical factor for which when $r = 0$ the composite plate is subjected to the uniformly compressive load whereas $r = 1$ or $r = 2$ indicate that the composite plate is subjected to the hydrostatic pressure and pure bending, respectively. Fig. 4 indicates that the magnitude of the local buckling coefficient increases with the increase of the numerical factor, r, while the highest values of k occur for a fully viscoelastic plate ($\gamma = 0$) and the lowest values occur for a fully elastic plate ($\gamma = 1$).

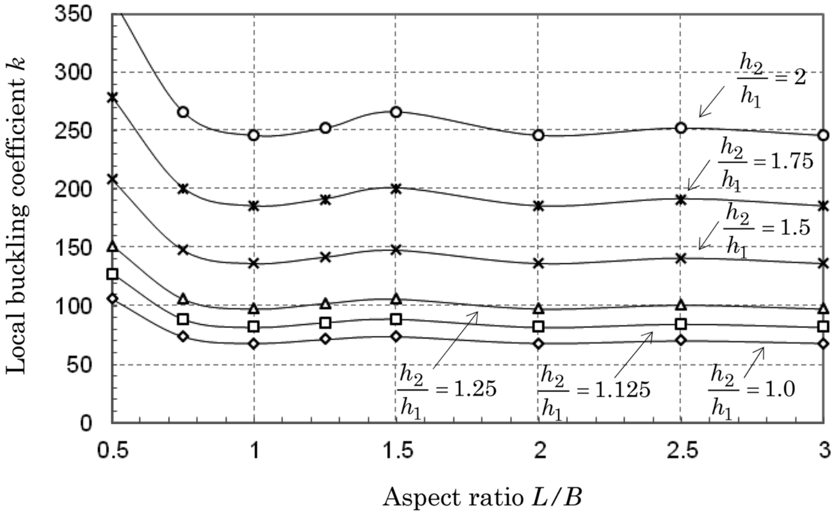

Fig. 5 Local buckling coefficient, k, of simply supported uniaxial viscoelastic thin plates ($\gamma = 0$) with variable thickness against the variation of aspect ratio ($\alpha = 0$)

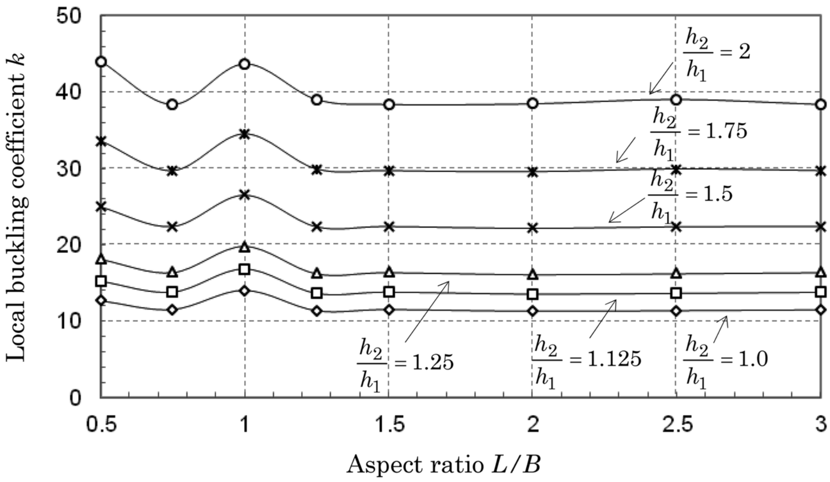

Fig. 6 Local buckling coefficient, k, of simply supported uniaxial viscoelastic thin plates ($\gamma = 0.5$) with variable thickness against the variation of aspect ratio ($\alpha = 0.5$)

The magnitudes of local buckling coefficients of uniaxial rectangular viscoelastic thin or moderately thick plates versus aspect ratio for different values of h_2/h_1 have been depicted in Figs. 5 to 8. It can be seen that for a given value of aspect ratio, the magnitude of local buckling coefficient k decreases as the ratio of h_2/h_1 decreases so that the lowest value of k occurs when the thickness of the plate is constant in the transverse direction. As expected, the local buckling coefficients of fully viscoelastic plates ($\gamma = 0$) are greater than the obtained results for composite viscoelastic plates ($0 < \gamma < 1$); however in all cases the magnitude of local buckling coefficient k reaches a constant value when $L/B > 2$ whilst the obtained results fluctuate when $0.5 < L/B < 1.5$.

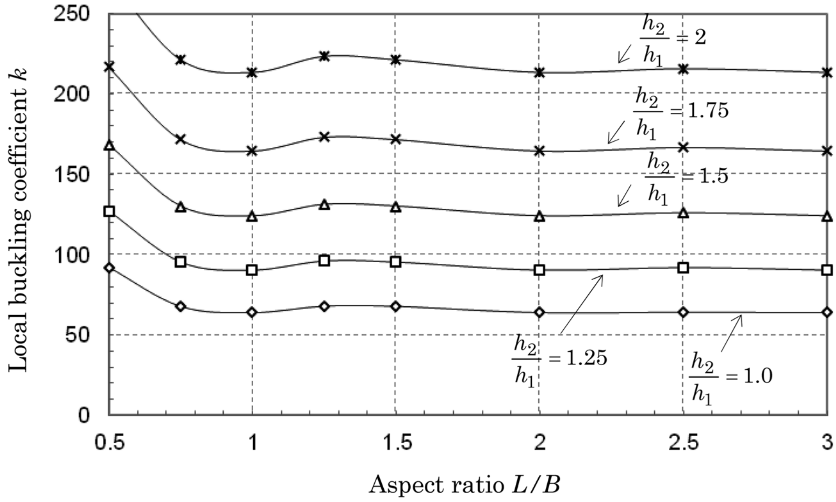

Fig. 7 Local buckling coefficient, *k*, of simply supported uniaxial viscoelastic moderately thick plates ($\gamma = 0$) with variable thickness against the variation of aspect ratio ($\alpha = 0$)

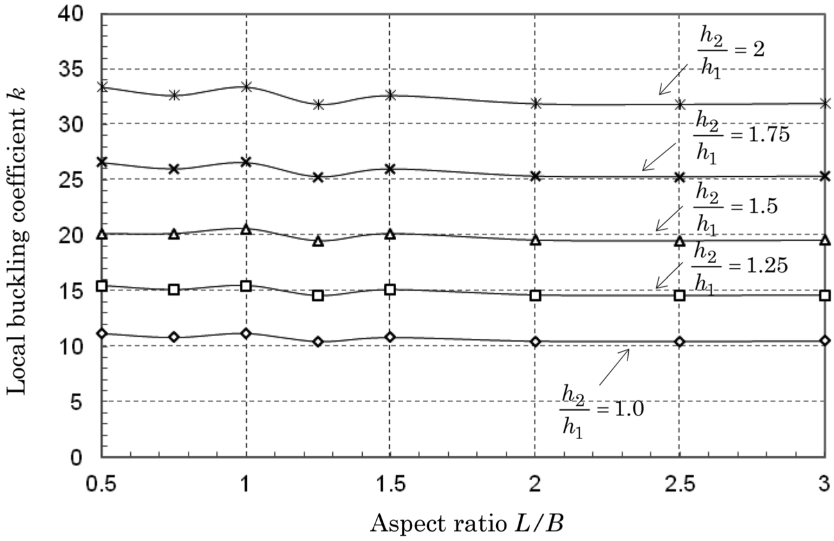

Fig. 8 Local buckling coefficient, *k*, of simply supported uniaxial viscoelastic moderately thick plates ($\gamma = 0.5$) with variable thickness against the variation of aspect ratio ($\alpha = 0.5$)

As depicted in Figs. 7 and 8, the magnitude of local buckling coefficient k for viscoelastic moderately thick plates ($H/L > 0.1$) does not significantly vary with the value of aspect ratio $L/B \geq 1$; however, as expected, the viscoelastic moderately thick plates experience higher values of buckling loads than viscoelastic thin plates.

The effect of boundary condition on the magnitude of local buckling coefficient k of uniaxial rectangular viscoelastic moderately thick plates ($\gamma = 0.5$) for different values of aspect ratio has been

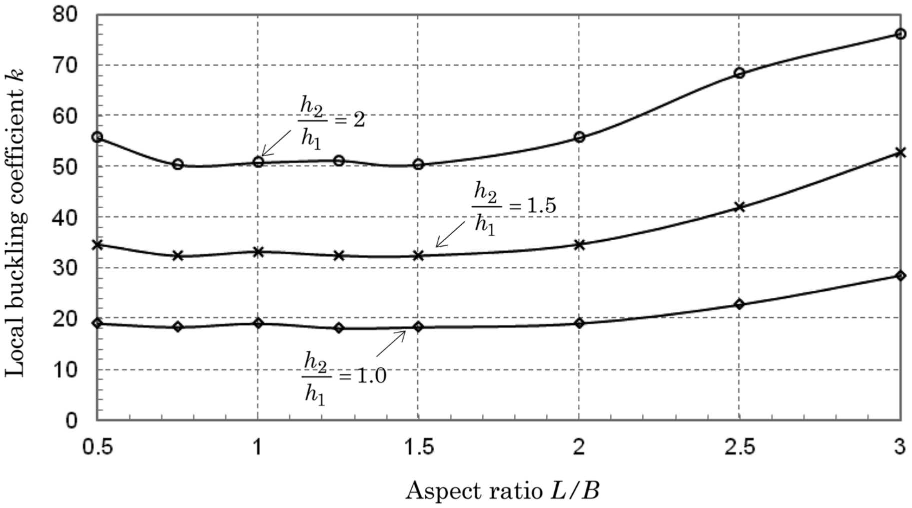

Fig. 9 Local buckling coefficient, *k*, of SSCC uniaxial viscoelastic moderately thick plates ($\gamma = 0.5$) with variable thickness against the variation of aspect ratio

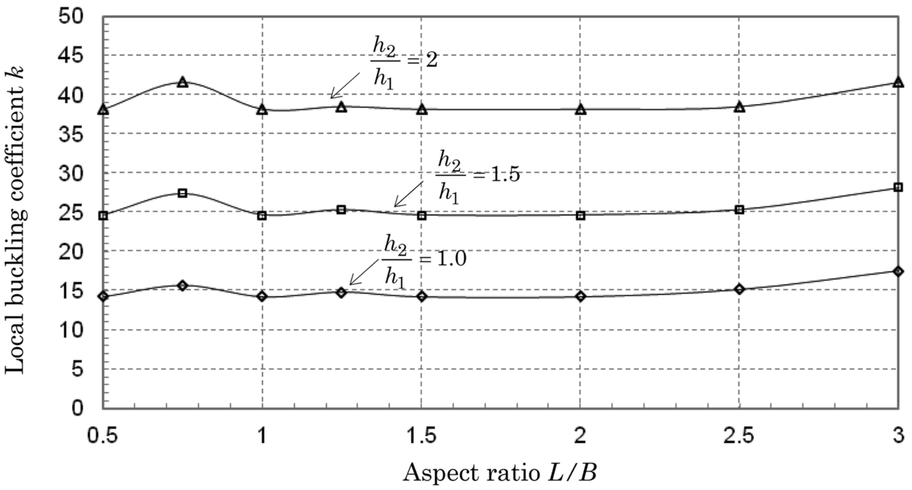

Fig. 10 Local buckling coefficient, *k*, of SSCS uniaxial viscoelastic moderately thick plates ($\gamma = 0.5$) with variable thickness against the variation of aspect ratio

depicted in Figs. 9 to 13 in which the loaded edges are assumed to be simply supported while the unloaded edges in the longitudinal direction of the viscoelastic composite plate are assumed to be clamped-clamped (denoted as SSCC), clamped-simply supported (denoted as SSCS), clamped-free (denoted as SSCF), simply supported-free (denoted as SSSF) or free-free (denoted as SSFF). These figures indicate that the value of local buckling coefficient decreases as the rigidity of the boundary condition along the unloaded edges decreases, and therefore the highest local buckling coefficient occurs when the longitudinal edges are clamped-clamped (Fig. 9) and the lowest local buckling coefficient occurs when the unloaded edges are free-free (Fig. 13).

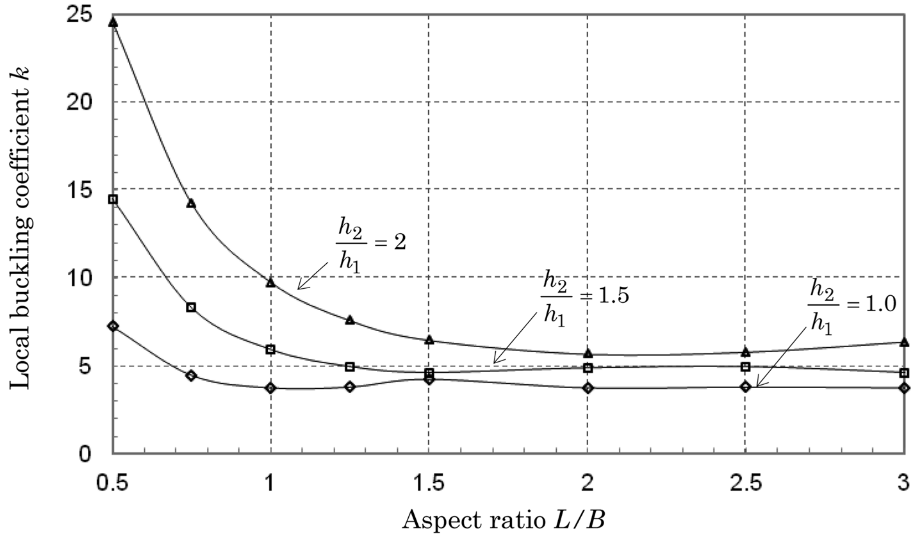

Fig. 11 Local buckling coefficient, *k*, of SSCF uniaxial viscoelastic moderately thick plates ($\gamma = 0.5$) with variable thickness against the variation of aspect ratio

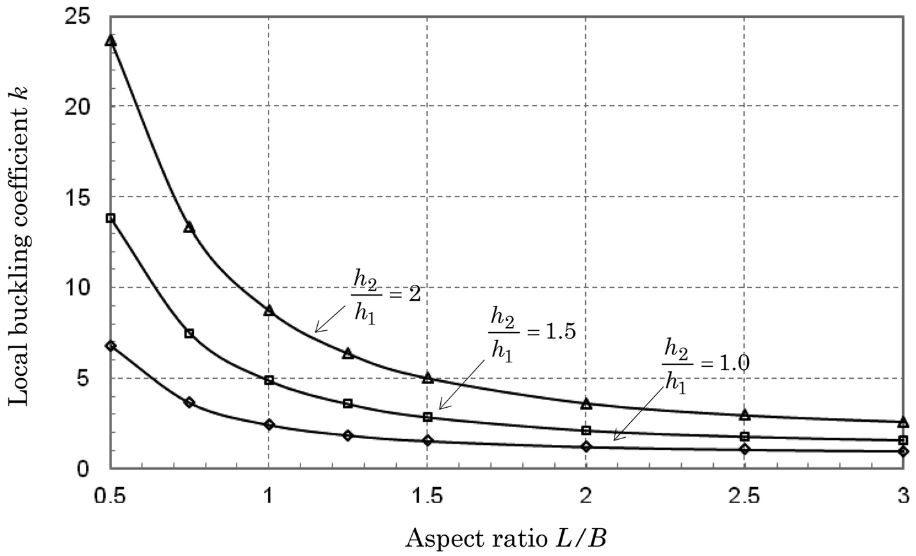

Fig. 12 Local buckling coefficient, *k*, of SSSF uniaxial viscoelastic moderately thick plates ($\alpha = 0.5$) with variable thickness against the variation of aspect ratio

Unlike the fully simply supported moderately thick viscoelastic composite plates (Fig. 8), the magnitude of local buckling coefficient k varies significantly when at least one longitudinal edge of the composite plate is free and $L/B \leq 2$, as depicted in Figs. 11 to 13; however for higher values of aspect ratio, the variation of plate thickness or the value of aspect ratio have no significant effect on the magnitude of buckling load as depicted in Figs. 11 to 13.

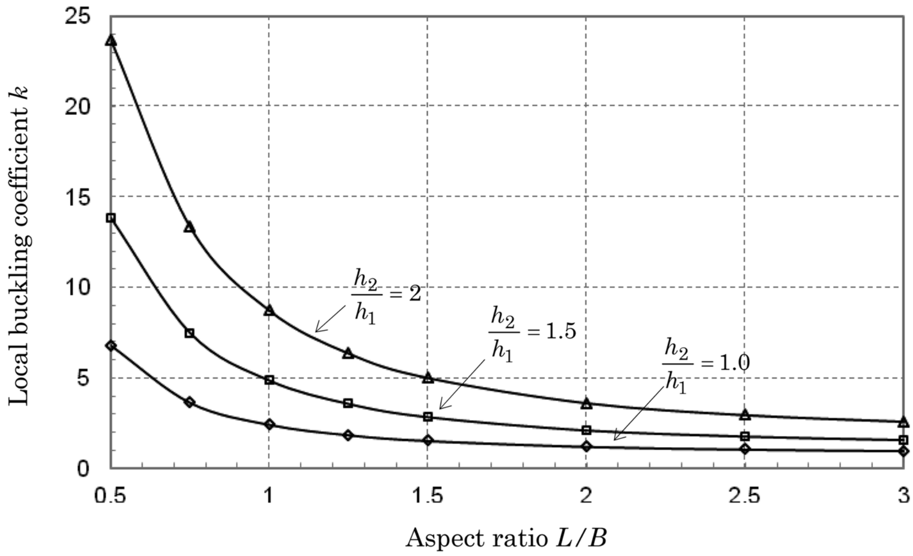

Fig. 13 Local buckling coefficient, k, of SSFF uniaxial viscoelastic moderately thick plates ($\gamma = 0.5$) with variable thickness against the variation of aspect ratio

4. Conclusions

The stability analysis of viscoelastic composite plates with variable thickness in the transverse direction was studied using the finite strip method. The governing equations are based on higher-order shear deformation theory and the effective moduli method. The results of analysis showed that the maximum local buckling coefficients occurred in the case of fully viscoelastic plates. Local buckling coefficients of plates with variable thickness become constant for large values of aspect ratio. It was also obtained that local buckling coefficients of thin plates were larger than those for moderately thick plates with the same conditions.

The effect of the rigidity of the longitudinal edges of the viscoelastic composite plate on the value of buckling load was also investigated. It was shown that the value of local buckling coefficient of moderately thick plates varies significantly for lower values of length to width ratio when at least one edge of the plate is free. The model developed in this paper has a potential to be used for buckling analysis of viscoelastic composite plates with variable thickness which are extensively used in the industrial structures such as aircraft, marine vessels, etc.

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Appendix

The coefficients f_i in Eq. (39) are determined from

$$\sum_{j=1}^{5} f_i \varphi_{ij} = \Gamma_i \qquad i = 1, 2,..5$$

where

$$\varphi_{ij} = \int_0^1 \phi_i(\varpi)\phi_j(\varpi)\mathrm{d}\varpi \quad \text{and} \quad \Gamma_i = \int_0^1 \phi_i(\varpi)\,\sigma_{crt}\mathrm{d}\varpi$$

in which σ_{crt} is given by Eq. (37) and ϕ_i can be represented by (Zenkour 2004)

$$\phi_1 = 1; \quad \phi_2 = \overline{\varpi}; \quad \phi_3 = \frac{1}{1+\chi_3\overline{\varpi}}; \quad \phi_4 = \frac{1}{1+\chi_4\overline{\varpi}}; \quad \phi_5 = \frac{1}{1+\chi_5\overline{\varpi}}$$

where

$$\chi_3 = \frac{1}{2}; \quad \chi_4 = \frac{1}{2}\left(1+\frac{9\gamma\varsigma}{1-\gamma}\right); \quad \chi_5 = \frac{3\gamma\varsigma(1+\nu_r)}{1-\gamma}$$