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Dynamic response of a beam on multiple supports with a moving mass

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Abstract. The dynamic behavior of an Euler beam with multiple point constraints traversed by a moving concentrated mass, a "moving-force moving-mass" problem, is analyzed and compared with the corresponding simplified "moving-force" problem. The equation of motion in matrix form is formulated using Lagrangian approach and the assumed mode method. The effects of the presence of intermediate point constraints in reducing the fluctuation of the contact force between the mass and the beam and the possible separation of the mass from the beam are investigated. The equation of motion and the numerical results are expressed in dimensionless form. The numerical results presented are therefore applicable for a large combination of system parameters.

Key words: moving mass; contact force; multiple supports; beam.

1. Introduction

The dynamic responses of a beam subjected to moving loads or moving masses have been studied extensively in connection with the design of railway tracks and bridges and machining processes. Timoshenko (1922) reported the classical solution of a beam subjected to a constant moving load. The book by Frýba (1972) presented a comprehensive survey of the references and methods for solving the problems involving moving loads on structures. Studies by Nelson and Conover (1971), Benedetti (1974), Steele (1967), Florence (1965), Katz *et al.* (1987), and Lee (1994) took into account the effects of elastic foundation, moving mass, multiple supports, and deflection dependent moving loads. Sloss *et al.* (1988) studied the feedback control of a simply supported Euler beam subjected to loads moving at constant speed with constant and harmonic magnitude. The dynamic behavior of a beam acted upon by a moving mass, the original "moving-mass moving-force" problem was approximated by the simplified "moving-force" problem by Timoshenko *et al.* (1974). Sadiku and Leipholz (1987), using Green's function, showed that the approximate solution for the "moving-force" problem was not always an "upper-bound" solution in terms of the deflection under the moving mass for the related "moving-mass moving-force" problem. It was found that the inertial effect of the moving mass could not be neglected in comparison with the gravitational effect even if the velocity of the moving mass was relatively small. However, the mass was assumed to be always in contact with the beam and the contact force between the mass and the beam had not been examined to verify this assumption. More-

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over, there is no reported study on the behavior of a beam with multiple supports subjected to a moving mass.

In the present study, the equation of motion in matrix form for an Euler beam on multiple supports acted upon by a concentrated mass moving at a constant speed is formulated using Lagrangian approach and the assumed mode method. The multiple supports are in the form of intermediate point constraints modeled by linear springs of very large stiffness. The equation of motion is non-dimensionalized so that the numerical results presented are applicable for a large combinations of system parameters instead of just applicable for specific cases in the reported studies. The possibility of the mass separating from the beam during the course of the motion is examined by monitoring the contact force between the mass and the beam during the motion. The effects of the presence of the intermediate point constraints in reducing the fluctuation of this contact force and therefore diminishing the possibility of the mass separating from the beam during the motion are to be investigated.

2. Theory and formulations

A slender beam of length L and mass m per unit length acted upon by a concentrated mass of mass M moving at a constant speed v shown in Fig. 1 is considered for the present study. A set of coordinate system, shown in Fig. 1, is assumed to be fixed in the inertial frame with the $\boldsymbol{i}$ unit vector parallel to the undeformed beam and the $\boldsymbol{j}$ unit vector pointing downward in the same direction of the gravitational field g. The two ends of the beam are either simply supported or clamped, which are the most frequently encountered boundary conditions in the actual applications. The transverse deflection for a point located at $x\boldsymbol{i}$ along the beam is denoted by w which is a function of both the time t and x. Fig. 1 also shows the beam with an intermediate point constraint. Multiple point constraints with locations indicated by $x=s_1, \cdots, s_N$ are considered in the present formulation. The position of the mass at any instant is indicated by s in the $\boldsymbol{i}$ direction. The transverse deflection of the beam is assumed to be small for the behavior of the beam to be governed by Euler beam theory.

The kinetic energy T of the beam is

$$T_B=\frac{1}{2}m\int_0^L\left(\frac{\partial w}{\partial t}\right)^2 dx \tag{1}$$

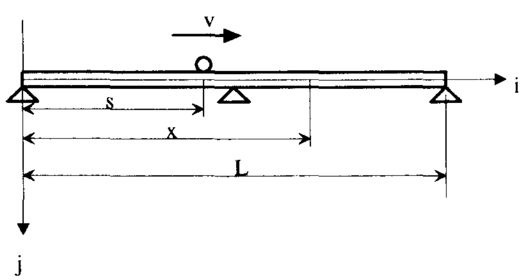

Fig. 1 An Euler beam on multiple supports acted upon by a moving mass.

The bending elastic strain energy of the beam is

$$V_\epsilon=\frac{1}{2}\int_0^L EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx \tag{2}$$

where E and I are the Young's modulus and the principal second moment of area of the cross section of the beam.

The intermediate supports at $x=s_p(p=1,\cdots,N)$ are modeled as linear springs of large stiffness k. The potential energy due to the intermediate supports is

$$V_s=\frac{1}{2}k(w^2(s_1)+\cdots+w^2(s_N)) \tag{3}$$

where $w(s_p)$ is the deflection of the beam at $x=s_p$.

The equation of motion can be formulated using Lagrangian approach either by considering the mass to be part of the system with the external force acting on the system given by the gravitational force acted on the mass, or by considering the presence of the moving mass in terms of the contact force acting on the beam. The two approaches result in the same equation of motion. The first approach which assumes the mass to be part of the system and treats the contact force between the mass and the beam as an internal force in elaborated in the following section.

The kinetic energy of the moving mass is

$$T_M=\frac{1}{2}M\left(v^2+\left(\frac{\partial w}{\partial t}+v\frac{\partial w}{\partial x}\right)^2\right) \tag{4}$$

The second term of the kinetic energy of the moving mas is due to the transverse component of the velocity of the mass caused by the deflection of the beam as well as the horizontal motion of the mass. The virtual work of the gravitational force on the system is

$$\delta W=Mg\delta w_{(x=s)} \tag{5}$$

The quantity δw, the virtual displacement of w, is to be evaluated at $x=s$. Using the Lagrangian approach, the Lagrangian of the system is defined by

$$\mathcal{L}=T_B+T_M-V_\epsilon-V_s \tag{6}$$

In the above formulation as well as in all the reported studies, the moving mass and the beam is assumed to be in contact at all times. The possibility of the mass separating from the beam during the course of motion can be detected by the change in sign of the contact force given by

$$F_c=Mg-M\left(\frac{\partial^2 w}{\partial t^2}+2v\frac{\partial^2 w}{\partial x\partial t}+v^2\frac{\partial^2 w}{\partial x^2}\right)_{(x=s)} \tag{7}$$

The contact force is defined to be positive if the force acting on the beam is pointing downward in the positive j direction. A change of sign from positive to negative indicates that the mass has separated from the beam and the equation of motion is no longer valid to describe the ensuing motion.

For simplicity in the subsequent computations, the following dimensionless quantities are introduced

$$\tau=t\sqrt{\frac{EI}{mL^4}}\ \xi=\frac{x}{L}\ \bar{w}=\frac{w}{L}\ \bar{s}=\frac{s}{L}\ \bar{s}_v=\frac{s_v}{L}\ \bar{M}=\frac{M}{mL}\ \bar{v}=v\sqrt{\frac{mL^2}{EI}}\ \bar{g}=\frac{gmL^3}{EI}\ \bar{k}=\frac{kL^3}{EI} \tag{8}$$

Using the assumed mode method, the dimensionless quantity $\bar{w}$ can be expressed as

$$\bar{w}=\sum_{i=1}^{n} q_i(t)\,\phi_i(\xi) \tag{9}$$

The resulting Euler-Lagrange's equation in vector form is

$$(\bar{\boldsymbol{M}}+\bar{M}\bar{\boldsymbol{H}})\ddot{\boldsymbol{q}}+2\bar{M}\bar{v}\bar{\boldsymbol{A}}\dot{\boldsymbol{q}}+(\bar{\boldsymbol{K}}+\bar{M}\bar{v}^2\bar{\boldsymbol{C}}+\bar{k}\bar{\boldsymbol{S}})\boldsymbol{q}=\bar{M}\bar{g}\bar{\boldsymbol{\Phi}} \tag{10}$$

where $\bar{\boldsymbol{S}}=\bar{\boldsymbol{S}}_1+\cdots+\bar{\boldsymbol{S}}_N$ and

$$(\bar{\boldsymbol{M}})_{ij}=\int_0^1 \phi_i\phi_j\,d\xi \tag{11}$$

$$(\bar{\boldsymbol{K}})_{ij}=\int_0^1 \phi_i''\,\phi_j''\,d\xi \tag{12}$$

$$(\bar{\boldsymbol{H}})_{ij}=\phi_i(\bar{s})\,\phi_j(\bar{s}) \tag{13}$$

$$(\bar{\boldsymbol{A}})_{ij}=\frac{1}{2}(\phi_i(\bar{s})\,\phi_j'(\bar{s})+\phi_i'(\bar{s})\,\phi_j(\bar{s})) \tag{14}$$

$$(\bar{\boldsymbol{C}})_{ij}=\phi_i(\bar{s})\,\phi_j''(\bar{s}) \tag{15}$$

$$(\bar{\boldsymbol{S}}_p)_{ij}=\phi_i(\bar{s}_p)\,\phi_j(\bar{s}_p) \tag{16}$$

$$(\bar{\boldsymbol{\Phi}})_i=\phi_i(\bar{s}) \tag{17}$$

The vector $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$ are $n\times 1$ column vectors consisting of q_i and $\dot{q}_i$ respectively. The matrix $\bar{M}$ is equal to the identity matrix due to the orthogonality of the assumed normalized beam functions. All the matrices exept $\bar{\boldsymbol{C}}$ are symmetric. The numerical integrations are performed using the fourth-order Runge-Kutta method on a personal computer. For a simply supported beam, the normalized assumed functions are

$$\phi_i(\xi)=\sqrt{2}\sin\ i\pi\xi \tag{18}$$

For a clamped-clamped beam, the normalized assumed functions are

$$\phi_i(\xi)=\sinh\beta_i\xi-\sin\beta_i\xi+\gamma_i(\cosh\beta_i\xi-\cos\beta_i\xi) \tag{19}$$

where β_i are the solutions of the corresponding characteristic equation

$$1-\cos\beta\cosh\beta=0 \tag{20}$$

and

$$\gamma_i=\frac{\sinh\beta_i-\sin\beta_i}{\cos\beta_i-\cosh\beta_i} \tag{21}$$

3. Numerical results and discussion

The convergence of the numerical solutions in terms of the number of terms for the assumed functions and the stiffness of the linear springs is first examined for both the simply supported and the clamped beams with the system parameters given by Sadiku and Leipholz (1987):

$$L = 6\text{m} \quad v = 6\text{m/s} \quad EI/m = 275.4408\text{m}^4/\text{s}^2 \quad M/mL = 0.2 \tag{22}$$

These parameters are first non-dimensionalized as indicated in the preceding section. The gravitational acceleration g is taken to be 9.81 m/s^2. The deflection under the moving mass and the contact force are presented in dimensionless form denoted by $\bar{U} = w(x=s)/L$ and $\bar{F}_c = F_c L^2/EI$. It was found that the numerical results for both the dimensionless deflection under the moving mass and the dimensionless contact force are almost converged for a simply supported beam using a two-term approximation for $\bar{w}$. A clamped-clamped beam would require more terms ($n \approx 8$) for the convergence of the numerical results. The convergence of the numerical results in terms of the stiffness of the linear springs are shown in Fig. 2 for a simply supported beam and in Fig. 3 for a clamped-clamped beam for a beam with an intermediate support at the midspan. It is apparant from the equation of motion that a very large value of $\bar{k}$ will give rise to difficulties for numerical computations. The computational time is in general longer for a larger $\bar{k}$ especially for $\bar{k}$ larger than 10^7. Moreover, for very large $\bar{k}$, the numerical results

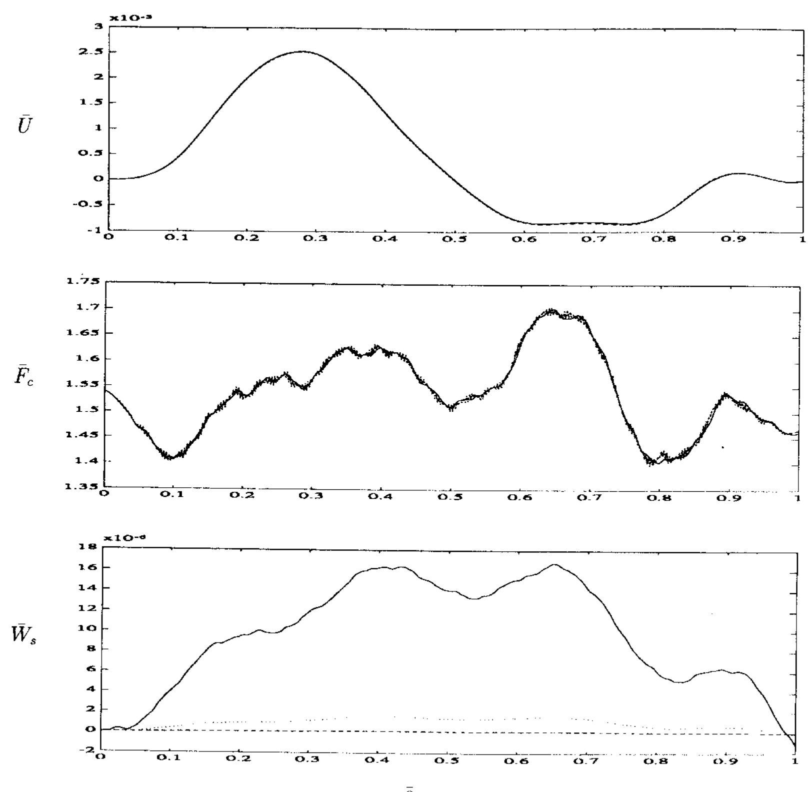

Fig. 2 Convergence study for a simply supported beam. '——', $\bar{k}=10^5$, '......', $\bar{k}=10^6$, '------', $\bar{k}=10^7$.

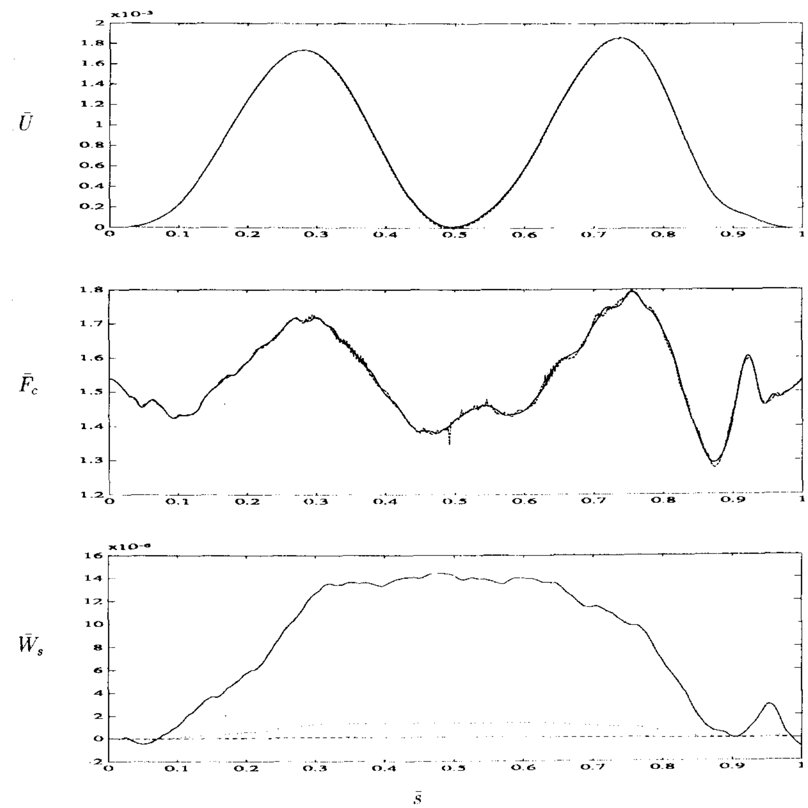

Fig. 3 Convergence study for a clamped-clamped beam. '——', $\bar{k}=10^5$, '······', $\bar{k}=10^6$, '------', $\bar{k}=10^7$.

for the dimensionless contact force is found to be superimposed by small-amplitude perturbations. As both the dimensionless deflection under the moving mass as well as the dimensionless contact force are almost converged for $\bar{k}=10^6$, this value of $\bar{k}$ is used in the following numerical computations. This value of $\bar{k}$ is also found to be sufficiently large to ensure that the deflection at the intermediate support, indicated by $\bar{W}_s$ in Figs. 2 and 3, is many orders smaller than the maximum dimensionless deflection under the moving mass.

The dimensionless deflection under the moving mass as well as the dimensionless contact force for a simply supported beam and a clamped-clamped beam with multiple supports ($N=0$, 1, 2) are shown respectively in Figs. 4 and 5. It can be seen from these figures that the presence of just a single intermediate support located at the midspan of the beam drastically reduces the deflection under the moving mass. A further increase of the number of intermediate supports does not produce any significant reduction in the magnitude of the dimensionless deflection under the moving mass. The presence of a single intermediate support at the midspan also reduces the fluctuation of the contact force between the mass and the beam. A further increase in the number of intermediate supports once again does not produce any significant reduction in the fluctuation of the dimensionless contact force.

The use of the presence of intermediate supports to suppress the separation of the mass from the beam during the course of motion is examined in Fig. 6 for a mass moving along a clamped-

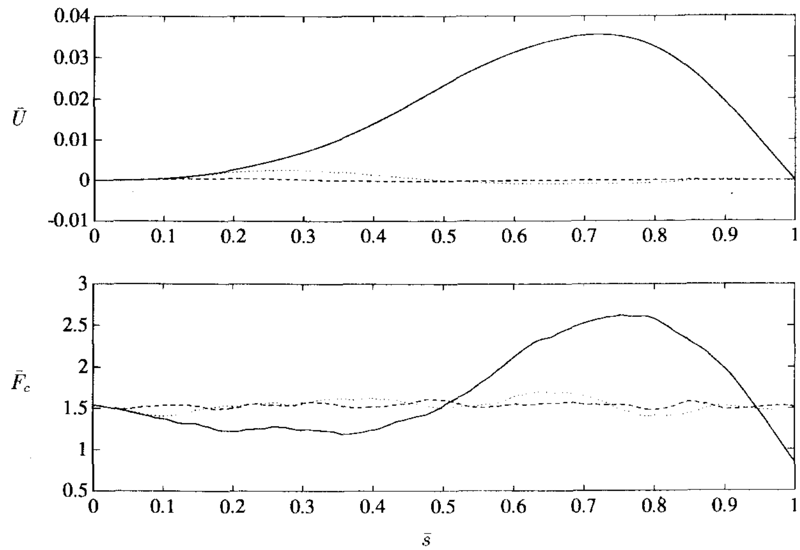

Fig. 4 The dimensionless deflection under the moving mass and the dimensionless contact force for a simply supported beam. '——', $N=0$, '······', $N=1$, $\bar{s}_1=1/2$, '------', $N=2$, $\bar{s}_1=1/3$, $\bar{s}_2=2/3$.

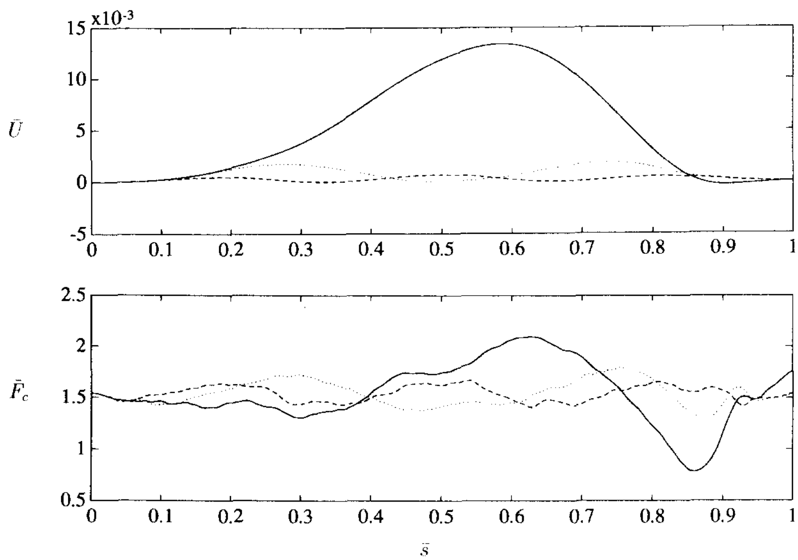

Fig. 5 The dimensionless deflection under the moving mass and the dimensionless contact force for a clamped-clamped beam. '——', $N=0$, '······', $N=1$, $\bar{s}_1=1/2$, '------', $N=2$, $\bar{s}_1=1/3$, $\bar{s}_2=2/3$.

clamped beam with $v=9$ m/s. For this case, the sign of the contact force changes from positive to negative towards the end of the motion in the absence of any intermediate support, indicating that the mass has separated from the beam and the equation of motion is no longer valid to describe the ensuing motion. The presence of intermediate supports reduces the fluctuation of the dimensionless contact force and makes the magnitude to be closer to the gravitational component, keeping the contact force to remain positive throughout the duration of the prescribed

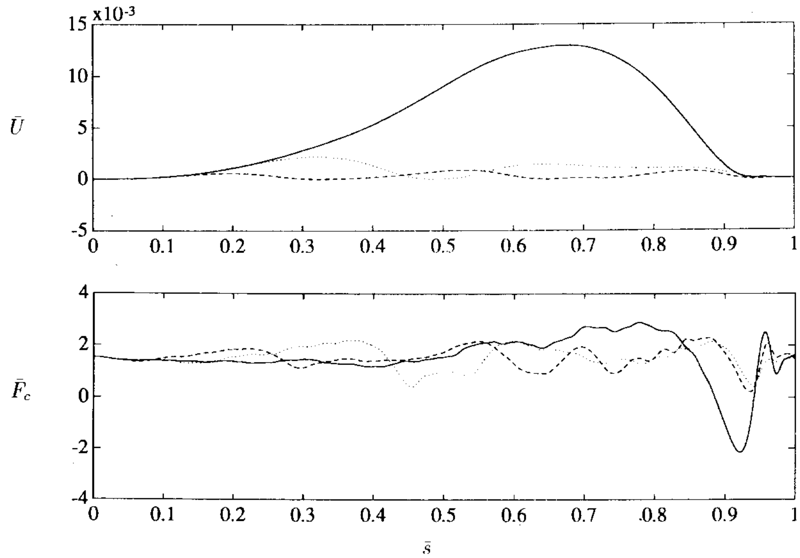

Fig. 6 The dimensionless deflection under the moving mass and the dimensionless contact force for a clamped-clamped beam with $v=9$ m/s, '——', $N=0$, '······', $N=1$, $\bar{s}_1=1/2$, '------', $N=2$, $\bar{s}_1=1/3$, $\bar{s}_2=2/3$.

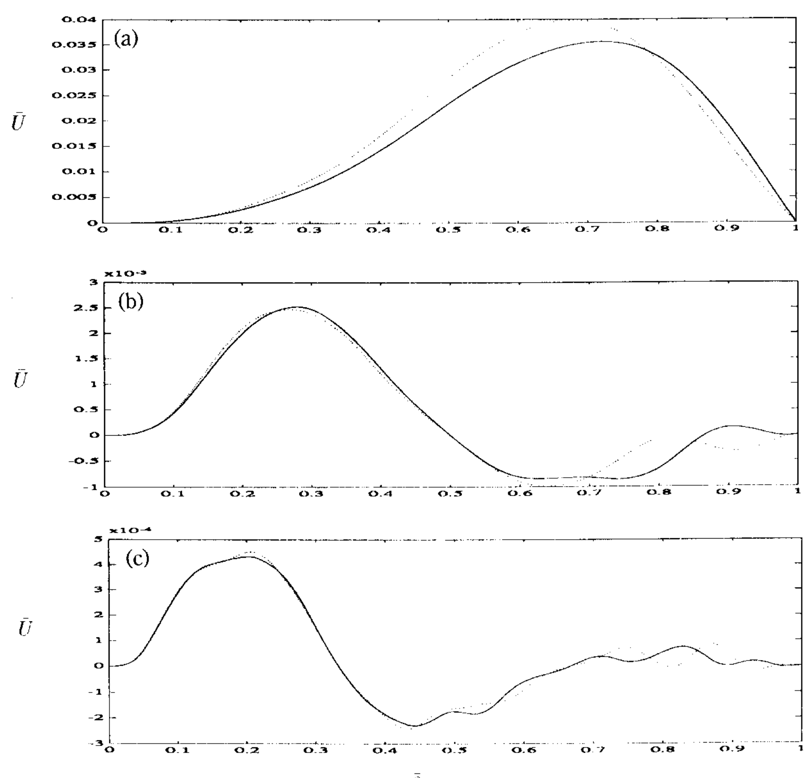

Fig. 7 Comparison between the dimensionless deflections under the moving load computed using the "moving-force moving-mass" formulation ('—') and the "moving-force" formulation ('------') for a simply supported beam. (a) $N=0$, (b) $N=1$, $\bar{s}_1=1/2$, (c) $N=2$, $\bar{s}_1=1/3$, $\bar{s}_2=2/3$.

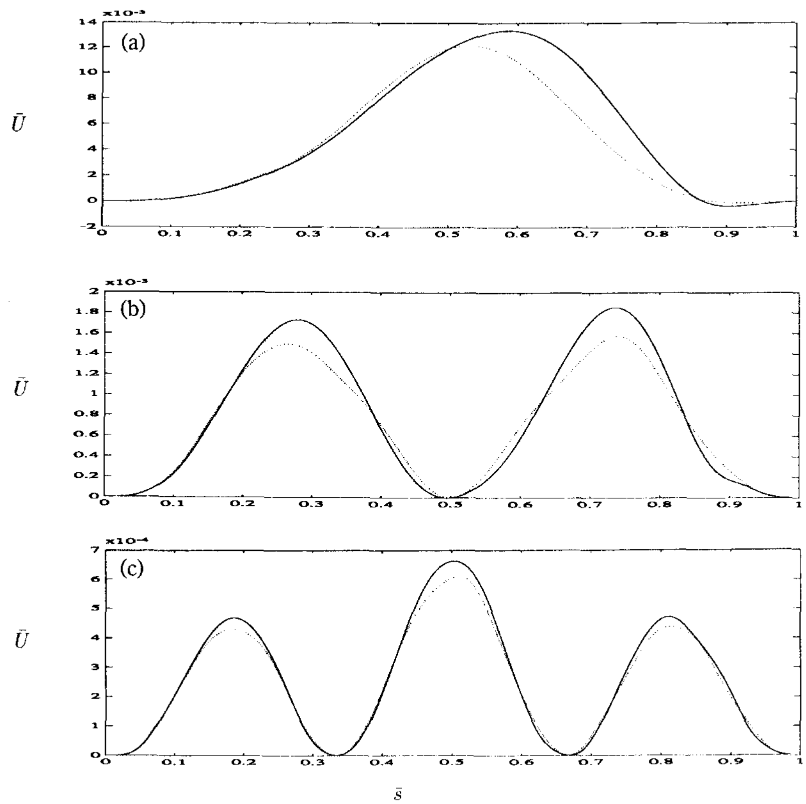

Fig. 8 Comparison between the dimensionless deflections under the moving load computed using the "moving-force moving-mass" formulation ('—') and the "moving-force" formulation ('------') for a clamped-clamped beam. (a) $N=0$, (b) $N=1$, $\bar{s}_1=1/2$, (c) $N=2$, $\bar{s}_1=1/3$, $\bar{s}_2=2/3$.

motion. The possible separation of the mass from the beam due to a high travelling speed of the mass can therefore be avoided by the use of multiple supports. However, if the travelling speed of the mass is too large, the presence of some intermediate supports may not be able to avoid the possible separation of the mass from the beam.

The differences between the numerical results for the present "moving-force moving-mass" formulation and the related "moving-force" formulation by removing the inertial terms of the moving mass are examined in Fig. 7 for a simply supported beam and in Fig. 8 for a clamped-clamped beam. The numerical results for both formulations, when converted to dimensional form, are found to be in excellent agreement with the reported results by Sadiku and Leipholz (1987). It has been concluded in Sadiku and Leipholz (1987) that the "moving-force" approximation is not an upper bound solution for the related "moving-force moving-mass" problem for the prescribed system parameters. This is indeed the case for the numerical results shown in Figs. 7 and 8 even for a beam with multiple supports. The numerical curves for both the formulations cross each other on several occasions during the course of the prescribed motion. However, the magnitude of differences between the numerical results for the two formulations diminishes with the increase in the number of intermediate supports.

4. Conclusion

The equation of motion in matrix form has been formulated for the dynamic response of a beam on multiple supports acted upon by a moving mass using the Lagrangian approach and the assumed mode method. The multiple supports are modeled by linear springs of large stiffnss. Convergence of numerical results is found to be achieved with just a few terms for the assumed deflection function and the stiffness of the linear spring moderately large. The present numerical results in dimensionless for enable the results to be applicable for a large combination of system parameters. Numerical results for other combinations of beam length, magnitude and speed of the moving mass can therefore be easily computed. It is found that the presence of multiple supports can suppress the separation of the mass from the beam by reducing the fluctuation of the contact force between the mass and the beam. The present numerical results also confirm the finding by Sadiku and Leipholz (1987) that numerical results computed using the "moving-force" formulation are not always upper-bound solutions for the results computed using the corresponding "moving-force moving-mass" formulation.

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