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Modelling of strain localization in a large strain context

S. Cescotto†
M.S.M Department, University of Liège, Quai Banning, 6, B-4000 Liege, Belgium

X. K. LI
National Laboratory for Structural Analysis of Industrial Equipments, Dalian University of Technology, Dalian 116024, China

Abstract. In order to avoid pathological mesh dependency in finite element modelling of strain localization, an isotropic elasto-plastic model with a yield function depending on the Laplacian of the equivalent plastic strain is implemented in a 4-node quadrilateral finite element with one integration point based on a mixed formulation derived from Hu-Washizu principle. The evaluation of the Laplacian is based on a least square polynomial approximation of the equivalent plastic strain around each integration point. This non local approach allows to satisfy exactly the consistency condition at each integration point. Some examples are treated to illustrate the effectiveness of the method.

Key words: localization; large strains; gradient plasticity.

1. Introduction

The theoretical and experimental study of strain localization as well as its numerical modelling have been given much attention in the past. From the theoretical point of view, for rate insensitive materials, localization is interpreted as a bifurcation phenomenon (Rice and Rudnicki 1980): if, for a given homogeneous strain state, the material constitutive equation allows for a non homogeneous solution to develop, a material instability occurs and strain localization can take place. The mathematical theory of bifurcation has been used extensively to analyse strain localization, first for relatively simple elastic, rigid plastic or classical elasto-plastic constitutive laws and subsequently for more complex material behaviour such as dilatant and pressure sensitive materials, non smooth yield surface, non standard materials, non linear incremental constitutive laws, etc...

The most fundamental results of theoretical studies are:

- with the help of bifurcation theory, it is possible to analyse the capacity of a given constitutive equation to allow for strain localization or not.
- for rate intensitive materials under quasi-static loading, if a classical (local) continuum formulation is used for the material constitutive law, the governing equations lose ellipticity when localization occur leading to a mathematically ill-posed problem; in particular, the thickness of the localized zone tends to zero and the energy dissipation vanishes (Bazant and Belyts-

† Professor

chko 1985).

This is in contradiction with experiments which have been extensively performed, specially on geomaterials (Desrues 1984, Vardoulakis 1979 and many others).

In finite element simulations, this loss of ellipticity induces a pathological mesh dependency of the numerical solution: according to the finite element type and the mesh refinement, shear bands may or may not occur and, when they occur, their orientation and their width can be dramatically modified by the meshing (Wang 1993, Sluys 1992).

Many attempts have been made to remedy this situation.

One of them consists in developing "enhanced" or "enriched" finite elements (Steinman and Williams 1991, Belyschko and Fish 1981) but this does not tackle the basic problem of loss of ellipticity.

A sound solution is to modify the formulation of the constitutive law of the material in such a way that, even when localization occurs, the problem remains well posed.

Different ways to achieve this have been followed.

The non-local approach basically consists in defining non local variables q_{nl} by a weighted average over a volume V_l of material with characteristic dimension l

$$q_{nl}=\int_{V_l} q\, w\, dV$$

with w the weighting function.

The non local variable is then used in the usual constitutive equation instead of the local one q. This method naturally introduces a characteristic length l in the formulation, which proves to be an essential feature to preserve the well posedness of the continuum problem during localization.

The choice of the variable q on which averaging is applied differs from one author to another: it can be the strains, the equivalent plastic strain (Lasry and Belyschko 1988), a damage variable (Bazant and Pijaudier-Cabot 1988).

The use of the Cosserat continuum theory is another interesting approach in which the introduction of couple-stress in addition to the classical stress components constitutes a very elegant way to induce an internal length in the material constitutive law (Mühlhaus 1989). A shortening of this approach is that there are no couple-stress effects in pure uniaxial loading so that the above mentioned drawbacks of the classical theory are still present for this type of situation.

Several authors have used the classical continuum theory but have included some viscous effects in the material constitutive equation to try to make the solution mesh independent. Sometimes, a classical Perzyna visco-plastic formulation is used; sometimes, a Duvaut-Lions regulation is added to the elasto-plastic constitutive law (Zhu 1992, Sluys, Block and de Borst 1992). It appears that the mesh independency can be achieved but is conditional to the amount of viscosity introduced in the model.

Finally, gradient plasticity is a very effective way to avoid pathological mesh dependency in case of localization (de Borst and Mühlhaus 1992, Mühlhaus and Aifantis 1991, etc.)

The basic idea is to introduce some gradients of the pertinent state variables (for example the equivalent plastic strain in pressure independent solids, the volumetric strain in dilatant materials, etc.) in the formulation of the elasto-plastic or elasto-visco-plastic equations. These gradients introduce a length scale in the material model and can restore the ellipticity of the governing equations when localization occurs.

It is to this category that the present paper belongs.

An isotropic elasto-plastic model with a yield function depending on the Laplacian of the equivalent plastic strain is implemented in a 4-node quadrilateral finite element with one integration point based on a mixed formulation derived from Hu-Washizu principle. The element is valid in the domain of large strains.

The evaluation of the Laplacian is derived from a least square polynomial approximation of the equivalent plastic strain around each integration point. This non local approach allows to satisfy exactly the consistency condition at each integration point and at each iteration of the step by step procedure used to solve the materially and geometrically non linear problem.

2. The mixed finite element

The finite element used in this work (Jetteur and Cescotto 1991) is based on the Hu-Washizu principle, has only one integration point and, thanks to a corotational formulation, is valid for both large rotations and large plane strains.

The velocity of a material point inside an element is given by a classical bilinear function of ξ, η (natural coordinates)

$$\dot{u}_i = a_o^i + a_j^i x_j + a_3^i \frac{A}{4} \xi \eta \tag{1}$$

Repeated subscripts imply summation over the range of those subscripts. A is the area of the element, x_j ($j=1, 2$) or equivalently (x, y) are the current coordinates in the deformed configuration of the element.

For strain rates and stresses, the following assumptions are chosen.

$$\begin{bmatrix} \dot{\varepsilon}_x \\ \dot{\varepsilon}_y \\ \dot{\gamma} \end{bmatrix} = \begin{bmatrix} \bar{\dot{\varepsilon}}_x \\ \bar{\dot{\varepsilon}}_y \\ \bar{\dot{\gamma}} \end{bmatrix} + \begin{bmatrix} h_{,x} & -h_{,y} \\ -h_{,x} & h_{,y} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\varepsilon}_1 \\ \dot{\varepsilon}_2 \end{bmatrix} = \underline{\bar{\dot{\varepsilon}}} + \underline{h}_{,\alpha} \underline{\dot{\varepsilon}}^x \tag{2}$$

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau \end{bmatrix} = \begin{bmatrix} \bar{\sigma}_x \\ \bar{\sigma}_y \\ \bar{\tau} \end{bmatrix} + \begin{bmatrix} h_{,x} & -h_{,y} \\ -h_{,x} & h_{,y} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix} = \underline{\bar{\sigma}} + \underline{h}_{,\alpha} \underline{\sigma}^x \tag{3}$$

A superposed bar indicates a constant field. The stresses are Cauchy stresses.

For classical plasticity, a von Mises yield surface with isotropic hardening is adopted:

$$\sigma_{VM} - Y(\varepsilon^p) = 0 \tag{4}$$

where $Y(\varepsilon^p)$ is the yield limit depending on the equivalent plastic strain only and σ_{VM} is the average von Mises stress on the element given by

$$\sigma_{VM}^2 = \frac{1}{A} \int \frac{3}{2} \underline{s}^T \underline{s} \, dA \tag{5}$$

with $\underline{s}$ the deviatoric stresses.

3. The gradient plasticity formulation

Instead of the local formulation of plasticity given by Eq. (4), we assume, as in de Borst and Mühlhaus (1992), that the yield limit Y is not only a function of ε^p but also of its Laplacian $\nabla^2 \varepsilon^p$

$$Y(\varepsilon^p,\ \nabla^2 \varepsilon^p) = \bar{Y}(\varepsilon^p) - c\ \nabla^2 \varepsilon^p \tag{6}$$

with c a positive material constant

In case of linear hardening for example, we have

$$\bar{Y} = Y_o + H'\ \varepsilon^p \tag{7}$$

(H' is positive for hardening and negative for softening).

4. Calculation of the Laplacian

To compute $\nabla^2 \varepsilon_J^p$ at an integration point J, we take account of the values of ε^p at the neighbouring integration points. Let s be the number of neighbouring points considered (including J). The Laplacian of ε^p at J is approximated by the formula

$$\nabla^2 \varepsilon_J^p = \sum_{I=1}^{s} g_{JI}\, \varepsilon_I^p \tag{8}$$

where the g_{JI} coefficients are computed as explained hereafter

For an interior element (Fig. 1), it is assumed that the evolution of ε^p around point J can be approximated by a second degree polynomial

$$\varepsilon^p = \underline{a}^T \underline{v} \tag{9}$$

with

$$\underline{a}^T = \langle a_1 a_2 a_3 a_4 a_5 a_6 \rangle \tag{10}$$

$$\underline{v}^T = \langle 1\ x\ y\ x^2\ xy\ y^2 \rangle \tag{11}$$

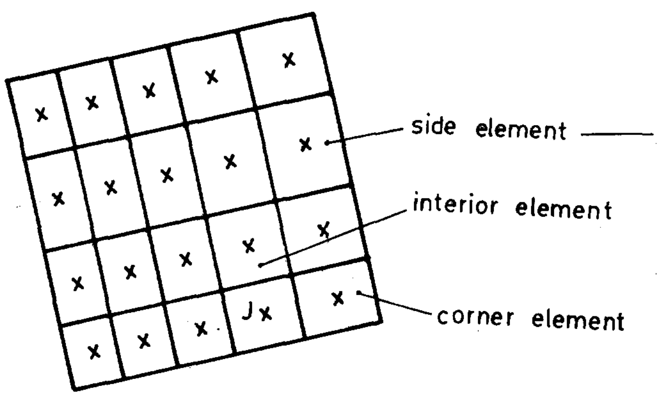

Fig. 1

The components of $\underline{a}$ are obtained by minimization of

$$\sum_{I=1}^{s} (\varepsilon^p - \varepsilon_I^p)^2 \tag{12}$$

This gives a result of the form

$$\underline{G}\,\underline{a} = \underline{M}\,\underline{b} \tag{13}$$

with

$$\underline{G} = \sum_{I=1}^{s} \underline{v}_I \underline{v}_I^T;\ \underline{M} = [\underline{v}_1 \underline{v}_2 \ \cdots\cdots\ \underline{v}_s];\ \underline{b} = \langle \varepsilon_1^p\, \varepsilon_2^p\ \cdots\ \varepsilon_s^p \rangle \tag{14}$$

Hence

$$\underline{a} = \underline{G}^{-1}\underline{M}\,\underline{b} = \underline{G}^{-1}[\varepsilon_1^p \underline{v}_1\, \varepsilon_2^p \underline{v}_2\ \cdots\ \varepsilon_s^p \underline{v}_s] \tag{15}$$

Since

$$\nabla^2 \varepsilon^p = \frac{\partial^2 \varepsilon^p}{\partial x^2} + \frac{\partial^2 \varepsilon^p}{\partial y^2} = 2a_4 + 2a_6 \tag{16}$$

let

$$\underline{g}^T = 2[(\text{row } 4 \text{ of } \underline{G}^{-1}) + (\text{row } 6 \text{ of } \underline{G}^{-1})] \tag{17}$$

Then

$$g_{JI} = \underline{g}^T \underline{v}_I \tag{18}$$

For a side or a corner element, a similar procedure is used.

The choice of the neighbouring integration points can be done automatically by a simple analysis of the mesh topology, which is very easy with the element used.

However, from Eq. (14), it is clear that matrix $\underline{G}$ must be regular. For an interior point, a necessary but not sufficient condition it that $s \geq 6$.

Usually (Fig. 2a), the integration points used in Eq. (8) will be point J and the integration point of the 8 direct neighbours.

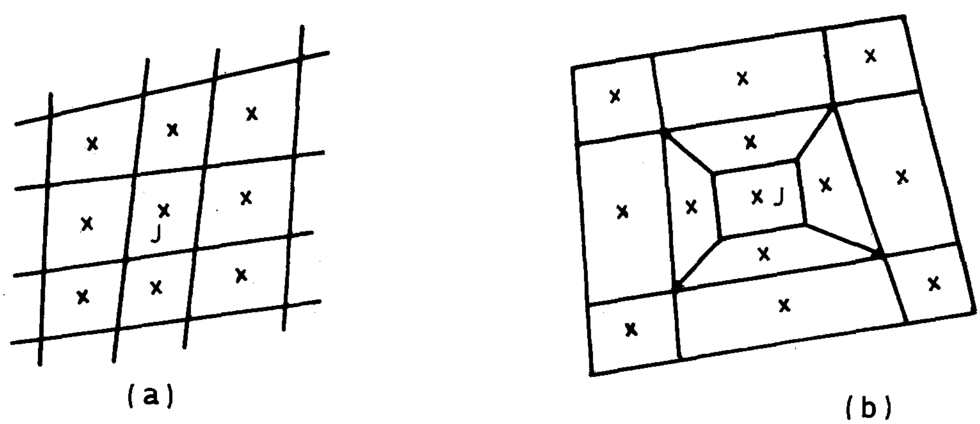

Fig. 2

So, the list of neighbouring integration points will contain $s=9$ points.

However, for the topology of Fig. 2b, there are only 4 direct neighbours so that condition $s \geq 6$ is not satisfied. In that case, the indirect neighbours (the neighbours of the direct neighbours) must also be included in the list. For the case of Fig. 2b, we will have $s=13$.

Note that it is possible reduce the size of this list by elimination of some of the indirect neighbours (for example by considering the distance to point J as a selection criterium).

5. Numerical considerations

From the basic ideas presented above, it is possible to develop the non local approach of strain localization.

Some numerical aspects deserve some attention.

A radial return technique is used to integrate stresses over a loading increment. However, since the yield limit at a given integration point is influenced by the plastic strains in the neighbourhood, a semi-local iteration procedure must be used to enforce the consistency condition (the word "semi-local" is used because these iterations involve a limited number of integration points around the one at which stresses are computed).

The consistent compliance matrix at a given integration point is also influenced by its neighbours. Since they belong to other elements, this means that the element stiffness matrix is no longer local. In fact, it can be shown that it involves all the degrees of freedom of all the elements but that it can be approximated in such a way that only the degrees of freedom of the direct neighbouring elements are involved. Clearly, this increases the bandwidth of the global equation system. For conciseness, all these numerical aspects are skipped in this paper. The interested reader can refer to Li and Cescotto (1995) for more details.

6. Examples of application

In order to show that pathological mesh dependence is overcome by the present method, we consider a rectangular bar with initial length $L_0=5$ and initial width B_0, meshed by $N \times M$ four noded rectangular mixed elements, shown in Fig. 3. The bar fixed at the bottom and loaded by an increasing prescribed horizontal displacement at the top, is analyzed as a plane strain problem. In order to have a pure shear problem, the vertical displacements of all the nodes are prevented and the horizontal displacements of nodes pertaining to the same horizontal line are imposed to be equal.

Four test cases with different dimensions in x coordinate and element meshes are specified as: Case 1: $B_0=3.0$, $M=5$, $N=3$; Case 2: $B_0=1.75$, $M=15$, $N=5$; Case 3: $B_0=1.0$, $M=25$, $N=5$; Case 4: $B_0=0.7$, $M=35$, $N=5$. The values of material elastic properties used for the entire bar are $E=210000$ and $\mu=0.0$. The initial yield strength is $\sigma_{y0}=240$ except for the elements which are between $y=-0.5$ and $y=0.5$. The initial yield strength for these elements has been weakened to the value of $\sigma_{y0}=230$. As the effective stress at a material point reaches to σ_{y0}, a softening modulus $H'=-10000$ is used for all elements, including the weakened elements.

The non local materiel parameter $c=2000$

Numerical results show that the thickness of the localization zone and the effective plastic

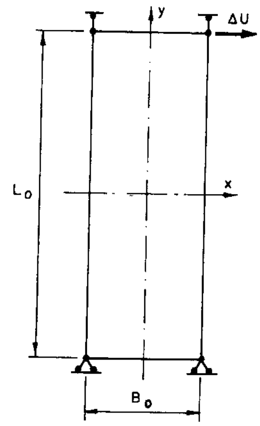
y
ΔU
x
L_0
B_0

Fig. 3

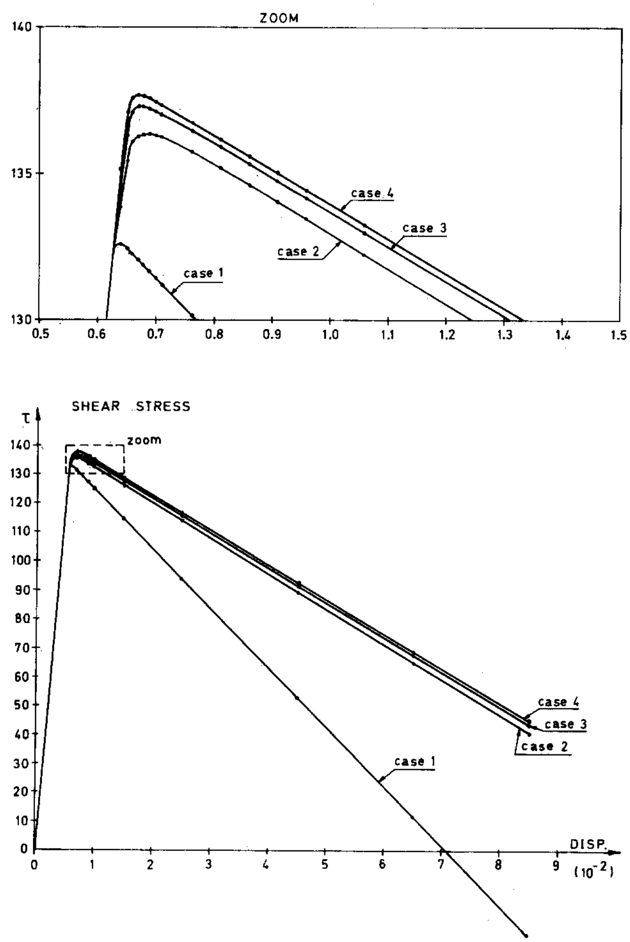
ZOOM
case 4
case 3
case 2
case 1
SHEAR STRESS
τ
zoom
DISP.
(10⁻²)

Fig. 4

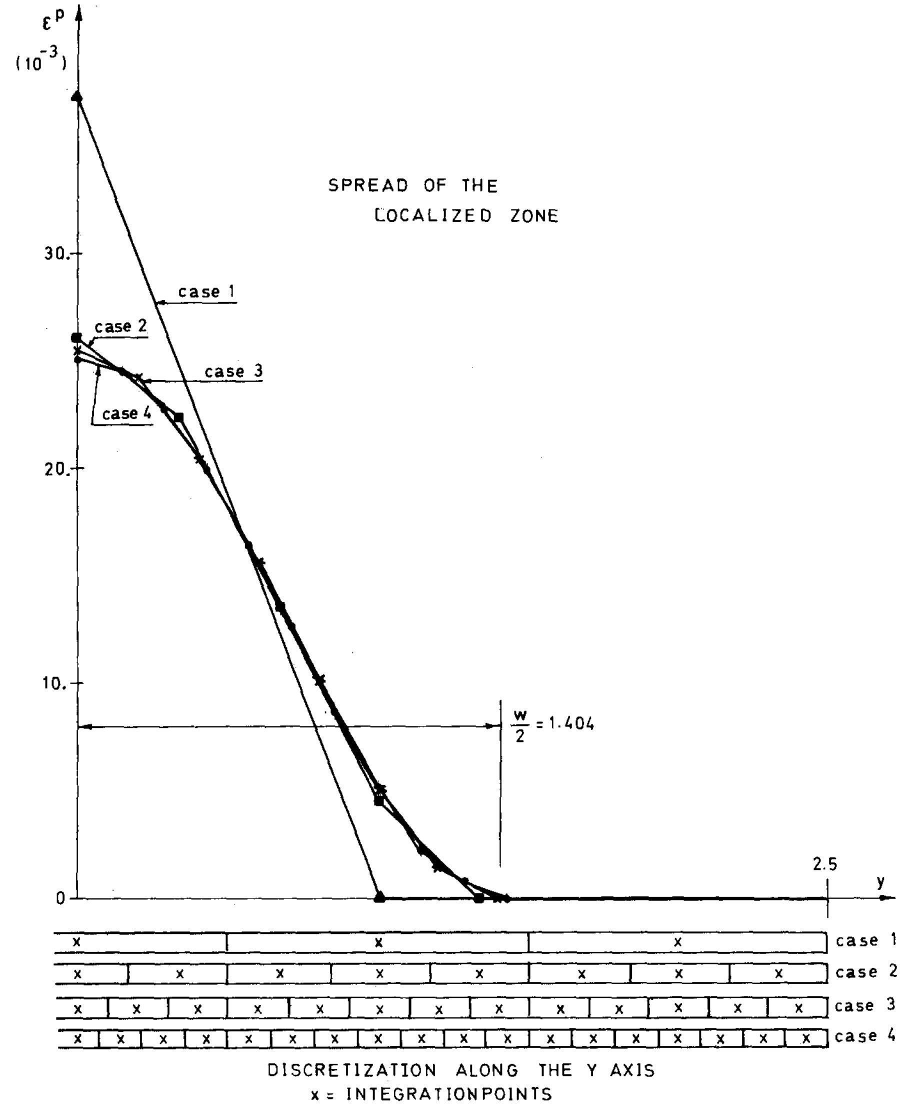

Fig. 5

strain distribution along y axis over the zone rapidly converge to a unique solution, which is independent of the finite element mesh and is only related to the softening modulus H' and the non-local material parameter c. This is illustrated in Fig. 4 and 5. Fig. 4 illustrates the convergence of the shear stress-displacement (at the nodes on the top face) curves. The solution for 3×5 element mesh is soft somewhat, however, the solutions for the other three cases with 5×15, 5×25, 5×35 elements converge to a unique solution. Particularly, the curves for 5×25 and 5×35 element meshes almost overlap. Fig. 5 illustrates the effective plastic strain distributions along y axis for a prescribed horizontal displacement $\Delta u=8.56\times10^{-2}$ at the top face of the bar. Because of the symmetry only the distribution on the half of the bar is illustrated. It is observed that the effective plastic strain profiles for 5×15, 5×25 and 5×35 element meshes almost coincide, though the profile for 3×5 element mesh deviates somewhat from the converged solution.

The half width of the localized zone has a theoretical value of $w/2=1.404$ also indicated on Fig. 5. The agreement with the numerical results is seen to be excellent.

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