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Use of homogenization theory to build a beam element with thermo-mechanical microscale properties

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Abstract. The homogenization method is used to develop a beam element in space for thermo-mechanical analysis of unidirectional composites. Local stress and temperature field in the microscale are described using the function of homogenization. The global (macroscopic) behaviour of the structure is supposed to be that of a beam. Beam-type kinematical hypotheses (including independent shear rotations) are hence applied and superposed on the microdescription. A macroscopic stiffness matrix for such a beam element is then developed which contains the microscale properties of the single cell of periodicity. The presented model enables us to analyse without too much computational effort complicated composite structures such as e.g. toroidal coils of a fusion reactor. We need only a FE mesh sufficiently fine for a correct description of the local geometry of a single cell and a few of the newly developed elements for the description of the global behaviour. An unsmearing procedure gives the stress and temperature field in the different materials of a single cell.

Key words: composite materials; beams; homogenization; finite element method; superconducting magnets.

1. Introduction

It is well known that the limitation of the computational power has usually an important influence on the models used for the description of mechanical behaviour of structures. This means that the applied theory, the structural model, and the discretisation should be carefully chosen to reflect correctly this behaviour and to make the numerical task not to heavy. Use of the double scale asymptotic theory of homogenization for the analysis of strongly heterogeneous materials has advantages which are important from the point of view of the economy of the computational process. The algorithm of the analysis naturally is split into three main steps: microanalysis, macroanalysis and unsmearing. These steps are independent in the sense that two different FE programs solve in turn two different boundary value problems operating on much smaller problems than the initial one. The geometrical domain of the microanalysis consists

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of a small representative volume cut out from the composite body. When the structure is very regular, as in the considered example of application, the cell of periodicity can be taken as such an elementary representative volume. The periodic boundary conditions are postulated for the analysis of the single cell of periodicity. The microanalysis defines the effective material coefficients starting from known geometry and given properties of components. The theory of homogenization yields the constitutive law valid on the macrolevel, starting from the microdescription of the problem (without any a-priori hypothesis about the macro-behaviour). This is the fundamental idea of the micromechanical approach adopted. On the macro-level (second step) we deal with the homogenized body subjected to the loads and to the boundary conditions defined for the global problem. At this stage of analysis any commercial FE code can be used. In this paper, however, we define a special beam type element which is suitable for unidirectional composites and allows us to reduce the number of elements used for the global analysis of the homogenized structure. The third step of the homogenization process, the unsmearing, results in stress and temperature recovery on the micro-level of the composite. The microdescription of the single cell of periodicity, the functions of homogenization obtained during the first step as well as the strains and temperature computed for the neighbourhood of the cell of interest resulting from the macroanalysis are taken into consideration to this end.

The microanalysis and unsmearing processes are briefly outlined in this paper. Attention is focused on the construction of the beam-like finite element for a layered beam or beam with parallel fibres compatible with the homogenization approach. The method is then applied to the full analysis of a particular superconducting coil used in thermonuclear fusion devices.

Such superconducting toroidal coils are made up of a large number of parallel fibres (superconductors), bonded by insulating material. In the cross-section of this superconducting beam we may distinguish a two dimensional cell of periodicity with characteristic, repeatable physical and geometrical properties. The whole material domain is constructed with such cells periodically repeated. The structure may hence be considered as an unidirectional composite with periodic structure.

1.1. Homogenization procedure

Let us consider the classical steady state problem of uncoupled thermoelasticity formulated for the composite body. The following equations define this problem:

$$\sigma_{ij,j}+f_i=0 \tag{1}$$

$$q_{i,i}-r=0 \tag{2}$$

$$\sigma_{ij}=a_{ijkl}e_{kl}-\alpha_{ij}\theta \tag{3}$$

$$q_i=-K_{ij}T_{,j} \tag{4}$$

$$u^{\varepsilon}(x)=0 \text{ on } \partial\Omega_1 \text{ and } \sigma^{\varepsilon}_{ij}(\boldsymbol{x})n_j=F_i \text{ on } \partial\Omega_2 \tag{5}$$

$$[u^{\varepsilon}(\boldsymbol{x})]=0 \ [\sigma^{\varepsilon}_{ij}(\boldsymbol{x})n_j]=0 \text{ on } S_J \tag{6}$$

$a_{ijkl}(\boldsymbol{x})$ tensor of elasticity
$K_{ij}(\boldsymbol{x})$ tensor of thermal conductivity
$\alpha_{ij}(\boldsymbol{x})$ thermal expansion

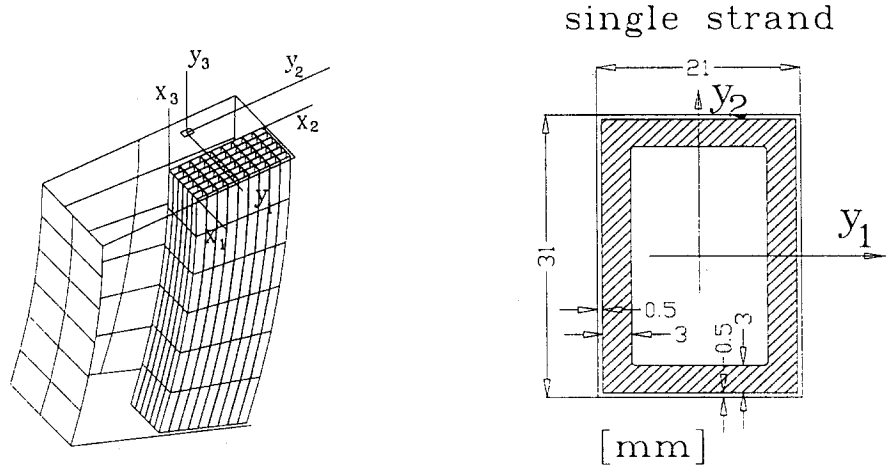

Fig. 1 Beam cross-section and the single cell of periodicity.

$\sigma_{ij}(x)$, $u_i(x)$	stress tensor and displacement vector
$T(x)$, $q_i(x)$	temperature and heat flux respectively
$r(x)$, $f_i(x)$	thermal sources and mass force respectively.
$[f]$	represents the jump of a function f.

Of course all the differentiations (above and throughout the paper) are to be understood in the weak sense and will be replaced by equivalent variational formulations for detailed analysis. We assume also that all usual requirements for the correctness of formulation (1)-(6) hold.

The separation of scales between macro and micro phenomena assumed above is formally expressed by the following relation between two systems of coordinates x and y (see Fig. 1):

$$y_\alpha = \varepsilon^{-1} x_\alpha \text{ and } y_3 = x_3 \tag{7}$$

We have placed the y_3 and x_3 axes in the direction parallel to the central line of the single conductor which forms the two dimensional cell of periodicity in space. Here and in the sequel the greek subscripts take values 1 or 2. We assume that ε can play in our problem the role of the small parameter, required by the homogenization theory.

We suppose further that the periodicity of material characteristics imposes an analogous periodical perturbation on the studied quantities describing the mechanical and thermal behaviour of the body. We will use hence the following representation for displacements, stresses, temperature and heat flux:

$$\boldsymbol{u}^\varepsilon(x) \equiv \boldsymbol{u}^0(x) + \varepsilon \boldsymbol{u}^1(x, y) + \varepsilon^2 \boldsymbol{u}^2(x, y) + \cdots + \varepsilon^k \boldsymbol{u}^k(x, y) \tag{8}$$

$$\sigma^\varepsilon(x, y) \equiv \sigma^0(x, y) + \varepsilon \sigma^1(x, y) + \varepsilon^2 \sigma^2(x, y) + \cdots + \varepsilon^k \sigma^k(x, y) \tag{9}$$

$$T^\varepsilon(x) \equiv T^0(x) + \varepsilon T^1(x, y) + \varepsilon^2 T^2(x, y) + \cdots + \varepsilon^k T^k(x, y) \tag{10}$$

$$\boldsymbol{q}^\varepsilon(x, y) \equiv \boldsymbol{q}^0(x, y) + \varepsilon \boldsymbol{q}^1(x, y) + \varepsilon^2 \boldsymbol{q}^2(x, y) + \cdots + \varepsilon^k \boldsymbol{q}^k(x, y) \tag{11}$$

where $\boldsymbol{u}^k$, σ^k, T^k, $\boldsymbol{q}^k$ for $k>0$ are Y-periodic i.e., take the same values on the opposite sides of the cell of periodicity. We describe now briefly the procedure of homogenization, which results from the assumed idealization.

By the simple application of the chain rule of differential calculus, we can see that the main

terms of σ and q (zero order terms) depend not only on $\boldsymbol{u}^0$, T^0 but also on $\boldsymbol{u}^1$, T^1 respectively. These terms may be written as follows:

$$\sigma_{ij}^0(\boldsymbol{x}, \boldsymbol{y}) = a_{ijkl}(\boldsymbol{y})(u^0_{k,l(x)} + u^1_{k,l(y)}) \tag{12}$$

$$q_k{}^0(\boldsymbol{x}, \boldsymbol{y}) = K_{kl}(\boldsymbol{y})(T^0_{,l(x)} + T^1_{,l(y)}) \tag{13}$$

We use here and in the sequel a shorthand explained below:

$$\frac{d}{dx_i} f = \left(\frac{\partial}{\partial x_i} + \frac{1}{\varepsilon} \frac{\partial}{\partial y_i} \right) f = f_{,i(x)} + \frac{1}{\varepsilon} f_{,i(y)} \tag{14}$$

By introducing into the principle of virtual work associated with the problem (1)-(6), Eqs. (8), (9) restricted up to the first term and taking a suitable form for the trial function we arrive at the following variational formulation:

$$\text{find } \chi_i^{pq} \in V_Y \text{ such that: } \forall\, v_i \in V_Y$$

$$\int_Y a_{ijkl}(\boldsymbol{y})(\delta_{ip}\delta_{jq} - \chi^{pq}_{i,j(y)}(\boldsymbol{y}))\, v_{k,l(y)}(\boldsymbol{y})\, d\Omega = 0 \tag{15}$$

where V_Y is the subset of V containing functions with equal traces on the opposite sides of the cell of periodicity. In the above formula the six functions $\chi_i^{pq}(y)$ depending only on the geometry of the cell of periodicity and on the values of jumps of a_{ijkl} across the interfaces between materials, are called functions of homogenization (detailed theory can be found in (Sanchez-Palencia E. 1980).

For a beam with parallel fibres this local problem is decoupled into the classical plane strain problem of linear elasticity and the antiplane problem. This decomposition is described in (Lefik and Schrefler 1992). It can be shown that u^1, the solution of problems Eq. (15), is of the form:

$$u_3{}^1(\boldsymbol{x}, \boldsymbol{y}) = 1/2(u^0_{\alpha,3(x)}(\boldsymbol{x}) + u^0_{3,\alpha(x)}(\boldsymbol{x}))\chi_3{}^{\alpha 3}(\boldsymbol{y}) + C(\boldsymbol{x}) \tag{16}$$

$$u_\gamma{}^1(\boldsymbol{x}, \boldsymbol{y}) = 1/2(u^0_{\alpha,\beta(x)}(\boldsymbol{x}) + u^0_{\alpha,\alpha(x)}(\boldsymbol{x}))\, \chi_\gamma{}^{\alpha\beta}(\boldsymbol{y}) + u^0_{3,3(x)}(\boldsymbol{x})\, \chi_\gamma{}^{33}(\boldsymbol{y}) + C_\gamma(\boldsymbol{x}) \tag{17}$$

The variational formulation for the first order term of the temperature in Eq. (10) is the folowing:

$$\text{find} \quad {}^p \in V_Y \text{ such that: } \forall\, \varphi \in V_Y$$

$$\int_Y K_{ij}(\boldsymbol{y})(\delta_{ip} + \vartheta^p_{,j(y)}(\boldsymbol{y}))\, \varphi_{,l(y)}(\boldsymbol{y})\, d\Omega = 0 \tag{18}$$

The above three scalar functions ϑ^p depend only on the geometry of the cell of periodicity and on the values of the jumps of material coefficients across S_J.

Eqs. (15) and (18) define the first subproblem of the analysis, namely microbehaviour. Introducing the Eqs. (12), (13), into Eqs. (3), (4) and integrating over Y we get the effective material coefficients:

$$D^h_{ijpq} = |Y|^{-1} \int_Y a_{ijkl}(\boldsymbol{y})(\delta_{kp}\delta_{lq} - \chi^{pq}_{k,l(y)}(\boldsymbol{y}))\, dY \tag{19}$$

$$k^h_{ij} = |Y|^{-1} \int_Y k_{ip}(\boldsymbol{y})(\delta_{jp} + \quad {}^j_{,p}(\boldsymbol{y}))\, dY \tag{20}$$

$$\alpha_{ij}^{h}=|Y|^{-1}\int_{Y}\alpha_{ij}(y)\,dY \tag{21}$$

In this way the parameters for the macrodescription are found.

These effective coefficients are valid for a 3D problem of uncoupled thermoelasticity. In the following part of the paper we will specialise the outlined procedure to the case of beam-like behaviour of the considered body. The simple integration over Y will then not be enough to define the effective behaviour in that case.

2. Definition of the equivalent homogeneous model of the beam

2.1. Assumptions

We show here briefly the macromechanical approach to derive an equivalent, homogeneous model of a beam (Lefik and Schrefler 1994). Since we assume that the global behaviour of the structure (macroscopically) is that of a beam we impose the beam-type kinematical constraints on the global displacements u and superpose it with the local perturbation obtained above. We further assume that the cell dimensions are small with respect to the cross-section diameter.

On the other hand, however, their number allows still to identify each particular cell with the coordinates of its centre without numerical troubles, i.e., the geometric description of the whole cross-section cell by cell-wise is possible.

We presume that displacements of the single fibre are constrained nearly everywhere inside of the beam cross-section. Y-periodicity conditions defined in preceding paragraphs are hence valid for most of the strands. Finally we assume that the local stresses and strains can be described using the above derived functions of homogenization with the exception, at most, for the boundary layer (for boundary layer see (Lefik and Schrefler 1996)).

2.2. Construction of the equivalent beam finite element for the composite structure

Let us suppose, that the 3D beam domain Ω is described by the definition of its main line $L=\langle 0, L\rangle\subset R^1$ (central axis of the beam) and the plane geometry of the cross-section $S\subset R^2$ in each point of the axis.

We consider the field of displacements in the form of Eq. (8) up to the term of first order:

$$u_k(x_i, y_\alpha)=u_k^0(x_i)+\varepsilon u_k^1(x_i, y_\alpha) \tag{22}$$

This first order approximation will allow us to define correctly the zero order term in the asymptotic expansion of stress. The global displacements u^0 in Eq. (22) may further be represented by a set of unknown functions defined along the axis of the beam:

$w_\alpha(x_3)$ the transversal displacement of the centroid of the beam,
$w_3(x_3)$ the axial displacement of the centroid of the beam
$\theta_\alpha(x_3)$ the independent component of the slope of the transversal displacement of the centroid of the beam (Timoshenko rotations)
$\theta_3(x_3)$ the twist angle
$b_i(x_3)$ three additional unknown functions defining the influence of w on the variation of

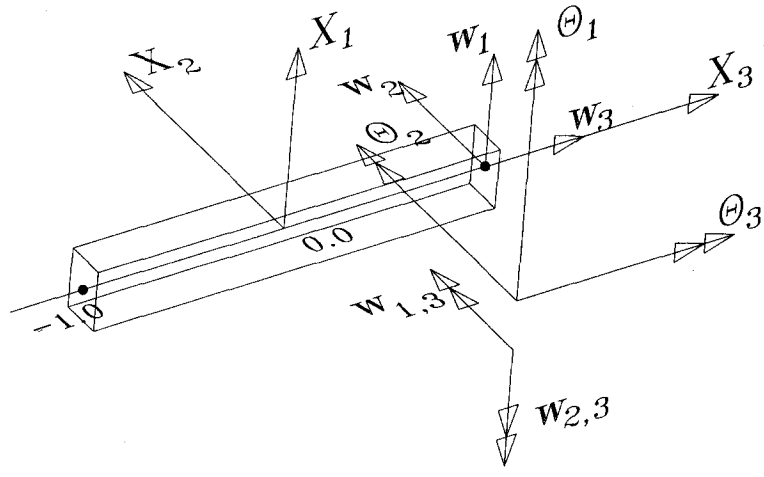

Fig. 2 Definition of the positive values of the unknowns.

w_α across the cross-section of the beam.

This representation is given by the following formulae:

$$u_\alpha^0(x_\alpha, x_3) = -\in_{\alpha\beta} x_\beta \theta_3(x_3) + w_\alpha(x_3) + f_{i\alpha}(x_\alpha)\, b_i(x_3) \tag{23}$$

$$u_3^0(x_\alpha, x_3) = \in_{\alpha\beta} x_\beta \theta_\alpha(x_3) - x_\alpha w_{\alpha,3}(x_3) + w_3(x_3) + \theta_{3,3}(x_3)\, \vartheta(x_\alpha) \tag{24}$$

where:

$$\in_{12} = 1 \quad \in_{21} = -1 \quad \in_{11} = 0 \quad \in_{22} = 0 \tag{25}$$

In Eq. (23) $\vartheta(x_\alpha)$ describes warping of the cross-section of the beam. The warping function is supposed to be depending only on the beam cross-section, thus can be calculated separately and is treated now as given. The reader is referred to (Lefik and Schrefler 1994) for an example of such a computation.

The set of functions $f_{i\alpha}$ will be chosen to make σ_{11} and σ_{22} minimal. This is commented in the sequel. Fig. 2 shows positive values of all the functions introduced above.

We define the three dimensional strain field as a full first order term in the asymptotic expansion analogous to Eq. (9), using the above derived homogenization functions:

$$e_{ij}^0(x_i, y_\alpha) = (\delta_{ip}\delta_{jq} - \chi_{i,j}^{pq}(y_\alpha))\, u_{(p,q)}^0(x_i) \tag{26}$$

First order of accuracy for stress and strain requires only the zero order temperature field to be prescribed for uncoupled thermoelasticity.

Since we assume steady state, it is enough to consider a temperature field constant over the cross-section or flux vector with components depending on x_3 only as a given thermal load. The three dimensional stress field is defined for each cell of periodicity as follows:

$$\sigma_{ij}^0(x_i, y_\alpha) = a_{ijkl}(y_\alpha)\, e_{kl}^0(x_i, y_\alpha) - \alpha_{ij}(y_\alpha)\, T^0(x_i) \tag{27}$$

The components of $\boldsymbol{e}^0$ are easy to obtain having χ the solution of local problem, Eq. (15). Both σ and e defined above depend only on $\boldsymbol{w}$, θ, $\boldsymbol{b}$. These functions are chosen to satisfy the stationary point of the full, three dimensional potential energy functional:

$$\Pi(\boldsymbol{w},\ \theta,\ \boldsymbol{b})=0.5\int_{\Omega} a_{ijkl}(y)\,e^{0}_{ij}(\boldsymbol{w},\ \theta,\ \boldsymbol{b})\,e^{0}_{kl}(\boldsymbol{w},\ \theta,\ \boldsymbol{b})\,d\Omega-R(\boldsymbol{w},\ \theta,\ \boldsymbol{b}) \tag{28}$$

where

$$R(\boldsymbol{w},\ \theta,\ \boldsymbol{b})=\int_{\Omega} f_i u_i^0(\boldsymbol{w},\ \theta,\ \boldsymbol{b})\,d\Omega+\int_{S_f} F_i u_i^0(\boldsymbol{w},\ \theta,\ \boldsymbol{b})\,dS+\int_{\Omega} \alpha_{ij}(y_\alpha)\,T^0(x_i)\,e^{0}_{ij}(\boldsymbol{w},\ \theta,\ \boldsymbol{b})\,d\Omega \tag{29}$$

The assumed field of displacements $\boldsymbol{u}$ is continuous over the whole beam cross-section by the Y-periodicity of χ.

2.3. The form of the additional term b

For the composite body the term $e^{0}_{\alpha\beta}(\boldsymbol{x},\ \boldsymbol{y})$ which enters into the full 3D functional of potential energy is not zero. It is composed of two constituents: the first one, due to the global displacement, is zero by assumption. The second one, an additive term resulting from the local perturbation on the cell of periodicity is not zero. Its value is:

$$e^{02}_{\alpha\beta}(\boldsymbol{x},\ \boldsymbol{y})=\chi^{33}_{\alpha\beta}(\boldsymbol{y})\,u^{0}_{3,3}(\boldsymbol{x}) \tag{30}$$

In this situation the work of the stress $\sigma_{(\alpha\alpha)}$ (perpendicular to the axis of the beam) which results from e_{33} via $a_{33\alpha\alpha}$ is not zero. Additional degrees of freedom have been chosen here to handle this extra term in the expression for the work of internal forces. The proposed method has some advantages (the obvious disadvantage is that we have two unknown functions more). Using these functions we have the possibility to introduce into the description of the problem in a natural way the effective orthotropy of the beam, which may be significant. Another advantage is that we can easily impose lateral constraints: e.g., by putting b_1, b_2 equal to zero we deal with zero values of strains in the direction perpendicular to the central axis.

To derive a suitable form for b we use here only the global material coefficients and neglect the local dependence on y hence:

$$\sigma_{\alpha\beta}=a^{h}_{\alpha\beta\gamma\delta}\,e^{0}_{\gamma\delta}+a^{h}_{\alpha\beta33}\,e^{0}_{33} \tag{31}$$

Hence, to have $\sigma_{\alpha\beta}=0$ we must write:

$$e^{0}_{\alpha\beta}=A^{h}_{\alpha\beta\gamma\delta}\,a^{h}_{r\delta33}\,e^{0}_{33}\quad A^{h}_{\alpha\beta\gamma\delta}\,a^{h}_{\gamma\delta33}=E^{h}_{\alpha\beta}\quad A^{h}_{\alpha\beta\gamma\delta}=(a^{h})^{-1}_{\alpha\beta\gamma\delta} \tag{32}$$

With these notations, one may easily verify that a suitable form for $f_{i\alpha}(x_\alpha)$ is the following:

$$f_{11}=-E^{h}_{11}x_1x_2\quad f_{12}=E^{h}_{11}\frac{x_1^{\,2}}{2}-E^{h}_{22}\frac{x_2^{\,2}}{2}\quad f_{13}=E^{h}_{11}x_1 \tag{33}$$

$$f_{22}=E^{h}_{22}x_1x_2\qquad f_{23}=E^{h}_{22}x_2 \tag{34}$$

For zero transversal stresses, i.e., $\sigma^{01}_{\alpha\beta}=0$, b takes the following values:

$$b_1x_3)=w_{2,33}(x_3)-\theta_{1,3}(x_3) \tag{35}$$

$$b_2(x_3)=-\theta_{2,3}(x_3)-w_{1,33}(x_3) \tag{36}$$

$$b_3(x_3)=w_{3,3}(x_3) \tag{37}$$

Other values of b allow to simulate a variety of intermediate states (Lefik and Schrefler 1994).

2.4. Numerical formulation of the proposed beam model

The code developed for the beam type element enables us to define an individual type of interpolation for each of the unknown functions. For $w_\alpha(x_3)$ we use the Hermitian interpolations over a one dimensional finite element parametrized with $-1<\xi<1$. For $w_3(x_3)$, $\theta_i(x_3)$ linear shape functions are used, but it is also possible to apply in this case the interpolation with higher order of continuity. To mark the different orders of approximations for different components of $w_i(x_3)$, $\theta_i(x_3)$ or $b_i(x_3)$ we label the corresponding shape function with superscripts, for example: $N^{wi}(\xi)$, $N^{\theta i}(\xi)$. The nodal degrees of freedom are:

$$\boldsymbol{v}^T=\{\{w_1^e\}\ \{\theta_2^e\}\ \{w_2^e\}\ \{\theta_1^e\}\ \{w_3^e\}\ \{\theta_3^e\}\ \{b_i^e\}\} \tag{38}$$

where the superscript e is used for the vector of the nodal values of unknowns in nodes of the element.

Vectors $\boldsymbol{w}^e$, $\boldsymbol{b}^e$, θ^e are of the form:

$$\{w_i^e\}=\{w_i^1\, w_{i,\xi}^1\ w_i^2\ \ w_{i,\xi}^2\},\ \{\theta_i^e\}=\{\theta_i^1\ \ \theta_i^2\},\ \{b_j^e\}=\{b_i^1\ \ b_i^2\} \tag{39}$$

where superscript 1 or 2 is used for the degrees of freedom in node 1 or 2 respectively. If needed the higher order derivatives in nodes are included in the same way. The interpolation via shape functions is defined over the element as follows:(there is no summation over repeated indices in the following)

$$\boldsymbol{w}_i(\xi)=\boldsymbol{N}^{wi}(\xi)\,\boldsymbol{w}_i^e\ \ \theta_i(\xi)=\boldsymbol{N}^{\theta i}(\xi)\,\theta_i^e\ \ \boldsymbol{b}_i(\xi)=\boldsymbol{N}^{bi}(\xi)\,\boldsymbol{b}_i^e \tag{40}$$

Derivatives are expressed by derivatives of shape functions which are all defined until the needed level of approximation:

$$\boldsymbol{N}_{,\xi}^{w\alpha}(\xi)\ \boldsymbol{N}_{,\xi\xi}^{w\alpha}(\xi)\ \boldsymbol{N}_{,\xi}^{\theta i}(\xi)\ \boldsymbol{N}_{,\xi}^{bi}(\xi) \tag{41}$$

2.5. Description of strains

We write Eqs. (23), (24) now for a single cell indicated with number s. The local set of coordinates y is placed in the plane of the cross-section of this cell of periodicity. Coordinates of the origin of y are x^s for this cell. For the single cell we have:

$$x_\alpha=x_\alpha^s+\varepsilon y_\alpha \tag{42}$$

We approximate also the warping function (and $f_{i\alpha}$ in the same way) on each conductor cross-section by a truncated series expansion:

$$\vartheta(x_\alpha,\ y_\alpha)=\ \vartheta(x_\alpha^s)+\varepsilon y_\gamma\,\vartheta_{,\gamma}(x_\alpha^s) \tag{43}$$

The assumed form of the global displacements (11), (12) permits us to make the following decomposition:

$$\boldsymbol{e}^0=(\mathbf{1}+\boldsymbol{L}\boldsymbol{X}(y_\alpha))(\boldsymbol{x}_{1e}(x_\alpha^s)\,\boldsymbol{B}_1(x_3)+\varepsilon\,\boldsymbol{r}_{ye}(y_\alpha,\ x_\alpha^s)\,\boldsymbol{B}_2(x_3))\,\boldsymbol{v}^e \tag{44}$$

Matrices $\boldsymbol{L}$, $\boldsymbol{X}$, $\boldsymbol{x}$, $\boldsymbol{r}$, $\boldsymbol{B}_1$, $\boldsymbol{B}_2$ are given in the appendix.

For the evaluation of Eq. (28) we introduce the stiffness matrix $\boldsymbol{K}$ by taking into account the form of strains Eq. (44) and performing the integration over Y:

$$\boldsymbol{K}=\int_L (\boldsymbol{K}_{01}+\varepsilon^2\boldsymbol{K}_{02}) \tag{45}$$

where:

$$\boldsymbol{K}_{01}=\boldsymbol{B}_1{}^T(x_3)\left(|Y|\sum_{s=1}^{S}\boldsymbol{x}_{1e}^T(x_\alpha^s)\,\boldsymbol{D}^h\boldsymbol{x}_{1e}(x_\alpha^s)\right)\boldsymbol{B}_1(x_3) \tag{46}$$

$$\boldsymbol{K}_{02}=\boldsymbol{B}_2{}^T(x_3)\left(\sum_{s=1}^{S}\int_Y \boldsymbol{r}_{ye}^T(y,\ x_\alpha^s)(\mathbf{1}+\boldsymbol{X}^T(y)\,\boldsymbol{L}^T)\,\boldsymbol{D}(y)(\mathbf{1}+\boldsymbol{L}\boldsymbol{X}(y))\,\boldsymbol{r}_{ye}(y,\ x_\alpha^s)\,dY\right)\boldsymbol{B}_2(x_3) \tag{47}$$

Integrals in Eqs. (46), (47) are calculated as the sum of integrals over the single cells, and have the meaning of linear energy density. The total number of strands is S, s is the index for the single strand, and the area of the cross-section is $S|Y|$. The work of internal forces is hence:

$$2W=\boldsymbol{v}^T\boldsymbol{K}\,\boldsymbol{v} \tag{48}$$

In deriving the above equations we have chosen the origin of the global set of coordinates in such a way that the sums with only x or y vanish. For the symmetrical cell of periodicity (geometry and mechanical properties) vanish also the integrals over Y which contain either $\boldsymbol{x}$ or $\boldsymbol{y}$. The matrix $\boldsymbol{D}^h$ of Eq. (46) contains the effective material coefficients which have been previously defined by homogenization procedure. We emphasize here that in the derivation of the beam element it was not assumed previously that the global behaviour is governed by the effective material coefficients calculated above.

The term depending on the temperature distribution in R in Eq. (28) can be expressed as follow:

$$R_t=\int_Y \alpha(y_\alpha)(\mathbf{1}-\boldsymbol{L}\boldsymbol{X}(y_\alpha))\,dY\sum_{s=1}^{S}(t^0+\boldsymbol{q}\boldsymbol{x}_s)\int_L(\boldsymbol{x}_{1e}(x_\alpha^s)\,\boldsymbol{B}_1(x_3))\,\boldsymbol{v}+$$

$$+\varepsilon^2 S\int_Y(q_\alpha\ y_\alpha)\,\alpha(y_\alpha)(\mathbf{1}-\boldsymbol{L}\boldsymbol{X}(y_\alpha))\,dY\int_L(\boldsymbol{r}_{ye}(y_\alpha,\ x_\alpha^s)\,\boldsymbol{B}_2(x_3))\boldsymbol{v} \tag{49}$$

where t^0 is the given global (mean) temperature, constant over the cross-section of the beam, $\boldsymbol{q}$ is the given constant vector of heat flux perpendicular to the beam axis. Integrals over Y in Eq. (49) are computed once by the subprogram that solves local problems (homogenization).

2.6. *Computation of stress and temperature in an arbitrary point of the beam cross-section*

The presented model permits us to analyse some local effects of the distribution of stress and temperature. Having beam type finite element solution, thus $\boldsymbol{e}^0(\boldsymbol{x},\ y_\alpha)$ via Eq. (44) we go back to Eq. (27) for σ^0 to obtain the local approximation of stress on the given cell. We can repeat this procedure for each cell of periodicity.

The local stress field Eq. (27) fulfils the equations of equilibrium everywhere in Y. It should be understood that this stress field is associated with uniform state of strains over the cell of periodicity $\boldsymbol{Y}$ given by the value of $\boldsymbol{e}^0(\boldsymbol{x}^s,\ y_\alpha)$. The local approximation of heat flux for this

cell is the following:

$$q_j^0(y)=k_{ij}(y)(\delta_{ip}+\vartheta^p_{i(y)}(y))\,T^0_{,p(x)} \tag{50}$$

When computed cell by cell over the array of cells, the graphs of stress or heat flux can exhibit some discontinuities due to discrete values of global strains and global gradient of temperature. To avoid this, when necessary, we replace in the above formulae the global strains and global gradient of temperature with the extrapolated values listed below:

$$\check{e}_{pq(x)}(y)=y_k C_k^{pq}(e_{pq(x)})+C^{pq}(e_{pq(x)}) \tag{51}$$

$$\check{\theta}^0(y)=y_k E_k(T^0)+E(T^0) \tag{52}$$

$$\check{\theta}_{,p(x)}(y)=y_k D_k^p(T^0_{,p(x)})+D^p(T^0_{,p(x)}) \tag{53}$$

In the above the coefficients C, D and E (depending on the values of $e_{pq(x)}$ and θ^0_p respectively) are computed to minimise the square error between extrapolated value $\hat{f}$ restricted to the particular cell in the neighbourhood of cells and its value f on this cell resulting from the global solution. The procedure described in this section will be called in the sequel unsmearing.

3. Examples of application

3.1. Substantial reduction of the problem size

The presented model enables us to analyse rather complex structures on a small computer. A realistic example of a superconducting beam shows the potential of our procedure. A comparison is made with a general pupose program.

For the homogenisation the section of a single superconductor (shown in Fig. 1) is subdivided into 120 brick elements with 24 degrees of freedom per element. This discretization is sufficiently fine for the calculation of the effective material coefficients which result from our code for homogenization. These effective material coefficients are used as the input data for global computation. For the description of the full cross-section of the beam the number of repetition of the cell of periodicity in rows and in columns is needed. For the higher order term in Eq. (45) we need the values of the functions of homogenization in the nodes of the local mesh (see Eq. (47)).

In this way, once more the detailed microdescription is sufficient to compute the stiffness of the whole cross-section. The geometry of the D shaped structure is shown in Fig. 3. We use 20 beam elements to approximate the central line of this coil. The total number of degrees of freedom is 400. An alternative for global computation is a 3D Finite Element mesh with 2874 nodes made of 1744 brick elements. The total number of variables (including any Lagrange multipliers needed to construct the model) is 7848.

The difference in the deflection of the central line for these two models never exceeds 5%. It is possible to obtain the local stress from ABAQUS or beam model (which gives only the mean stress, meaningless for engineering purposes) only via our unsmearing procedure which is a simple inversion of that used to construct the beam element.

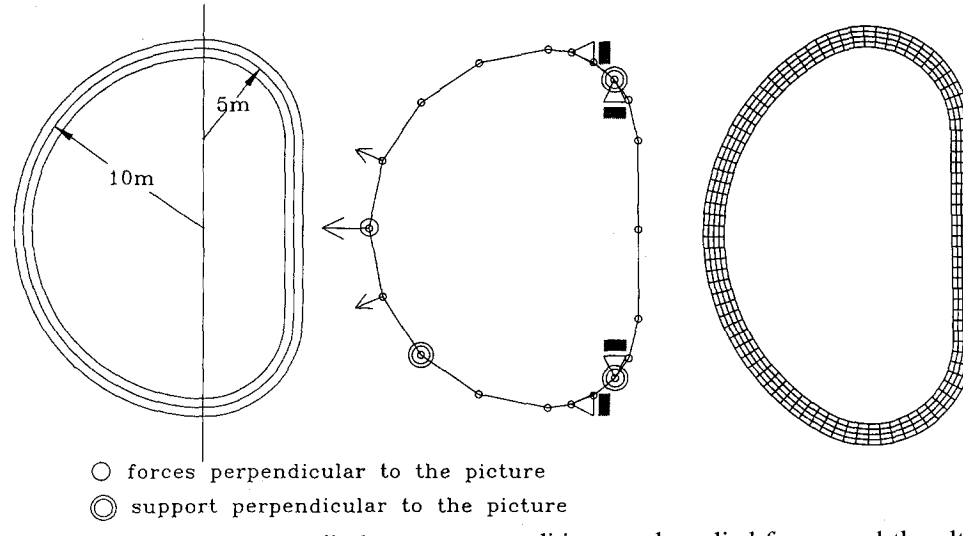

Fig. 3 Scheme of the superconducting coil, the support conditions and applied forces and the alternative FE mesh for ABAQUS.

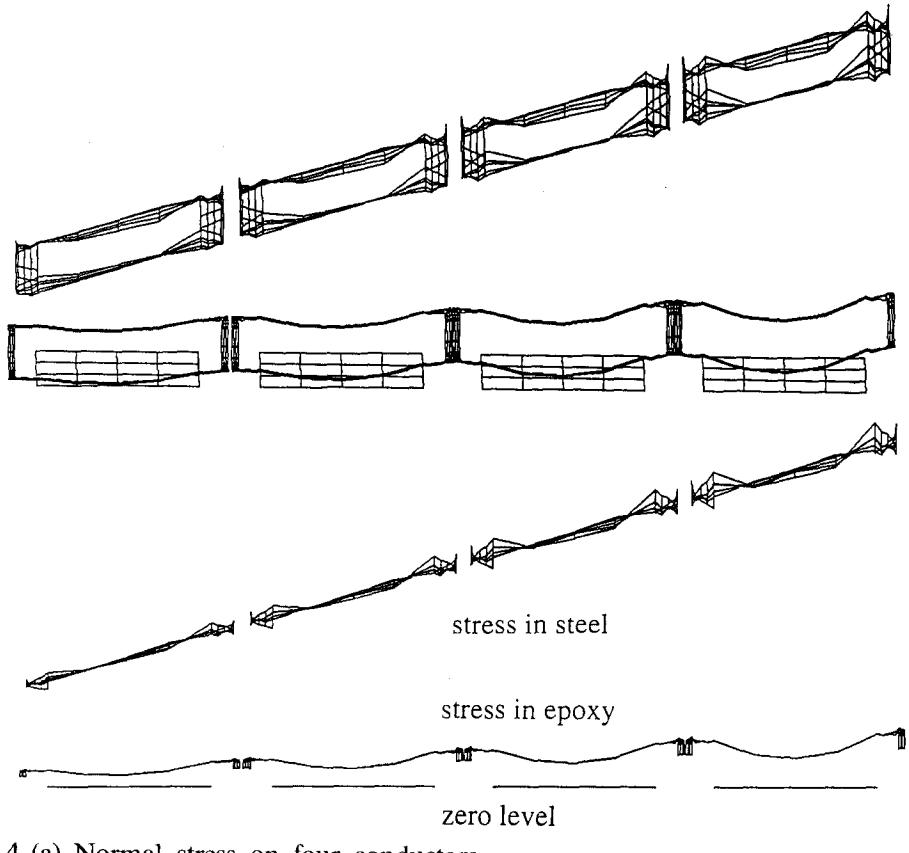

Fig. 4 (a) Normal stress on four conductors
(b) Normal stress on four conductors-section of the graph shown in Fig. 4a.

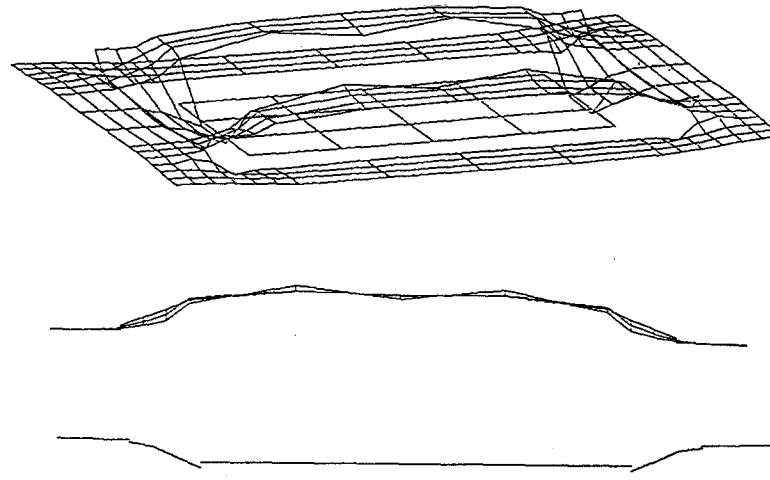

Fig. 5 Shear stress on one conductor.

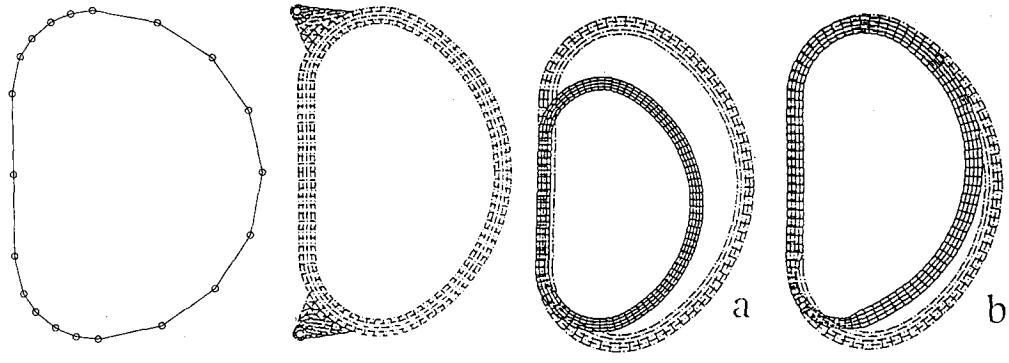

Fig. 6 Deformed configurations of the finite element mesh for both cases of thermal load.

3.2. Global stresses image obtained via cell by cell local unsmearing process

The normal stress diagram for several cells, each one obtained via an independent numerical procedure, proves that the unsmearing routine results in a quite realistic global graph. Fig. 4 shows such a collection of independently calculated distributions of normal stresses for a few neighbouring cells. The stresses are smoothed material by material. The shear stresses σ_{23} diagrams obtained via unsmearing procedure and smoothed material by material are shown in Fig. 5. These graphs are continuous in the x_2 direction.

3.3. Examples of thermoelastic computation for the superconducting coils

As an example of thermoelastic computation we have analysed, under different loading conditions, the coils mentioned in the introduction.

The whole D shaped frame is immersed in liquid helium to assure the temperature needed for superconductivity. We have considered two simple idealisations of possible cases of thermal condition:

(1) The situation when the coil is fully immersed in the liquid medium of a temperature

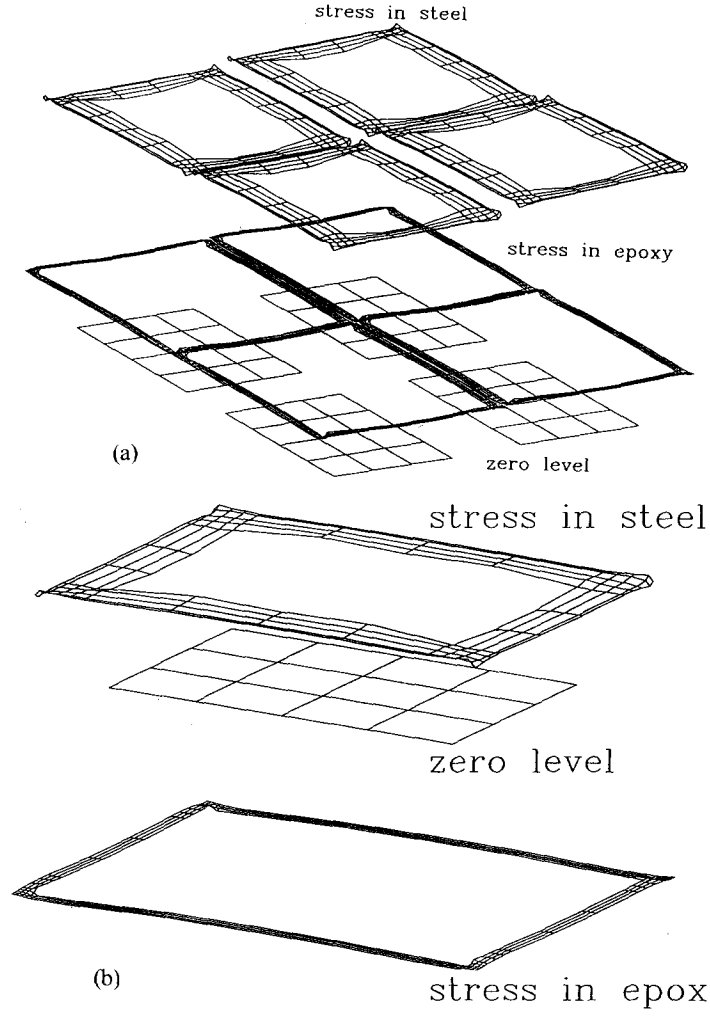

Fig. 7 (a) Graphs of stress σ_{33} in the cross-section perpendicular to fibres for uniform cooling
(b) Graphs of stress σ_{33} in the cross-section perpendicular to fibres for nonuniform cooling.

of 4 *K* and mechanically fixed. In this case the supports are designed to eliminate only the rigid motion of the structure.

(2) The situation when the coil is partially immersed in the liquid medium of a temperature of 4 *K*, mechanically fixed as in the above example.

In all cases a time independent problem is considered.

The homogenization procedure is carried out with our program. The program for homogenization is run twice. First effective elastic coefficients and six vectors of homogenization functions for unsmearing are computed (graphs of these functions are avalaible in [5]). Then the thermal effective characteristics are calculated: effective conductivity and effective thermal expansion. The coefficients needed for the beam model are also computed.

For macroanalysis the beam element is used and then the global problems is solved using ABAQUS Finite Element code for cases a) and b). In Fig. 6. deformed configurations of the finite element mesh are shown for both cases. For these analyses homogenized material is used with the characteristics previously computed. For the uncoupled problem ABAQUS needs to

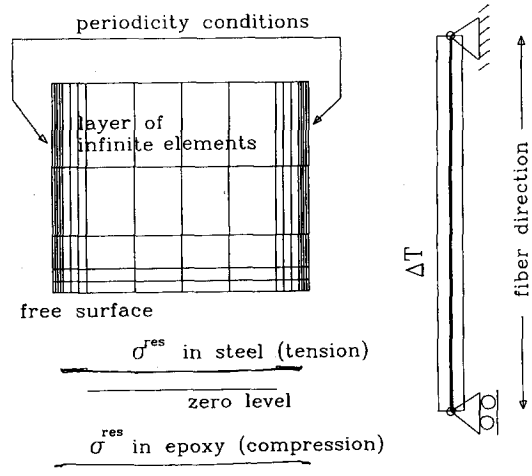

Fig. 8 Finite element mesh and boundary conditions for the boundary correction of the global solution.

be run twice. First-to find the temperature field, then the corresponding displacements and forces. We need the gradients of temperature and strains in the regions of interest of the coils as input data for unsmearing. Strains are directly obtained from standard postprocessing, gradients of temperature can be replaced by their local aproximation with finite differences. For the two considered cases the unsmearing procedure has been applied to recover the stress field in the cross-section of the single conductor, using again the apropriate option of our code for homogenization. In Fig. 7. the graphs of stress in the cross-section perpendicular to fibres are shown. The elevations of faces over the cell of periodicity are proportional to σ_{33}. Case a) is easy to verify by simple hand calculations. Case b) shows the full complex state of stress in the neighbourhood of a single conductor.

3.4. *Local behaviour near the end of the stress-free edge-correction of the global solution*

The presented solution obtained for the heterogeneous body, being a relatively good approximation in the interior of the beam is generally not acceptable near the boundary. The main reason is that only the global solutions can verify the boundary conditions (the local one is still of the same form and proportional to the global deformation in each local cell of the structure, including those on the border). The most dangerous case occurs when the limiting surface cuts across the fibres. We consider here this situation. The global solution is computed for a simply supported composite beam in a uniform steady field of temperature T which exceeds the temperature of the neutral state of the composite structure, the difference being ΔT. The beam is supposed to be made of an array of superconductors shown in Fig. 1. Global axial force vanishes, global strains are constant and different from zero. We have the non vanishing fluctuation of normal stress on the free surface on each cell of periodicity; this forms the residual:

$$\sigma_{33}^{res}(\boldsymbol{x}, \boldsymbol{y}) = -a_{33kl}(\boldsymbol{y})\,\chi_{kl}^{pq}(\boldsymbol{y})\,u^{0}_{(p,q)}(\boldsymbol{x}) \tag{54}$$

We can define a boundary corrector near this surface. This corrector should have the following properties:

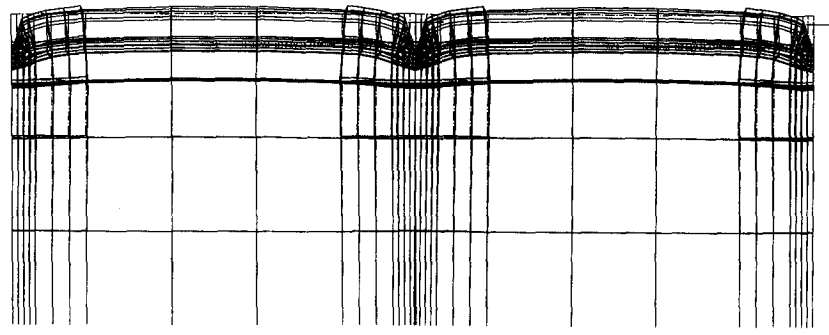

Fig. 9 Deformed mesh configuration on the cell of periodicity at the free edge of the beam.

$$\sigma_{ij}^{corr}(z, y) = -\sigma_{ij}^{res}(y) \text{ for } z_3 = 0 \tag{55}$$

$$\sigma_{ij,j(z)}^{corr}(z, y) = 0 \tag{56}$$

$$\sigma_{\alpha\beta}^{corr}(z, y) \text{ } Y\text{-periodic with respect to } y. \tag{57}$$

$$\sigma^{corr}(z, y) \text{ decreases exponentially for } z_3 \to \infty \tag{58}$$

In the above Eqs. (55), (56), (57) and Eq. (58) z denotes an another position vector in the local coordinate system with its origin at the free surface and one axis perpendicular to it:

$$z_3 = -\frac{x_3}{\varepsilon} \tag{59}$$

For detailed study of similar problems the reader is referred to (Ladeveze 1985). The problem Eqs. (55)-(58) can be solved using a commercial FE code once the effective material coefficients, functions of homogenizations and the global solution are known. In this example the program ABAQUS is used. To compute values of the above prescribed boundary conditions we take the value of global strains from our beam model, we use our own FE program which calculates the set of homogenization functions and determine then unsmeared stress $\sigma_{33}^{res}(y)$ on the cell of periodicity. A graph of this stress is similar to that in Fig. 7a. We assure identical definition of the FE meshes in our local model and that for ABAQUS models.

In the definition of the model 3D brick elements are used. This choice is determined by the model used in our homogenization. The periodicity of the solution is assured by imposing kinematical constraints between nodes on the opposite faces of the cell of periodicity (ABAQUS command *EQUATION). Condition (58) is accounted for by use of infinite elements (see for details (Lefik, Schrefler (1996)). Only one cell of periodicity is modelled by finite elements with the heterogeneous material properties. The rest of the semi-infinite region is discretised by infinite elements as shown in Fig. 8. For these infinite elements effective material coefficients are used which were computed previously by our homogenization program. In Fig. 9 we show the real, corrected displacements on one cell of periodicity at the free edge of the beam.

4. Conclusions

We have developed via the homogenization procedure a beam element for unidirectional com-

posites. This model allows for a substantial reduction in size of the problem and is hence easily applicable to large engineering structures such as superconducting coils. Unsmearing procedures yield realistic stress diagrams with local features needed for engineering design. Some residual thermal stresses, present even in the case of statically determinated supports, are recovered with unsmearing. These stresses do not appear in the global homogenized model.

Since the computational process splits into several numerically disjoint steps and resulting graphs are constructed piecewise, the algorithm is numerically not very costly.

Acknowledgements

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Appendix

In this appendix we give some matrices which are used in the text of the paper.

We introduce a matrix of homogenization functions:

$$\boldsymbol{X}^T(y)=[\{\chi_i^{11}(y)\}\{\chi_i^{22}(y)\}\{\chi_i^{33}(y)\}\{\chi_i^{12}(y)\}\{\chi_i^{23}(y)\}\{\chi_i^{13}(y)\}]_{3\times 6}$$

Problem Eq. (15) can be rewritten in matrix form:

$$\textbf{find } \chi_i^{pq} \in V_Y \textbf{ such that: } \forall \boldsymbol{v} \in V_Y$$

$$\int_Y \boldsymbol{e}(\boldsymbol{v}(y))\,\boldsymbol{D}(y)(\boldsymbol{1}-\boldsymbol{L}\boldsymbol{X}(y))\,\boldsymbol{e}(\boldsymbol{u}^0)\,dY=0$$

where $\boldsymbol{L}$ is the strain matrix operator and $\boldsymbol{D}$ contains the material coefficients a_{ijkl}. Using a finite element discretization of the cell of periodicity, $\boldsymbol{X}$ is interpolated as:

$$X(\xi,\ \eta)=\sum N_e(\xi,\ \eta)X^e$$

Other shape functions used in the paper are described by Eq. (40)

The vector of the global displacements $\boldsymbol{u}$ is fixed as follows (the superscript will be dropped in the sequel):

$$\boldsymbol{u}^0=\{u_1^0\ u_2^0\ u_3^0\}$$

The assumed form of the global displacements Eq. (23), (24) permits us to make the following decomposition:

$$\boldsymbol{u}(x_\alpha,\ x_3)=(\boldsymbol{N}_u'(x_3)+\boldsymbol{x}_u(x_\alpha)\boldsymbol{N}_u''(x_3)+\boldsymbol{r}_u(x_\alpha)\boldsymbol{N}_u'''(x_3))\boldsymbol{v}^e$$

In the above we have introduced the following matrices:

$$\boldsymbol{N}_u'=\begin{bmatrix} N^{w1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \\ \mathbf{0} & \mathbf{0} & N^{w2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & [\mathbf{0}]_{3\times3} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & N^{w3} & \mathbf{0} & \end{bmatrix}$$

$$\boldsymbol{x}_u^s=\begin{bmatrix} x_1 & x_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_1 & x_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_1 & x_2 \end{bmatrix} \tag{45}$$

$$\boldsymbol{N}_u''=\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & E_{11}N^{b3} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -N^{\theta3} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & N^{\theta3} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & E_{22}N^{b3} \\ -N_3^{w1} & -N^{\theta2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -N_3^{w1} & N^{\theta1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{46}$$

$$\boldsymbol{r}_u=\begin{bmatrix} -E_{11}x_1x_2 & E_{11}\dfrac{x_1x_1}{2}-E_{22}\dfrac{x_2x_2}{2} & 0 \\ E_{11}\dfrac{x_1x_1}{2}-E_{22}\dfrac{x_2x_2}{2} & E_{22}x_1x_2 & 0 \\ 0 & 0 & (x_\alpha) \end{bmatrix} \tag{47}$$

$$\boldsymbol{N}_u'''=\begin{bmatrix} [\mathbf{0}]_{3\times5}\begin{bmatrix} \mathbf{0} & N^{b1} & \mathbf{0} \\ 0 & 0 & N^{b2} \\ N_3^{\theta3} & \mathbf{0} & \mathbf{0} \end{bmatrix}[\mathbf{0}]_{3\times1} \end{bmatrix}$$

We write equations for $\boldsymbol{u}$ now for a single cell (conductor) indicated with superscript s. The local set of coordinates $\boldsymbol{y}$ is placed in the plane of the cross-section of this cell of periodicity. Coordinates of the origin of $\boldsymbol{y}$ are $\boldsymbol{x}^s$ for this cell.

$$\boldsymbol{u}(x_\alpha^a,\ y_\alpha,\ x_3)=(\boldsymbol{N}_u'(x_3)+\boldsymbol{x}_u(x_\alpha^s)\boldsymbol{N}_u''(x_3)+\boldsymbol{r}_u(x_\alpha^s)\boldsymbol{N}_u'''(x_3))\boldsymbol{v}^e+$$
$$+\varepsilon(\boldsymbol{x}_u(y_\alpha)\boldsymbol{N}_u''(x_3)+y_\alpha\boldsymbol{r}_{u,\alpha}(\boldsymbol{x}_\alpha^s)\boldsymbol{N}_u'''(x_3))\boldsymbol{v}^e$$

Strains

The ordering of the strains vectors is the following:

$$\boldsymbol{e}^{01}=\{e_{22}^{01}\ e_{33}^{01}\ e_{12}^{01}\ e_{23}^{01}\ e_{13}^{01}\}$$

The symbol $\boldsymbol{e}^{01}$ is used to distinguish the part of the zero order strains which is given by derivative

of the global displacement vector.

Using the interpolation functions and the vector of nodal unknowns $\boldsymbol{v}$, $\boldsymbol{e}$ may be written as:

$$\boldsymbol{e}^{01}(x_\alpha, x_3)=(\boldsymbol{B}'(x_3)+\boldsymbol{x}_e(x_\alpha)\,\boldsymbol{B}''(x_3)+\boldsymbol{r}_e(x_\alpha)\,\boldsymbol{B}'''(x_3))\,\boldsymbol{v}^e$$

where:

$$\boldsymbol{x}_e=\begin{bmatrix} 0 & 0 & 0.5E_{11}x_2 & -0.5E_{11}x_1 & 0.5E_{11}x_1 & 0.5E_{11}x_2 \\ 0 & 0 & 0.5E_{22}x_2 & -0.5E_{22}x_1 & 0.5E_{22}x_1 & 0.5E_{22}x_2 \\ x_1 & x_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5x_1 & 0.5E_{22}x_2 & 0.5E_{22}x_2 & -0.5x_1 \\ 0 & 0 & -0.5x_2 & 0.5E_{11}x_2 & 0.5E_{11}x & 0.5x_2 \end{bmatrix}$$

$$\boldsymbol{r}_e(x_\alpha)=\begin{bmatrix} [\mathbf{0}]_{2\times 2} & & [\mathbf{0}]_{2\times 4} & & \\ & (x_\alpha) & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 \\ [\mathbf{0}]_{4\times 2} & 0 & {}_{,2}(x_\alpha) & f_{12}(x_\alpha) & f_{22}(x_\alpha) \\ & 0 & {}_{,1}(x_\alpha) & f_{11}(x_\alpha) & f_{12}(x_\alpha) \end{bmatrix}$$

$$\boldsymbol{B}'=\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\,\mathbf{0}\,\mathbf{0} & E_{11}N^{b3} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\,\mathbf{0}\,\mathbf{0} & E_{22}N^{b3} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & N_{33}^{w3} & \mathbf{0}\,\mathbf{0}\,\mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\,\mathbf{0}\,\mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 0.5N^{\theta 1} & \mathbf{0} & \mathbf{0}\,\mathbf{0}\,\mathbf{0} & \mathbf{0} \\ \mathbf{0} & -0.5N^{\theta 2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\,\mathbf{0}\,\mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$\boldsymbol{B}''=\begin{bmatrix} -N_{,33}^{w1} & -N_{,3}^{\theta 2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -N_{,33}^{w2} & N_{,3}^{\theta 1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 0.5N_{,3}^{\theta 3} & -N^{b1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -N^{b2} & 0.5N_{,3}^{\theta 3} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & N^{b2} & 0.5N_{,3}^{\theta 3} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -0.5N_{,3}^{\theta 3} & -N^{b1} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$\boldsymbol{B}'''\begin{bmatrix} [\mathbf{0}]_{2\times 5} & & [\mathbf{0}]_{2\times 4} & & \\ & N_{,33}^{\theta 3} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ & 0.5N_{,33}^{\theta 3} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ [\mathbf{0}]_{4\times 5} & \mathbf{0} & 0.5N_{,3}^{b1} & \mathbf{0} & \mathbf{0} \\ & \mathbf{0} & \mathbf{0} & 0.5N_{,3}^{b2} & \mathbf{0} \end{bmatrix}$$

In local coordinates $\boldsymbol{y}$ for the single cell s, following the approximation Eq. (43) for $\boldsymbol{\vartheta}$ and similar for $f_{i\alpha}$ we have:

$$\begin{aligned}\boldsymbol{e}^{01}(x_\alpha^s, y, x_3)=&(\boldsymbol{B}'(x_3)+\boldsymbol{x}_e(x_\alpha^s)\,\boldsymbol{B}''(x_3)+\boldsymbol{r}_e(x_\alpha^s)\,\boldsymbol{B}(x_3))\,\boldsymbol{v}^e+\\&+\varepsilon\,(\boldsymbol{x}_e(y_\alpha)\,\boldsymbol{B}''(x_3)+y_\alpha\boldsymbol{r}_{e,\alpha}(x_\alpha^s)\,\boldsymbol{B}'''(x_3))\,\boldsymbol{v}^e\end{aligned}$$

which in a more compact form becomes:

$$\boldsymbol{e}^{01}(x_\alpha^s, y, x_3)=(\boldsymbol{x}_{1e}(x_\alpha^s)\,\boldsymbol{B}_1(x_3)+\varepsilon\,\boldsymbol{r}_{ye}(y_\alpha, x_\alpha^s)\,\boldsymbol{B}_2(x_3))\,\boldsymbol{v}^e$$

where

$$\boldsymbol{x}_{1e}(x_\alpha^s)=[\boldsymbol{1},\ \boldsymbol{x}_e(x_\alpha^s),\ \boldsymbol{r}_e(x_\alpha^s)],\quad \boldsymbol{r}_{ye}(y_\alpha, x_\alpha^s)=[\boldsymbol{x}_e(y_\alpha),\ y_1\boldsymbol{r}_{e,2}(x_\alpha^s),\ y_2\boldsymbol{r}_{e,2}(x_\alpha^s)]$$

$$\boldsymbol{B}_1(x_3)=[\boldsymbol{B}'(x_3),\ \boldsymbol{B}''(x_3),\ \boldsymbol{B}'''(x_3)]^T,\quad \boldsymbol{B}_2(x_3)=[\boldsymbol{B}''(x_3),\ \boldsymbol{B}'''(x_3),\ \boldsymbol{B}'''(x_3)]^T$$

The correctness of the above Eqs. can be verified by consecutive substitutions.