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Closed form solutions for element equilibrium and flexibility matrices of eight node rectangular plate bending element using integrated force method

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Abstract. Closed form solutions for equilibrium and flexibility matrices of the Mindlin-Reissner theory based eight-node rectangular plate bending element (MRP8) using Integrated Force Method (IFM) are presented in this paper. Though these closed form solutions of equilibrium and flexibility matrices are applicable to plate bending problems with square/rectangular boundaries, they reduce the computational time significantly and give more exact solutions. Presented closed form solutions are validated by solving large number of standard square/rectangular plate bending benchmark problems for deflections and moments and the results are compared with those of similar displacement-based eight-node quadrilateral plate bending elements available in the literature. The results are also compared with the exact solutions.

Keywords: closed form solutions; flexibility matrix; equilibrium matrix; integrated force method; rectangular plate element

1. Introduction

Square/rectangular plate bending elements are used to solve thin or moderately thick plate bending problems with orthogonal boundaries. Though applications of these elements are limited in practice, closed form solutions of equilibrium and flexibility matrices of such elements produce, in general, more accurate results and in considerably less time compare to those obtained using numerical methods. In this paper closed form solutions for equilibrium and flexibility matrices of 8-node (MRP8) rectangular plate bending element using Integrated Force Method are presented. The Mindlin-Reissner plate theory has been employed in the formulation as it accounts the effect of shear deformation and the same model can be used for the analysis of both thin and moderately

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thick plate bending problems.

The Integrated Force Method (IFM) is a new novel matrix formulation developed by Patnaik (1973) for the analysis of civil, mechanical and aerospace engineering structures. In this method all internal forces of the structure are treated as unknown variables and which are computed by simultaneously imposing equilibrium equations and compatibility conditions. The IFM is independent of redundant forces and the basic determinate structure. While analyzing the structural mechanics problems, in general, equilibrium equations and compatibility conditions are to be satisfied in addition to the constitutive relations which describe the material behavior. The IFM integrates the system equilibrium equations and the global compatibility conditions in a fashion paralleling approaches in continuum mechanics (example, the Beltrami - Michel formulation of elasticity (Love 1944)). The IFM provides a natural way of integrating the equilibrium equations and the compatibility conditions while performing structural analysis. IFM is based on variational principle (Patnaik 1986) and its stationary condition of the functional yields the equilibrium equations, compatibility and natural boundary conditions. The Mindlin-Reissner theory based bilinear plate bending element using Integrated Force Method is reported by Dhananjaya *et al.* (2007).

Extensive research efforts are spent on modeling the behavior of the elements and later deriving the matrices which represent their characteristic behavior in the finite element method of analysis. The various matrices are formed with interpolation functions for displacement and sometimes force distribution within or on the boundary of the element. Later on algebraic manipulations, including differentiation and integration, are performed on describing the characteristics of the element stiffness, flexibility and equilibrium matrices. As the number of degrees of freedom of the element increases, the algebraic manipulations become huge and intractable. Therefore automatic generation and closed form of these matrices have been attempted by several researchers: Luff *et al.* (1971), Gunderson *et al.* (1971), Cecchi *et al.* (1977), Noor *et al.* (1979), Hoa *et al.* (1980), Chang *et al.* (1990), Yew *et al.* (1995), Eriksson *et al.* (1999), Nagabhushanam *et al.* (1992), Closed form of stiffness matrices for a four node quadrilateral element and commonly used hybrid finite elements are developed by Griffiths (1994), Lee *et al.* (1998) and Zhov *et al.* (2006). Zhou (2002), has developed the load induced stiffness matrix of a rectangular plate for the finite element method. Review article on symbolic computations in structural engineering is reported by Pavlovic (2003). Closed form dynamic stiffness matrix expressions for MITC4 element are reported in the reference: (Thompson 2003). Rectangular finite element formulation with its applications is given by Oztorun (2006).

Similar to the development of closed form solutions or automatic generation of stiffness matrices in the displacement-based finite element method as cited above, the IFM is also in need of development of closed form solutions of element equilibrium and flexibility matrices, and compatibility conditions for analyzing civil, mechanical and aerospace engineering structures. In this direction, Nagabhushanam *et al.* (1990), developed a general purpose program to generate compatibility matrix for the Integrated Force Method . Automatic generation of sparse and banded compatibility matrix using the Integrated Force Method is presented by Nagabhushanam *et al.* (1991). Generation of compatibility conditions for elasticity and discrete models have been reported by Patnaik *et al.* (2000). Dhananjaya *et al.* (2008) developed a software for automatic generation of element equilibrium and flexibility matrices using Integrated Force Method. In the present paper, IFM has been used to obtain closed form solutions for equilibrium and flexibility matrices of the Mindlin-Reissner theory based 8-node rectangular plate bending element for the analysis of thin

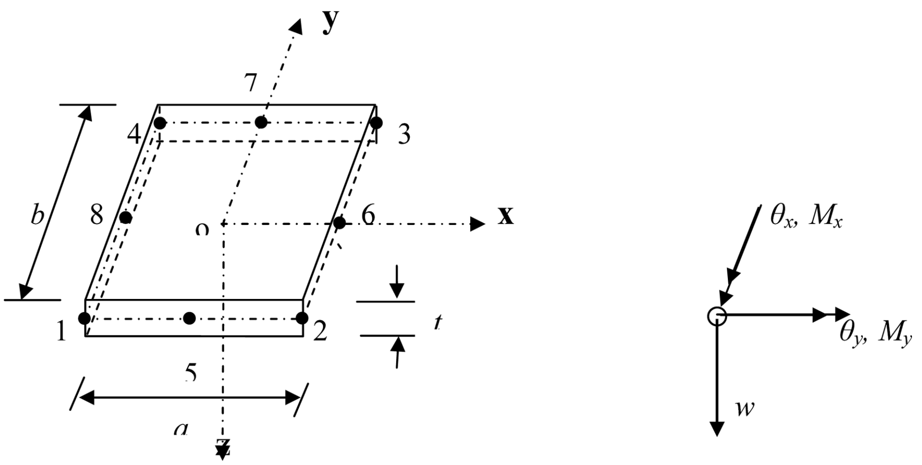

Fig. 1 A typical 8 node rectangular plate bending element of dimensions: $a \times b \times t$ and degrees of freedom (w, θ_x, θ_y) at each node

($t/L \leq 0.01$, where t = thickness of plate and L = span of plate) and moderately thick ($0.01 < t/L \leq 0.2$) square/rectangular plate bending problems.

In this paper, closed form solutions for equilibrium and flexibility matrices of 8-node rectangular plate bending element (MRP8) to analyze the thin/moderately thick plate bending problems using IFM are presented. The Mindlin-Reissner plate theory has been employed in the formulation which accounts the effect of shear deformation. Three degrees of freedom namely a transverse displacement w and two rotations θ_x, θ_y are considered at each node of 8-node element as shown in the Fig. 1. The shear correction factor as suggested by Reissner (1944) has been considered in the formulation. Suitable displacement and stress-resultants fields are chosen over the element and the corresponding element equilibrium and flexibility matrices in closed form are obtained by carrying out exact integration over the square or rectangular plate domain. Large number of standard square/rectangular plate bending benchmark problems are analyzed for deflections and moments to study the accuracy and convergence of 8-node element MRP8 using its closed form solutions of element equilibrium and flexibility matrices. The results obtained by the closed form solutions are compared with those obtained using displacement-based 8- node quadrilateral plate bending elements available in the literature (Spilker (1982), commercial software (ANSYS (version 5.6), NISA (version 9.3)). Results are also compared with the exact solutions. The closed form solutions presented in this paper, produce excellent results, in general, for both thin and moderately thick square/rectangular plate bending problems with simply supported/clamped boundary conditions.

2. Formulation of element equilibrium and flexibility matrices

Basic theory of the Integrated Force Method has been given in the appendix A. In this section brief formulation on the development of equilibrium and flexibility matrices of plate bending element in closed form is described. The Mindlin-Reissner theory has been employed in the formulation. In the Mindlin-Reissner theory, a line that is straight and normal to mid-surface of the un-deformed plate remain straight but not necessarily normal to the mid-surface of the deformed plate. This leads to the following definition of the displacement components u, v, w in the **x**, **y**, **z** Cartesian coordinates system

$$u = -z\theta_x(x,y); \quad v = -z\theta_y(x,y); \quad w = w(x,y) \tag{1}$$

where

- x, y are coordinates in the reference mid-surface
- z is the coordinate through the thickness of the plate t with $-t/2 \le z \le t/2$
- w is the transverse (lateral) displacement
- θ_x, θ_y represent the rotations of the normal in **x-z** and **y-z** planes (Fig. 1) respectively

Engineering strains for the Mindlin-Reissner plate theory can be written as

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = -\begin{Bmatrix} z\dfrac{\partial \theta_x}{\partial x} \\ z\dfrac{\partial \theta_y}{\partial y} \\ z\left(\dfrac{\partial \theta_x}{\partial y} + \dfrac{\partial \theta_y}{\partial x}\right) \\ \theta_y - \dfrac{\partial w}{\partial y} \\ \theta_x - \dfrac{\partial w}{\partial x} \end{Bmatrix} \tag{2}$$

The stress-strain relations for an isotropic two-dimensional plate material is given by

$$\{\sigma\} = [C_{con}]\{\varepsilon\} \tag{3}$$

where $\{\sigma\} = [\sigma_x \;\; \sigma_y \;\; \tau_{xy} \;\; \tau_{yz} \;\; \tau_{xz}]^T$ = vector of stress components

$\{\varepsilon\} = [\varepsilon_x \;\; \varepsilon_y \;\; \gamma_{xy} \;\; \gamma_{yz} \;\; \gamma_{xz}]^T$ = vector of strain components

$$[C_{con}] = \text{constitutive matrix} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 & 0 & 0 \\ \nu & 1 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{(1-\nu)}{2} & 0 & 0 \\ 0 & 0 & 0 & \dfrac{(1-\nu)}{2} & 0 \\ 0 & 0 & 0 & 0 & \dfrac{(1-\nu)}{2} \end{bmatrix}$$

E = Young's modulus; ν = Poisson's ratio

The stress-resultants for plates can be written as

$$M_x = \int_{-t/2}^{t/2} z\sigma_x dz$$

$$M_y = \int_{-t/2}^{t/2} z\sigma_y dz$$

$$M_{xy} = \int_{-t/2}^{t/2} z\tau_{xy} dz$$

$$Q_y = \int_{-t/2}^{t/2} \tau_{yz} dz$$

$$Q_x = \int_{-t/2}^{t/2} \tau_{xz} dz \tag{4}$$

Eqs. (2), (3) and (4) yield the moment-curvature relations as

$$\{M\} = [C_1]\{k\} \tag{5}$$

Where $\{M\}$ = vector of stress-resultants
$= [M_x \;\; M_y \;\; M_{xy} \;\; Q_y \;\; Q_x]^T$

$[C_1]$ = matrix relating stress-resultants to curvatures
$\{k\}$ = vector of curvatures
$= \left[\frac{\partial\theta_x}{\partial x} \;\; \frac{\partial\theta_y}{\partial y} \;\; \frac{\partial\theta_x}{\partial y}+\frac{\partial\theta_y}{\partial x} \;\; \theta_y-\frac{\partial w}{\partial y} \;\; \theta_x-\frac{\partial w}{\partial x}\right]^T$

From the Eq. (5), the curvature-moment relations can be written as

$$\{k\} = [C_1]^{-1}\{M\} = [H]\{M\} \tag{6}$$

where $[H] = [C_1]^{-1}$
= matrix relating curvatures to stress-resultants

The matrix $[H]$ for the Mindlin-Reissner plate with Reissner's shear correction factor (Reissner 1945) of 5/6 can be written as

$$[H] = \frac{1}{D_1}\begin{bmatrix} 1 & -\nu & 0 & 0 & 0 \\ -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & \frac{t^2(1+\nu)}{5} & 0 \\ 0 & 0 & 0 & 0 & \frac{t^2(1+\nu)}{5} \end{bmatrix} \tag{7}$$

where $D_1 = Et^3/12$
The strain energy U_p of the elastic plate in bending and shear is written as

$$U_p = \iint\frac{1}{2}\left[M_x\frac{\partial\theta_x}{\partial x}+M_y\frac{\partial\theta_y}{\partial y}+M_{xy}\left(\frac{\partial\theta_x}{\partial y}+\frac{\partial\theta_y}{\partial x}\right)+Q_y\left(\theta_y-\frac{\partial w}{\partial y}\right)+Q_x\left(\theta_x-\frac{\partial w}{\partial x}\right)\right]dxdy \tag{8}$$

The vectors $\{M\}$ and $\{k\}$ for a discrete plate bending element can be expressed in matrix notations in terms of assumed stress-resultants and displacement fields respectively as

$$\{M\} = [\psi]\{F_e\} \tag{9}$$

$$\{k\} = [D_{op}][\phi_1]\{\alpha\} = [D_{op}][\phi]\{X_e\} \tag{10}$$

where

$[\psi]$ = matrix of polynomial terms for stress-resultant fields

$\{F_e\}$ = vector of force components of the discrete element

$[\phi_1]$ = matrix of polynomial terms for displacement fields

$$[\phi] = [\phi_1][A]^{-1}$$

$[A]$ = matrix formed by substituting the coordinates of the element nodes into the polynomial of displacement fields

$\{\alpha\}$ = coefficients of the displacement field polynomial

$\{X_e\}$ = vector of displacements of the discrete element

$$[D_{op}] = \text{differential operator matrix} = \begin{bmatrix} 0 & \frac{\partial}{\partial x} & 0 \\ 0 & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & 0 & 1 \\ -\frac{\partial}{\partial x} & 1 & 0 \end{bmatrix}$$

Substituting Eqs. (9) and (10) into the Eq. (8), the strain energy for the discrete element can be expressed as

$$U_p = \frac{1}{2}\{X_e\}^T[B_e]\{F_e\} \tag{11}$$

where $[B_e]$ represents the element equilibrium matrix and is given by

$$[B_e] = \iint [\phi]^T[D_{op}]^T[\psi]dxdy \tag{12}$$

The complementary strain energy for the elastic plate in bending and shear is expressed as

$$U_c = \iint \frac{1}{2}\frac{1}{D_1}\left[M_x^2 + M_y^2 - 2\nu M_x M_y + 2(1+\nu)M_{xy}^2 + Q_y^2\frac{t^2(1+\nu)}{5} + Q_x^2\frac{t^2(1+\nu)}{5}\right]dxdy$$

Using the Eq. (7), the complementary strain energy for the discrete element is written as

$$U_c = \frac{1}{2}\{F_e\}^T[G_e]\{F_e\} \tag{13}$$

where $[G_e]$ represents the element flexibility matrix and is given by

$$[G_e] = \iint [\psi]^T [H][\psi] dxdy \tag{14}$$

The Eqs. (12) and (14) are used to obtain closed form solutions of equilibrium matrix $[B_e]$ and flexibility matrices $[G_e]$ respectively. Exact integration scheme is used with limits of integration $-a/2$ to $+a/2$ and $-b/2$ to $+b/2$, where 'a' and 'b' are the dimensions of the rectangular plate. These element equilibrium matrix $[B_e]$ and element flexibility matrix $[G_e]$ of all elements are assembled to obtain the global equilibrium matrix $[B]$ and global flexibility matrix $[G]$ of the structure and they are used to setup the IFM governing equation to analyze the structure by IFM.

2.1 Displacement and stress-resultant fields

In this plate bending element (MRP8), the displacement fields for w, θ_x and θ_y are assumed in terms of generalized coordinates α_1, α_2 …… α_{24} and they are given in the Eq. (15)

$$\begin{aligned} w &= \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \alpha_7 x^2 y + \alpha_8 xy^2 \\ \theta_x &= \alpha_9 + \alpha_{10} x + \alpha_{11} y + \alpha_{12} x^2 + \alpha_{13} xy + \alpha_{14} y^2 + \alpha_{15} x^2 y + \alpha_{16} xy^2 \\ \theta_y &= \alpha_{17} + \alpha_{18} x + \alpha_{19} y + \alpha_{20} x^2 + \alpha_{21} xy + \alpha_{22} y^2 + \alpha_{23} x^2 y + \alpha_{24} xy^2 \end{aligned} \tag{15}$$

Using independent generalized force parameters F_1, F_2 ……… F_{21}, stress-resultant fields in polynomial terms are written as given in the Eq. (16)

$$\begin{aligned} M_x &= F_1 + F_2 x + F_3 y + F_4 x^2 + F_5 xy + F_6 y^2 + F_7 x^2 y + F_8 xy^2 \\ M_Y &= F_9 + F_{10} x + F_{11} y + F_{12} x^2 + F_{13} xy + F_{14} y^2 + F_{15} x^2 y + F_{16} xy^2 \\ M_{xy} &= F_{17} + F_{18} x + F_{19} y + F_{20} x^2 + F_{21} y^2 \\ Q_y &= (F_{11} + F_{18}) + (F_{13} + 2F_{20})x + 2F_{14} y + 2F_{16} xy + F_{15} x^2 \\ Q_x &= (F_2 + F_{19}) + 2F_4 x + (F_5 + 2F_{21})y + 2F_7 xy + F_8 y^2 \end{aligned} \tag{16}$$

Eqs. (7), (12), (14), (15) and (16) are used to obtain closed form solutions of element equilibrium and flexibility matrices of eight-node rectangular plate bending element MRP8. These are given in the Tables 1 and 2 respectively.

3. Numerical tests and discussions

Simply supported/clamped thin or moderately thick and cantilever strip plate bending example problems are analyzed for different grid sizes using the closed form solutions of element MRP8 to estimate moments and deflections. The performance of the element MRP8 using the closed form solutions are studied with respect to accuracy and convergence. The results are compared with those of few 8-node displacement-based quadrilateral plate bending elements available in the literature (Spilker 1982) and in the commercial software (ANSYS (version 5.6), NISA (version 9.3)). The results of the element MRP8 are also compared with the exact solutions (Timoshenko *et al.* 1959, Jane *et al.* 2000) for thin and moderately thick plates respectively. The example problems considered here are:

Table 1 Closed form element equilibrium matrix for 8-node rectangular plate bending element (non-zero elements are shown)

$B(1,2) = b/6;$
$B(1,4) = -ab/3;$
$B(1,5) = -b^2/12;$
$B(1,7) = ab^2/18;$
$B(1,8) = b^3/40;$
$B(1,11) = a/6;$
$B(1,13) = -a^2/12;$
$B(1,14) = -ab/3;$
$B(1,15) = a^3/40;$
$B(1,16) = a^2b/18;$
$B(1,18) = a/6;$
$B(1,19) = b/6;$
$B(1,20) = -a^2/6;$
$B(1,21) = -b^2/6;$
$B(2,1) = -b/6;$
$B(2,2) = ab/12;$
$B(2,3) = b^2/12;$
$B(2,4) = -a^2b/24;$
$B(2,5) = -ab^2/24;$
$B(2,6) = -b^3/40;$
$B(2,7) = a^2b^2/48;$
$B(2,8) = ab^3/80;$
$B(2,17) = -a/6;$
$B(2,18) = a^2/12;$
$B(2,19) = ab/12;$
$B(2,20) = -a^3/40;$
$B(2,21) = -ab^2/24;$
$B(3,9) = -a/6;$
$B(3,10) = a^2/12;$
$B(3,11) = ab/12;$
$B(3,12) = -a^3/40;$
$B(3,13) = -a^2b/24;$
$B(3,15) = a^3b/80;$
$B(3,16) = a^2b^2/48;$
$B(3,17) = -b/6;$
$B(3,18) = ab/12;$
$B(3,19) = b^2/12;$
$B(3,20) = -a^2b/24;$
$B(3,21) = -b^3/40;$
$B(4,2) = b/6;$
$B(4,4) = -ab/3;$
$B(4,5) = b^2/12;$
$B(4,7) = -ab^2/18;$
$B(4,8) = b^3/40;$
$B(4,11) = -a/6;$
$B(4,13) = a^2/12;$
$B(4,14) = -ab/3;$
$B(4,15) = -a^3/40;$
$B(4,16) = a^2b/18;$
$B(4,18) = -a/6;$
$B(4,19) = b/6;$
$B(4,20) = a^2/6;$
$B(4,21) = b^2/6;$
$B(5,1) = -b/6;$
$B(5,2) = ab/12;$
$B(5,3) = -b^2/12;$
$B(5,4) = -a^2b/24;$
$B(5,5) = ab^2/24;$
$B(5,6) = -b^3/40;$
$B(5,7) = -a^2b^2/48;$
$B(5,8) = ab^3/80;$
$B(5,17) = a/6;$
$B(5,18) = -a^2/12;$

Table 1 Continuted

$B(5,19)=ab/12;$	$B(8,4)=ab/24;$
$B(5,20)=a^3/40;$	$B(8,5)=ab^2/24;$
$B(5,21)=ab^2/24;$	$B(8,6)=b^3/40;$
$B(6,9)=a/6;$	$B(8,7)=a^2b^2/48;$
$B(6,10)=-a^2/12;$	$B(8,8)=ab^3/80;$
$B(6,11)=ab/12;$	$B(8,17)=a/6;$
$B(6,12)=a^3/40;$	$B(8,18)=a^2/12;$
$B(6,13)=-a^2b/24;$	$B(8,19)=ab/12;$
$B(6,14)=ab^2/24;$	$B(8,20)=a^3/40;$
$B(6,15)=a^3b/80;$	$B(8,21)=ab^2/24;$
$B(6,16)=-a^2b^2/48;$	$B(9,9)=a/6;$
$B(6,17)=-b/6;$	$B(9,10)=a^2/12;$
$B(6,18)=ab/12;$	$B(9,11)=ab/12;$
$B(6,19)=-b^2/12;$	$B(9,12)=a^3/40;$
$B(6,20)=-a^2b/24;$	$B(9,13)=a^2b/24;$
$B(6,21)=-b^3/40;$	$B(9,14)=ab^2/24;$
$B(7,2)=-b/6;$	$B(9,15)=a^3b/80;$
$B(7,4)=-ab/3;$	$B(9,16)=a^2b^2/48;$
$B(7,5)=-b^2/12;$	$B(9,17)=b/6;$
$B(7,7)=-ab^2/18;$	$B(9,18)=ab/12;$
$B(7,11)=-b^3/40;$	$B(9,19)=b^2/12;$
$B(7,11)=-a/6;$	$B(9,20)=a^2b/24;$
$B(7,13)=-a^2/12;$	$B(9,21)=b^3/40;$
$B(7,14)=-ab/3;$	$B(10,2)=-b/6;$
$B(7,15)=-a^3/40;$	$B(10,5)=b^2/12;$
$B(7,16)=-a^2b/18;$	$B(10,4)=-ab/3;$
$B(7,18)=-a/6;$	$B(10,7)=ab^2/18;$
$B(7,19)=-b/6;$	$B(10,8)=-b^3/40;$
$B(7,20)=-a^2/6;$	$B(10,11)=a/6;$
$B(7,21)=-b^2/6;$	$B(10,13)=a^2/12;$
$B(8,1)=b/6;$	$B(10,14)=-ab/3;$
$B(8,2)=ab/12;$	$B(10,15)=a^3/40;$
$B(8,3)=b^2/12;$	

Table 1 Continuted

$B(10,16) = -a^2 b/18;$
$B(10,18) = a/6;$
$B(10,19) = -b/6;$
$B(10,20) = a^2/6;$
$B(10,21) = b^2/6;$
$B(11,1) = b/6;$
$B(11,2) = ab/12;$
$B(11,3) = -b^2/12;$
$B(11,4) = a^2 b/24;$
$B(11,5) = -ab^2/24;$
$B(11,6) = b^3/40;$
$B(11,7) = -a^2 b^2/48;$
$B(11,8) = ab^3/80;$
$B(11,17) = -a/6;$
$B(11,18) = -a^2/12;$
$B(11,19) = ab/12;$
$B(11,20) == -a^3/40;$
$B(11,21) = -ab^2/24;$
$B(12,9) = -a/6;$
$B(12,10) = -a^2/12;$
$B(12,11) = ab/12;$
$B(12,12) = -a^3/40;$
$B(12,13) = a^2 b/24;$
$B(12,14) = -ab^2/24;$
$B(12,15) = a^3 b/80;$
$B(12,16) = -a^2 b^2/48;$
$B(12,17) = b/6;$
$B(12,18) = ab/12;$
$B(12,19) = -b^2/12;$
$B(12,20) = a^2 b/24;$
$B(12,21) = b^3/40;$
$B(13,2) = 2/3b;$
$B(13,8) = b^3/30;$
$B(13,14) = 2/3ab;$
$B(13,16) = -a^2 b/9;$
$B(13,19) = 2/3b;$
$B(14,1) = -2/3b;$
$B(14,2) = ab/3;$
$B(14,4) = -a^2 b/6;$
$B(14,6) = -b^3/30;$
$B(14,8) = ab^3/60;$
$B(15,17) = -2/3b;$
$B(15,18) = ab/3;$
$B(15,20) = -a^2 b/6;$
$B(15,21) = -b^3/30;$
$B(16,4) = 2/3ab;$
$B(16,7) = ab^2/9;$
$B(16,11) = -2/3a;$
$B(16,15) = -a^3/30;$
$B(16,18) = -2/3a;$
$B(17,17) = 2/3a;$
$B(17,19) = ab/3;$
$B(17,20) = a^3/30;$
$B(17,21) = ab^2/6;$
$B(18,9) = 2/3a;$
$B(18,11) = ab/3;$
$B(18,12) = a^3/30;$
$B(18,14) = ab^2/6;$
$B(18,15) = a^3 b/60;$
$B(19,2) = -2/3b;$
$B(19,8) = -b^3/30;$
$B(19,14) = 2/3ab;$
$B(19,16) = a^2 b/9;$
$B(19,19) = -2/3b;$
$B(20,1) = 2/3b;$
$B(20,2) = ab/3;$
$B(20,4) = a^2 b/6;$
$B(20,6) = b^3/30;$
$B(20,8) = ab^3/60;$
$B(21,17) = 2/3b;$
$B(21,18) = ab/3;$
$B(21,20) = a^2 b/6;$
$B(v) = b^3/30;$
$B(22,4) = 2/3ab;$
$B(22,7) = -ab^2/9;$
$B(22,11) = 2/3a;$
$B(22,15) = a^3/30;$
$B(22,18) = 2/3a;$
$B(23,17) = -2/3a;$
$B(23,19) = ab/3;$
$B(23,20) = -a^3/30;$
$B(23,21) = -ab^2/6;$
$B(24,9) = -2/3a;$
$B(24,11) = ab/3;$
$B(24,12) = -a^3/30;$
$B(24,14) = -ab^2/6;$
$B(24,15) = a^3 b/60;$

Table 2 Closed form element flexibility matrix for 8-node rectangular plate bending element (non-zero elements are shown)

$G(1,1) = ab;$

$G(1,4) = a^3b/12;$

$G(1,6) = ab^3/12;$

$G(1,9) = -a\upsilon b$

$G(1,12) = -a^3\upsilon b/12;$

$G(1,14) = -a\upsilon b^3/12;$

$G(2,2) = a(5a^2 + 12t^2\upsilon + 12t^2)b/60;$

$G(2,8) = a^3b^3/144 + t^2\upsilon ab^3/60 + t^2ab^3/60;$

$G(2,10) = -a^3\upsilon b\ \ /12;$

$G(2,16) = -a^3\upsilon b^3/144;$

$G(2,19) = t^2a(1+\upsilon)b/5;$

$G(3,3) = ab^3/12;$

$G(3,7) = a^3b^3/144;$

$G(3,11) = -a\upsilon b^3/12;$

$G(3,15) = -a^3\upsilon b^3/144;$

$G(4,1) = a^3b/12;$

$G(4,4) = a^3(3a^2 + 16t^2 + 16t^2\upsilon)b/240;$

$G(4,6) = a^3b^3/144;$

$G(4,9) = -a^3\upsilon b/12;$

$G(4,12) = -\upsilon a^5b/80;$

$G(4,14) = -a^3\upsilon b^3/144;$

$G(5,5) = a^3b^3/144 + t^2\upsilon ab^3/60 + t^2ab^3/60;$

$G(5,13) = -a^3\upsilon b^3/144;$

$G(5,21) = t^2a(1+\upsilon)b^3/30;$

$G(6,1) = ab^3/12;$

$G(6,4) = a^3b^3/144;$

$G(6,6) = ab^5/80;$

$G(6,9) = -a\upsilon b^3/12;$

$G(6,12) = -a^3\upsilon b^3/144;$

$G(6,14) = -a\upsilon b^5/80;$

$G(7,3) = a^3b^3/144;$

$G(7,7) = a^5b^3/960 + t^2a^3b^3/180 + t^2\upsilon a^3b^3/180;$

$G(7,11) = -a^3\upsilon b^3/144;$

Table 2 Continued

$G(7,15) = -\upsilon a^5 b^3 / 960;$

$G(8,2) = a^3 b^3 / 144 + t^2\ \upsilon a b^3 / 60 + t^2 a b^3 / 60;$

$G(8,8) = a^3 b^5 / 960 + t^2 \upsilon a b^5 / 400 + t^2 a b^5 / 400;$

$G(8,10) = -a^3 \upsilon b^3 / 144;$

$G(8,16) = -a^3 \upsilon b^5 / 960;$

$G(8,19) = t^2 a(1+\upsilon) b^3 / 60;$

$G(9,1) = -a \upsilon b;$

$G(9,4) = -a^3 \upsilon b / 12;$

$G(9,6) = -a \upsilon b^3 / 12;$

$G(9,9) = ab;$

$G(9,12) = a^3 b / 12;$

$G(9,14) = a b^3 / 12;$

$G(10,2) = -a^3 \upsilon b / 12;$

$G(10,8) = -a^3 \upsilon b^3 / 144;$

$G(10,10) = a^3 b / 12;$

$G(10,16) = a^3 b^3 / 144;$

$G(11,3) = -a \upsilon b^3 / 12;$

$G(11,7) = -a^3 \upsilon b^3 / 144$

$G(11,11) = ab(5b^2 + 12t^2 + 12t^2 \upsilon) / 60;$

$G(11,15) = a^3 b^3 / 144 + t^2 \upsilon a^3 b / 60 + t^2 a^3 b / 60;$

$G(11,18) = t^2 a(1+\upsilon) b / 5;$

$G(12,1) = -a^3 \upsilon b / 12;$

$G(12,4) = -\upsilon a^5 b / 80;$

$G(12,6) = -a^3 \upsilon b^3 / 144$

$G(12,9) = a^3 b / 12;$

$G(12,12) = a^5 b / 80;$

$G(12,14) = a^3 b^3 / 144;$

$G(13,5) = -a^3 \upsilon b^3 / 144;$

$G(13,13) = a^3 b^3 / 144 + t^2 \upsilon a^3 b / 60 + t^2 a^3 b / 60;$

$G(13,20) = t^2 a^3 (1+\upsilon) b / 30;$

$G(14,1) = -a \upsilon b^3 / 12;$

Table 2 Continued

$G(14,4) = -a^3\upsilon b^3/144$

$G(14,6) = -a\upsilon b^5/80$

$G(14,9) = ab^3/12$

$G(14,12) = a^3b^3/144$

$G(14,14) = ab^3(3b^2 + 16t^2 + 16t^2\upsilon)/240$

$G(15,3) = -a^3\upsilon b^3/144$

$G(15,7) = -\upsilon a^5b^3/960$

$G(15,11) = a^3b^3/144 + t^2\upsilon a^3b/60 + t^2a^3b/60$

$G(15,15) = a^5b^3/960 + t^2\upsilon a^5b/400 + t^2a^5b/400$

$G(15,18) = t^2a^3(1+\upsilon)b/60$

$G(16,2) = -a^3\upsilon b^3/144$

$G(16,8) = -a^3\upsilon b^5/960$

$G(16,10) = a^3b^3/144$

$G(16,16) = a^3b^5/960 + t^2a^3b^3/180 + t^2\upsilon a^3b^3/180$

$G(17,17) = 2(1+\upsilon)ab$

$G(17,20) = a^3(1+\upsilon)b/6$

$G(17,21) = (1+\upsilon)ab^3/6$

$G(18,11) = t^2a(1+\upsilon)b/5$

$G(18,15) = t^2a^3(1+\upsilon)b/60$

$G(18,18) = a(5a^2 + 6t^2\upsilon + 5a^2\upsilon + 6t^2)b/30$

$G(19,2) = t^2a(1+\upsilon)b/5$

$G(19,8) = t^2a(1+\upsilon)b^3/60$

$G(19,19) = ab(5b^2 + 6t^2 + 5\upsilon b^2 + 6t^2\upsilon)/30$

$G(20,13) = t^2a^3(1+\upsilon)b/30$

$G(20,17) = a^3(1+\upsilon)b/6$

$G(20,20) = a^3(3a^2 + 3a^2\upsilon + 8t^2\upsilon + 8t^2)b/120$

$G(20,21) = a^3(1+\upsilon)b^3/72$

$G(21,5) = t^2a(1+\upsilon)b^3/30$

$G(21,17) = (1+\upsilon)ab^3/6$

$G(21,20) = a^3(1+\upsilon)b^3/72$

$G(21,21) = ab^3(3b^2 + 8t^2 + 3\upsilon b^2 + 8t^2\upsilon)/120$

1. A simply supported/clamped square thin plate subjected to uniform load. The parameters of the problem are : size of the plate = 100 × 100, t = 1, E = 10^7, ν = 0.3, q = 10 (Spilker 1982)
2. A simply supported rectangular thin plate with the aspect ratio 3 subjected to uniform load. The parameters of the problem are : L = 300, B = 100, t = 1, E = 10^7, ν = 0.3, q = 10 (Spilker 1982)
3. A long cantilever plate (Fig. 2. strip plate) subjected to point load at tip or uniform load over the entire plate. The parameters of the problem are: L = 1000, B = 30, t = 5, E = 2 × 10^5, ν = 0.0, P = 25, q = 0.01. Here the Poisson's ratio (ν) is considered to be zero to compare the results with the beam solution.
4. A simply supported/clamped moderately thick square plate (t/L = 0.1) with uniform load. The parameters of the problem are : L = 100, B = 100, t = 10, E = 2 × 10^5, ν = 0.3, q = 10, P = 400.

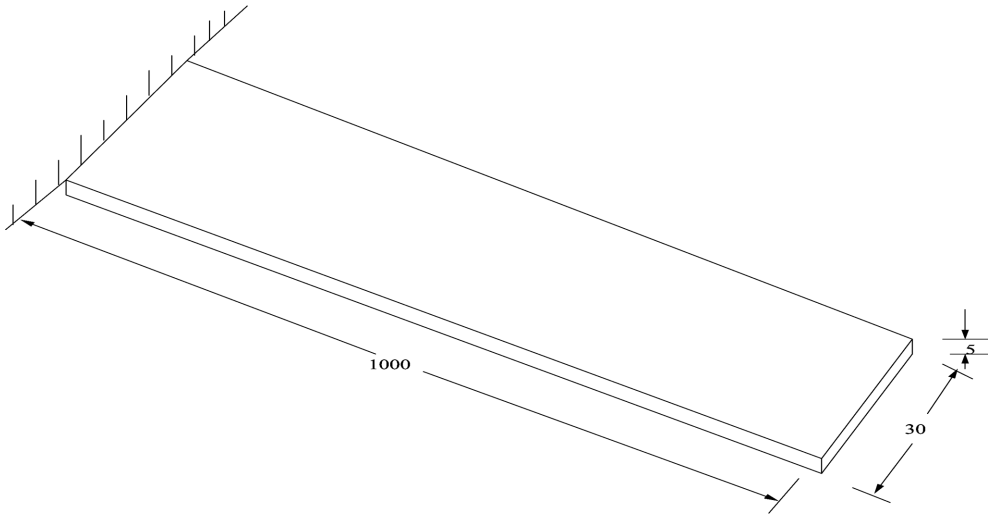

Fig. 2 Cantilever plate (strip plate) : L = 1000, B = 30, t = 5, E = 2 × 10^5, ν = 0.0, P = 25, q = 0.01

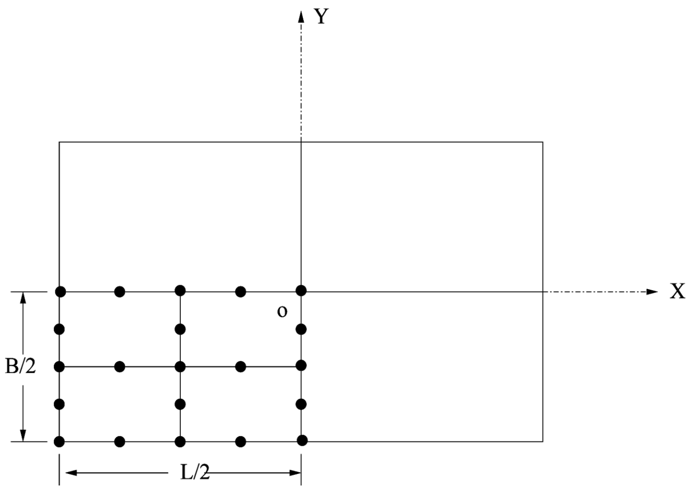

Fig. 3 A typical mesh (2 × 2) in one quadrant of the rectangular plate

Due to symmetry of the plate with respect to geometry, loading and boundary conditions in the above example problems 1, 2 and 4, one quadrant of the plate is considered for the analysis. The typical mesh (2 × 2) considered in one quadrant is shown in the Fig. 3.

Central deflections and moments for the above example problems for various mesh sizes are obtained using closed form solutions of eight-node rectangular plate bending element MRP8. The central deflections and moments for a simply supported square plate subjected to uniform load for various mesh sizes are given in the Tables 3 and 4 respectively. And corresponding converging

Table 3 Normalized central deflection for a simply supported square thin plate with uniform load (Example problem 1)

Elements	QH1	QH2	QH3	QH4	MRP8
1 × 1	0.956	0.890	0.900	0.835	0.959
2 × 2	1.000	0.985	0.990	0.980	1.005
3 × 3	1.000	0.990	0.990	0.990	1.003
4 × 4	1.000	0.990	0.990	0.990	1.001

Table 4 Normalized central moment for a simply supported square thin plate with uniform load (Example problem 1)

Elements	QH1	QH2	QH3	QH4	MRP8
1 × 1	0.580	0.560	0.640	0.700	0.361
2 × 2	0.980	1.015	0.965	1.010	0.991
3 × 3	0.990	1.030	0.985	1.006	0.982
4 × 4	0.990	1.020	0.990	1.006	0.992

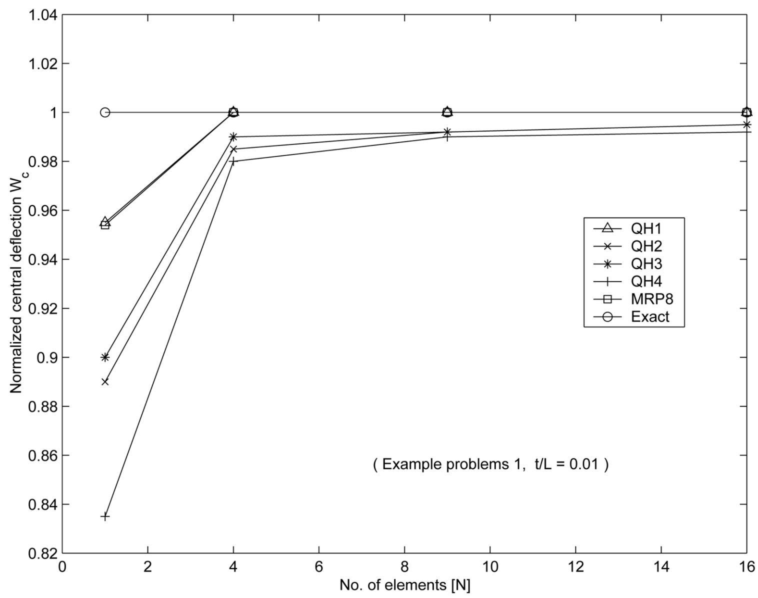

Fig. 4 Normalized central deflection for a simply supported square thin plate with uniform load

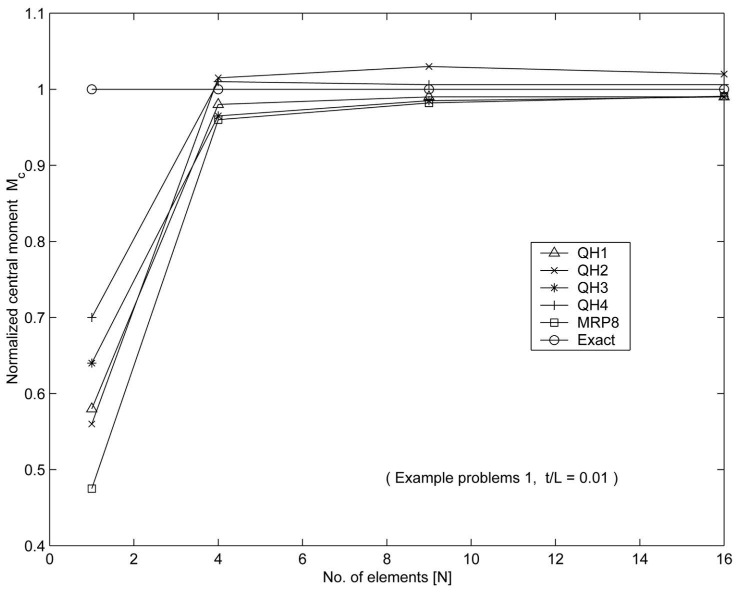

Fig. 5 Normalized central moment for a simply supported square thin plate with uniform load

Table 5 Normalized central deflection for a clamped square thin plate with uniform load (Example problem 1)

Elements	QH1	QH2	QH3	QH4	MRP8
1 × 1	1.170	0.900	0.990	0.440	1.153
2 × 2	0.990	0.955	0.940	0.830	0.994
3 × 3	1.000	0.980	0.985	0.935	1.003
4 × 4	1.000	0.990	0.990	0.970	1.003

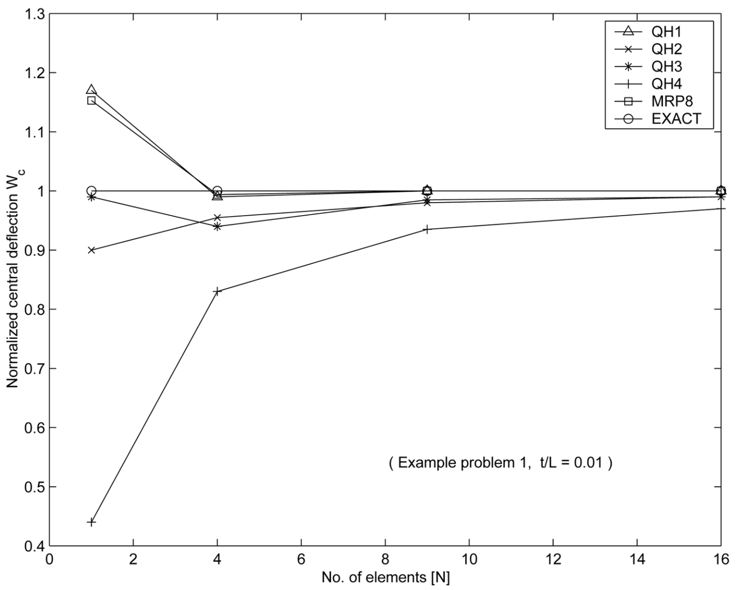

Fig. 6 Normalized central deflection for a clamped square thin plate with uniform load

Table 6 Normalized central deflection for a simply supported rectangular thin plate with uniform load (L/B = 3, Example problem 2)

Elements	QH1	QH2	QH3	QH4	MRP8
1 × 1	1.020	0.865	0.920	0.890	1.001
2 × 2	1.006	1.075	1.028	0.980	1.005
3 × 3	1.000	1.006	1.008	0.985	1.000
4 × 4	1.000	1.000	1.000	0.990	1.000

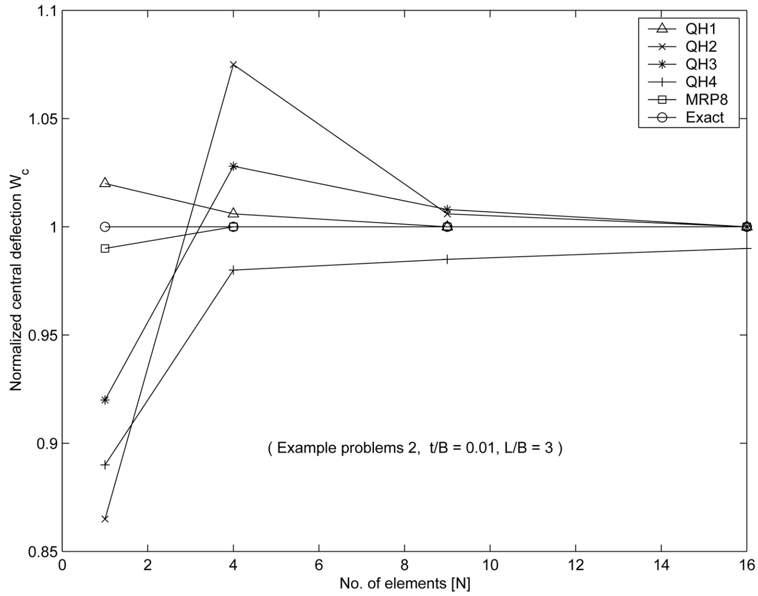

Fig. 7 Normalized central deflection for a simply supported rectangular thin plate with uniform load

trends are depicted in the Figs. 4 and 5 respectively. The Table 5 shows central deflections for a clamped square plate subjected to uniform load, and Fig. 6 depicts the corresponding converging trends. The central deflections for a simply supported rectangular plate with aspect ratio 3 subjected to uniform load are given in the Table 6 and corresponding converging trends are shown in the Fig. 7. The results are compared with those obtained from displacement-based 8-node quadrilateral plate bending elements available literature (Spilker 1982). The results are also compared with the exact solutions (Timoshenko *et al.* 1959). Tables 3-6 and Figs. 4-7 indicate that closed form solutions of element equilibrium and flexibility matrices of the eight-node element MRP8 have produced, in general, better results compare to those of similar displacement based 8-node quadrilateral plate bending elements considered here.

Tip deflections, and bending moments at the clamped edge of long cantilever plate (strip plate, example problem 3) for various grid sizes are summarized in Tables 7-10 and the corresponding convergence trends are shown in the Figs. 8-11. The results are compared with those obtained from similar 8-node quadrilateral plate bending elements (ANSYS8 and NISA8) from commercial software (ANSYS (version 5.6), NISA (version 9.3). The results are also compared with exact solutions. It is interesting to observe from the Tables 7-10 and Figs. 8-11 that the closed form

Table 7 Tip deflection for a cantilever strip plate with point load at the tip (t/L = 0.005, Example problem 3)

Elements	NISA8	ANSYS8	MRP8
2 × 1	133.33	129.72	133.34
4 × 1	133.33	131.25	133.34
8 × 1	133.33	132.20	133.34
16 × 1	133.33	132.73	133.34
32 × 1	133.33	133.00	133.34
64 × 1	133.33	133.33	133.34
Exact = 133.33			

Table 8 Moment at the clamped edge for a cantilever strip plate with point load at the tip (t/L = 0.005, Example problem 3)

Elements	NISA8	ANSYS8	MRP8
2 × 1	833.33	833.33	833.33
4 × 1	833.33	833.33	833.33
8 × 1	833.33	833.33	833.33
16 × 1	833.33	833.33	833.33
32 × 1	833.33	833.33	833.33
64 × 1	833.33	833.33	833.33
Exact = 833.33			

Table 9 Tip deflection for a cantilever strip plate with uniform load (t/L = 0.005, Example problem 3)

Elements	NISA8	ANSYS8	MRP8
2 × 1	600.01	579.23	600.01
4 × 1	600.01	587.61	600.01
8 × 1	600.01	593.23	600.01
16 × 1	600.01	596.40	600.01
32 × 1	600.01	598.01	600.01
Exact = 600.00			

Table 10 Moment at the clamped edge for a cantilever strip plate with uniform load (t/L = 0.005, Example problem 3)

Elements	NISA8	ANSYS8	MRP8
2 × 1	4791.67	4791.67	5000.00
4 × 1	4947.90	4947.92	5000.00
8 × 1	4986.96	4987.10	5000.00
16 × 1	4996.75	4996.67	5000.00
32 × 1	4999.17	4999.17	5000.00
Exact = 5000.00			

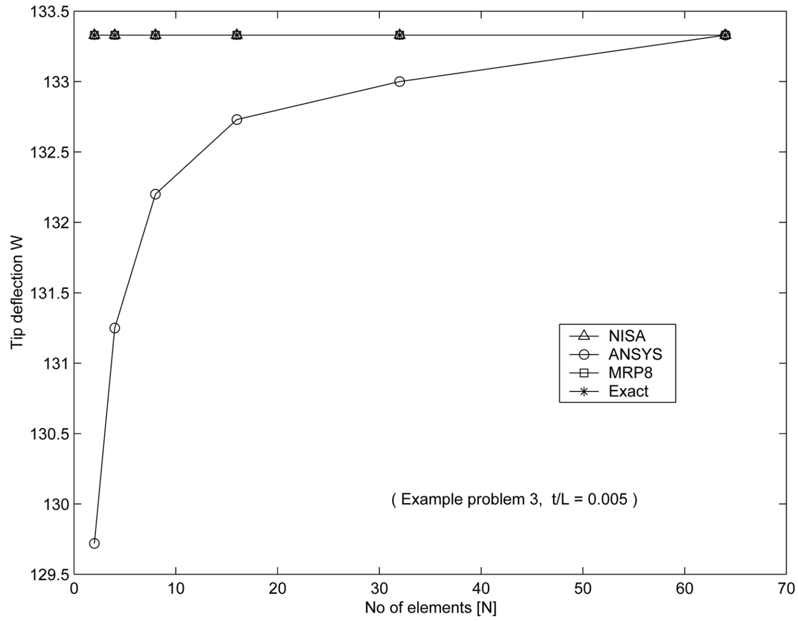

Fig. 8 Tip deflection for a cantilever strip plate with the point load at tip

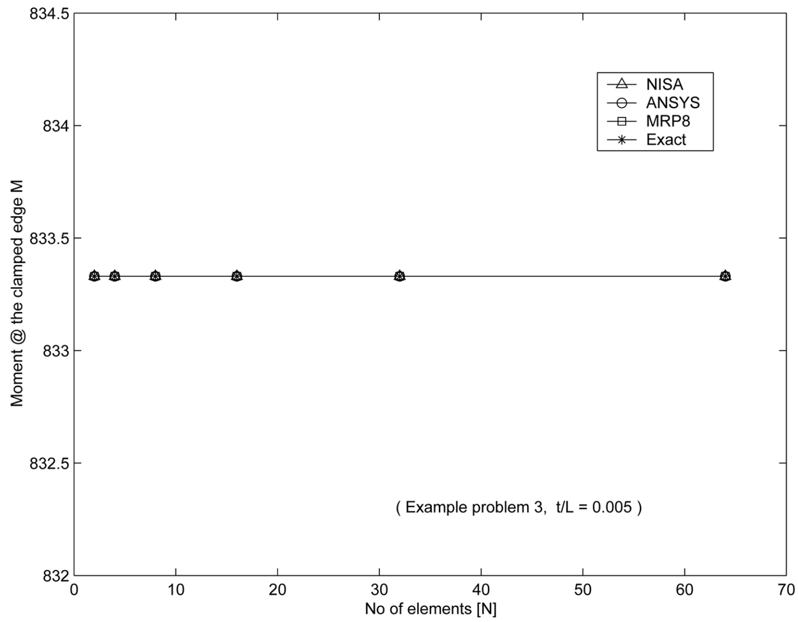

Fig. 9 Moment at the clamped edge for a cantilever strip plate with the point load at tip

solutions of element equilibrium and flexibility matrices of MRP8 estimates exact value for both tip deflections and the moments at the clamped edge for all mesh sizes including least mesh size (1×1) unlike displacement based 8-node quadrilateral plate bending elements considered here which are converging, in general, to exact values at large number of elements.

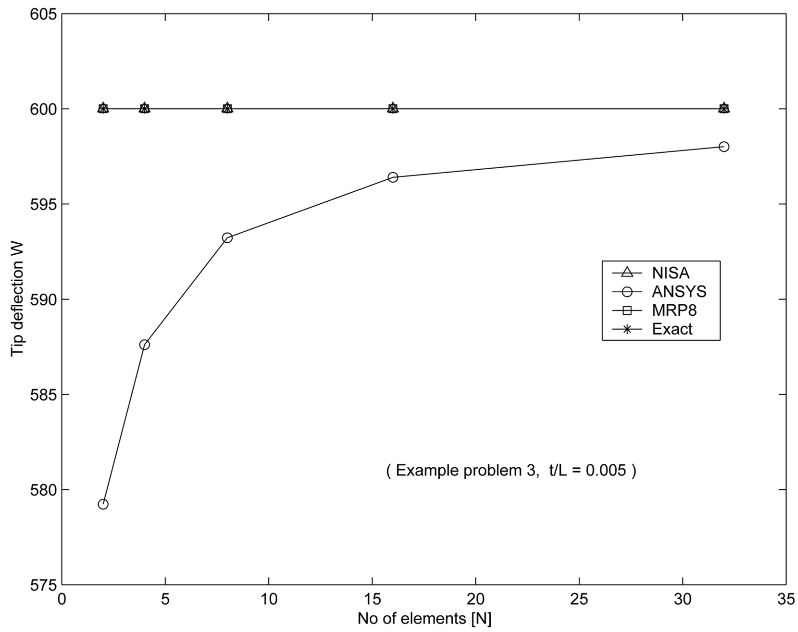

Fig. 10 Tip deflection for a cantilever strip plate with uniform load

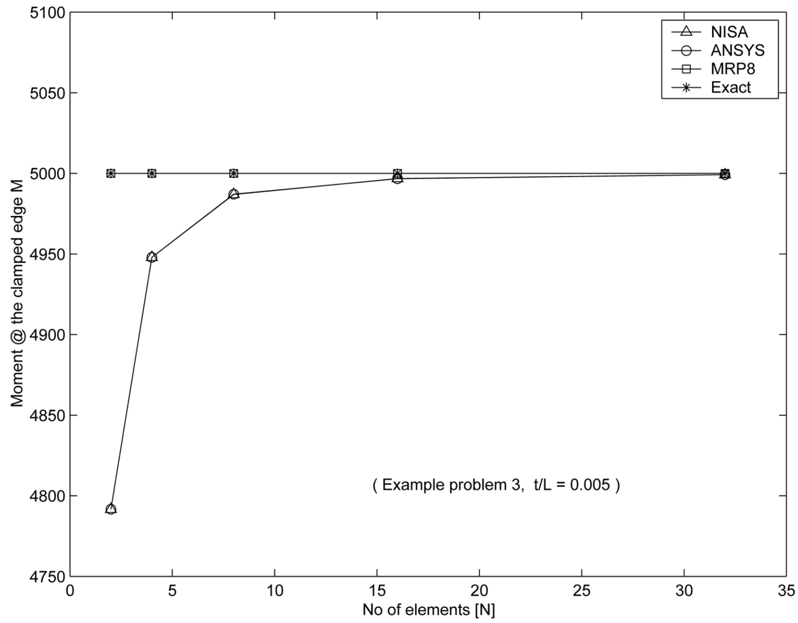

Fig. 11 Moment at clamped edge for a cantilever strip plate with uniform load

Tables 11 and 12 summarize the central deflections and moments respectively of a simply supported moderately thick square plate (t/L = 0.1, example problem 4) for various grid sizes and the Figs. 12 and 13 show corresponding convergence trends. Similarly Tables 13 and 14 summarize the central deflections and moments respectively of a clamped moderately thick square plate (t/L =

Table 11 Central deflection for a simply supported square moderately thick plate with uniform load W_c (t/L = 0.1, Example problem 4)

Elements	NISA8	ANSYS8	MRP8
1 × 1	0.2218	0.2218	0.2262
2 × 2	0.2345	0.2345	0.2357
3 × 3	0.2340	0.2340	0.2351
4 × 4	0.2337	0.2337	0.2345
Exact = 0.2331			

Table 12 Central moment for a simply supported moderately thick square plate with uniform load M_c (t/L = 0.1, Example problem 4)

Elements	NISA8	ANSYS8	MRP8
1 × 1	6683	6683	2180
2 × 2	5158	5158	4699
3 × 3	4959	4959	4740
4 × 4	4881	4881	4763
Exact = 4790			

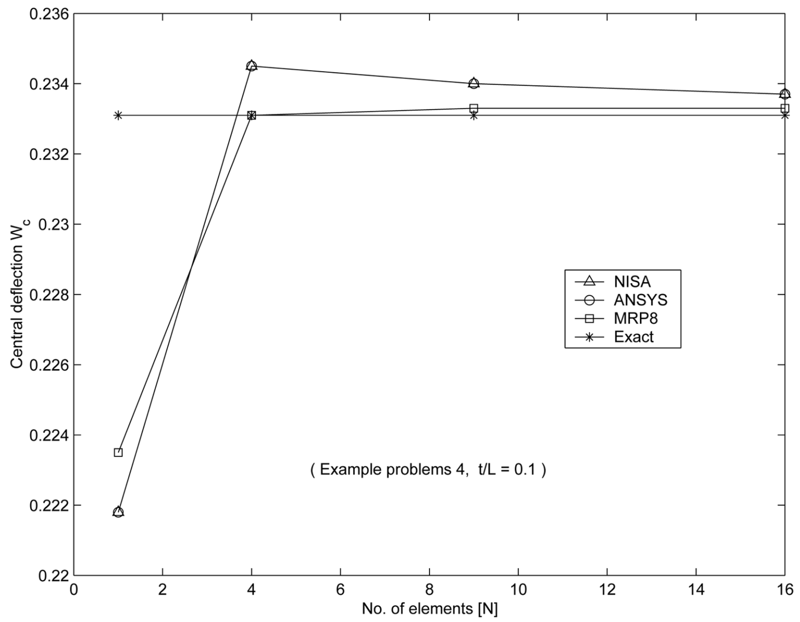

Fig. 12 Central deflection for a simply supported moderately square thick plate with uniform load

0.1, example problem 4) for various grid sizes and the Figs. 14 and 15 show corresponding convergence trends. These Tables 11-14 and Figs. 12-15 show that the closed form solutions of element equilibrium and flexibility matrices of MRP8 estimates central deflections and moments

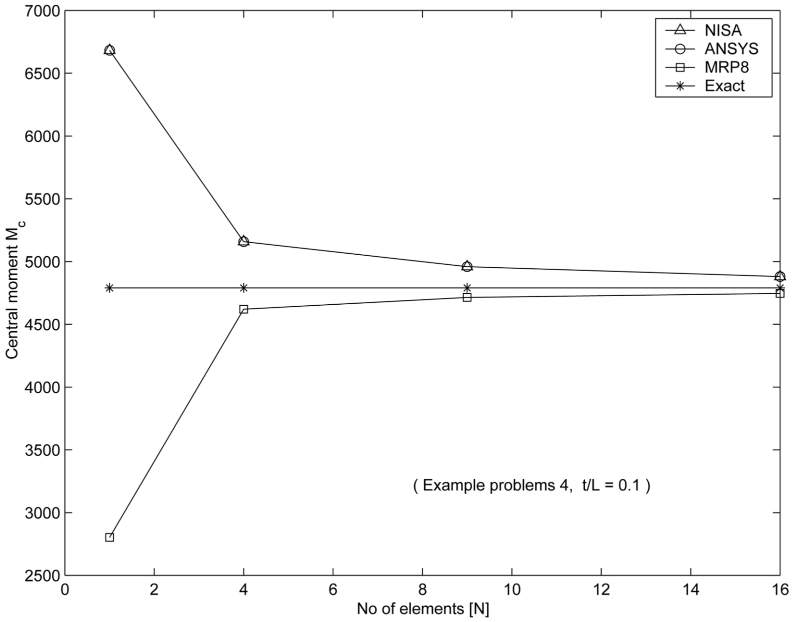

Fig. 13 Central moment for a simply supported moderately thick square plate with uniform load

Table 13 Central deflection for a clamped moderately thick square plate with uniform load W_c (t/L = 0.1, Example problem 4)

Elements	NISA8	ANSYS8	MRP8
1 × 1	0.0956	0.0956	0.0908
2 × 2	0.0820	0.0820	0.0819
3 × 3	0.0822	0.0822	0.0822
4 × 4	0.0822	0.0822	0.0822
Exact = 0.0819			

Table 14 Central moment for a clamped moderately thick square plate with uniform load M_c (t/L = 0.1, Example problem 4)

Elements	NISA8	ANSYS8	MRP8
1 × 1	4875	4875	1903
2 × 2	2735	2735	2111
3 × 3	2485	2485	2272
4 × 4	2409	2409	2289
Exact = 2310			

better, in general, compared to similar 8-node quadrilateral plate bending elements ANSYS8, NISA8 considered here from commercial software (ANSYS (version 5.6), NISA (version 9.3). The results are also compared with the exact solutions (Jane *et al*. 2000).

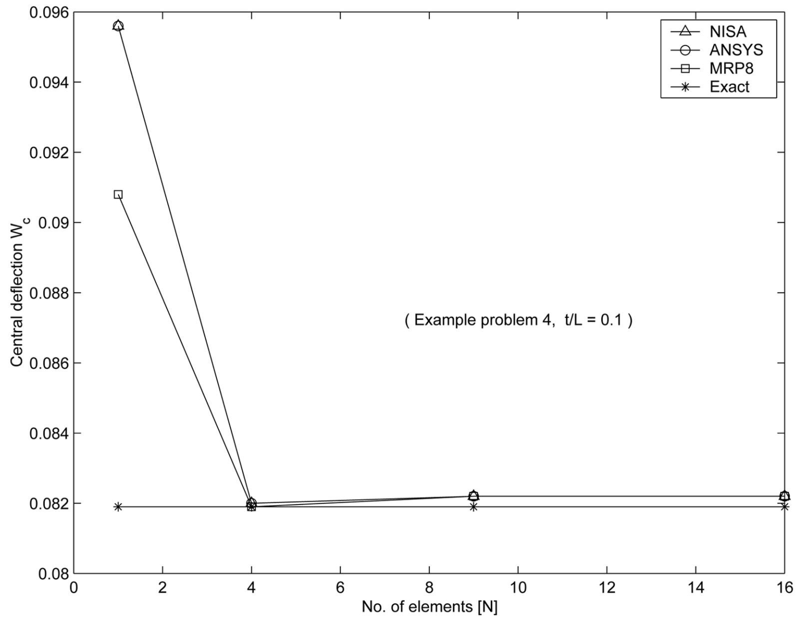

Fig. 14 Central deflection for a clamped moderately thick square plate with uniform load

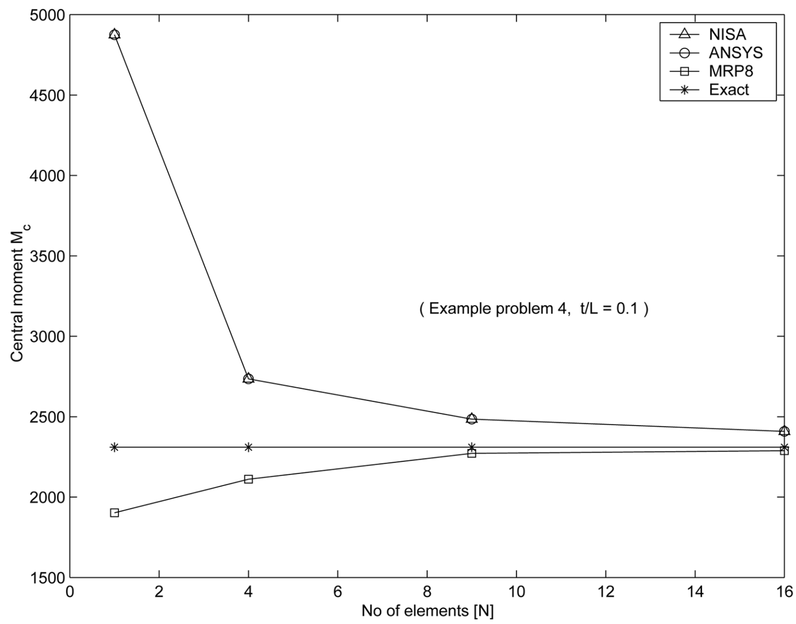

Fig. 15 Central moment for a clamped moderately thick square plate with uniform load

In all example problems 1-4, Tables 3-14 and Figs. 4-15 indicate that closed form solutions of element equilibrium and flexibility matrices of the eight-node rectangular plate bending element MRP8 have, in general, produced excellent results.

4. Conclusions

Closed form solutions for equilibrium and flexibility matrices of eight-node rectangular plate bending element (MRP8) using Integrated Force Method are presented. The Mindlin-Reissner theory has been employed in the formulation which accounts the effect of shear deformation.

These closed form solutions are validated by analyzing large number of standard thin and moderately thick plate bending benchmark problems to obtain deflections and moments. The results are compared with those obtained from similar displacement-based eight node quadrilateral plate bending elements available in the literature. The results are also compared with the exact solutions.

Closed form solutions for equilibrium and flexibility matrices of eight-node rectangular plate bending element (MRP8) have produced, in general, excellent results in all example problems considered here. Therefore these closed form solutions can be used in the analysis of square or rectangular thin and moderately thick plate bending situations without any shear locking problem.

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Appendix A: Basic theory of IFM

In the Integrated Force Method of analysis , a structure idealized by finite elements is designated as "*structure(n, m)*". Where (n, m) are force and displacement degrees of freedom of the discrete model, respectively. The structure (n, m) has m equilibrium Equations and $r = (n - m)$ compatibility conditions. Equilibrium equations (EE) represent the vectorial summation of the internal forces $\{F\}$ to the external loads $\{P\}$ at the nodes of the finite element descritization. It can be written in symbolized matrix notation as

$$\text{Equilibrium Equations[EE]} : [B]\{F\} = \{P\} \tag{A.1}$$

Where $[B]$ = global equilibrium matrix
$\{F\}$ = Vector of internal forces of the structure
$\{P\}$ = vector of external loads on the structure

The Compatibility Conditions (CC) are constraints on strains, and for finite element models they are also constraints on member deformations.

In IFM, St. Venant's approach has been extended for discrete mechanics to develop the compatibility conditions. Development of CC is briefly explained below:

The Deformation-Displacement Relationship (DDR) for discrete mechanics is equivalent to the strain-displacement relationship in elasticity. The DDR for discrete analysis was obtained during the development of the variational energy formulation for the IFM [12]

According to work energy conservation theorem, the internal energy stored in the body in the structure is equal to the work done by the external load, that is,

$$\frac{1}{2}\{F\}^T\{\beta\} = \frac{1}{2}\{P\}^T\{X\} \tag{A.2}$$

where $\{X\}$ represents nodal displacements. Eq. (A.2) can be rewritten by eliminating the load $\{P\}$ in favor of forces $\{F\}$, by using the Eq. (A.1) to obtain the following relation

$$\frac{1}{2}\{F\}^T[B]^T\{X\} = \frac{1}{2}\{F\}^T\{\beta\} \tag{A.3}$$

Eq. (A.3) can be simplified as

$$\frac{1}{2}\{F\}^T[[B]^T\{X\} - \{\beta\}] = 0 \tag{A.4}$$

Since $\{F\}$ is not a null vector, its coefficient must be equal to zero, which yields the DDR as

$$\{\beta\} = [B]^T\{X\} \tag{A.5}$$

Where $\{\beta\}$ are member deformations.

This equation represents the Deformation Displacement Relations (DDR) for the discrete structure. The elimination of m displacements from n deformations displacement relations given by the above equation yields $r = (n - m)$ compatibility conditions and the associated matrix [C].

Here while obtaining the matrix [C], no reference is made to redundant forces. Thus the concept of redundant force selection is eliminated in IFM.

It can be symbolized in matrix notations as

$$[C]\{\beta\} = 0 \tag{A.6}$$

Substituting Eq. (5) into the Eq. (A.6), we obtain

$$[C]\{\beta\} = [C][B]^T\{X\} = 0$$

$$\{X\}^T([B][C]^T) = \{0\}$$

Since the displacement vector $\{X\}$ is not a null vector, we have

$$[B][C]^T = 0 \tag{A.7}$$

where $[C]$ is the $(r \times n)$ compatibility matrix. It is a kinematics relationship, and it is independent of design parameters, material properties and external loads. This matrix is rectangular and banded. The deformation $\{\beta\}$ in the compatibility conditions (CC) given by the Eq. (6) represents the total deformation consisting of an elastic component $\{\beta_e\}$ and the initial component $\{\beta_o\}$ as

$$\{\beta\} = \{\beta_e\} + \{\beta_0\} \tag{A.8}$$

The CC in terms of elastic deformation can be written as

$$[C]\{\beta\} = [C]\{\beta_e\} + [C]\{\beta_0\} = 0 \tag{A.9}$$

$$[C]\{\beta_e\} = \{\delta R\}$$

Where $$\{\delta R\} = -[C]\{\beta_0\} \tag{A.10}$$

Using the flexibility characteristics, Eq. (A.6) with initial deformations can be rewritten as

$$[C][G]\{F\} = \{\delta R\} \tag{A.11}$$

Combining Eqs. (A.1) and (A.11), we lead to the IFM governing equation as

$$\begin{bmatrix} [B] \\ ([C][G]) \end{bmatrix} \{F\} = \begin{Bmatrix} P \\ \delta R \end{Bmatrix}$$

$$[S]\{F\} = \{P^*\} \tag{A.12}$$

Notations

$[A]$	: matrix relating nodal degrees of freedom and coefficients of the polynomial
$[B]$	: global equilibrium matrix ($m \times n$)
$[B_e]$	: element equilibrium matrix ($m_e \times n_e$)
$[C]$	: compatibility matrix ($r \times n$)
$[D_{op}]$	: differential operator matrix
E	: Young's modulus
$\{F\}$	: vector of internal forces of the structure ($n \times 1$)
$\{F_e\}$	: vector of internal forces of the discrete element ($n_e \times 1$)
$[G]$	: global flexibility matrix ($n \times n$)
$[G_e]$	: element flexibility matrix ($n_e \times n_e$)
$[H]$	: matrix relating the curvatures to stress resultants
$[J]$	: deformation coefficient matrix ($m \times n$)
L, B	: Length and breadth of the plate
M_c	: central moment of the plate
$\{M\}$	: vector of stress resultants
P	: point load at the center or tip of the plate
$\{P\}$	: vector of external loads ($m \times 1$)
q	: uniform load over the plate
$[S]$	: IFM governing matrix ($n \times n$)
W_c	: Central deflection of the plate
$\{X\}$	: vector of displacements of the structure ($m \times 1$)
$\{X_e\}$	: vector of displacements of the discrete element ($m_e \times 1$)
$\{k\}$	: vector of curvatures
n, m	: force and displacement degrees of freedom of the structures respectively
n_e, m_e	: element force and displacement degrees of freedom respectively
t	: thickness of the plate
$\{\alpha\}$	: generalized coordinates of the polynomial in the displacement field
$\{\beta\}$	: vector of elastic deformations
$\{\beta_0\}$	: vector of initial deformations
ν	: Poisson's ratio
$[\phi_1]$	: matrix of polynomial terms for displacement fields
$[\psi]$	: matrix of polynomial terms for stress-resultants fields