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Extension of the adaptive boundary element scheme for the problem with mixed boundary conditions

N. Kamiya†

Department of Informatics and Natural Science, School of Informatics and Science, Nagoya University, Nagoya 464-01, Japan

Y. Aikawa‡

Graduate School of Nagoya University

K. Kawaguchi‡†

Nippon HP Co., Suginamiku, Tokyo 168, Japan

Abstract. This paper presents a construction of adaptive boundary element for the problem with mixed boundary conditions such as heat transfer between heated body surface and surrounding medium. The scheme is based on the sample point error analysis and on the extended error indicator, proposed earlier by the authors for the potential and elastostatic problems, and extended successfully to multidomain and thermoelastic analyses. Since the field variable is connected with its derivative on the boundary, their errors are also interconnected by the specified condition. The extended error indicator on each boundary element is modified to meet with the situation. Two numerical examples are shown to indicate the differences due to the prescribed boundary conditions.

Key words: boundary element method; adaptive mesh; error analysis; h-version method; boundary conditions; mixed boundary conditions; potential problem; elastostatics

1. Introduction

The construction of appropriate meshes or elements is a fundamental requirement for success in numerical analyses using discretized elements such as the finite element method (FEM) and the boundary element method (BEM), which demands serious research for adaptive elements (e.g., Babuska *et al.* eds. 1986, Shephard and Weatherill eds. 1991, Kamiya ed. 1992). Accuracy of the results obtained through these softwares, specifically employed as blackbox by inexperienced users, will not be satisfied. Some FEM softwares are implemented to improve accuracy automatically, up to the desired level, with the adaptive elements.

The researches of the adaptive boundary elements date back to about ten years, e.g., Alarcon

† Professor
‡ Graduate Student
‡† Engineer

and Reverter (1986), Rencis and Jong (1989), Rank (1989), Guiggiani (1990), Parreira (1990), Postel and Stephan (1990), Mullen and Rencis (1985), Urekew and Rencis (1989), Umetani (1986), Ingber and Mitra (1986), Sun and Zamani (1990), Cerrolaza, Gomez-Lera and Alarcon (1988), Yuuki, Cao and Ueda (1990), see also for further References, Kita and Kamiya (1994). Among others, the authors proposed a method "sample point error analysis and h-version mesh refinement by extended error indicator". Its reliability has been proven by various applications to the potential problem (Kamiya and Kawaguchi 1992a), elastostatics (Kamiya and Kawaguchi 1992b), multi-domain problem (Kamiya and Koide 1993), thermo-elastostatics (Kamiya, Aikawa and Kawaguchi 1994a) and elastic problem with body force (Kamiya, Aikawa and Kawaguchi 1994b). The method uses the extended error indicator representing an *influence* of element-wise error on an inconsistency of the integral equation, in place of element-wise errors directly employed conventionally.

We here further extend the method to the problems with the mixed boundary conditions, for which element-wise errors of the field variable or its derivative are not defined explicity. The heat transfer conditions in the heat conduction phenomena and the spring support in the elasticity are the typical cases of the mixed boundary conditions. The extended error indicator, however, can treat such case without substantial modification, because it is defined by the boundary integral equation and the related integral representation connecting the element-wise errors. In what follows, we explain the method by using the potential-type problem, say, corresponding to the steady-thermal problem without internal heat generation accompanying the heat transfer boundary condition. Analysis will be conducted for the two-dimensional heat conduction problem and elastostatics.

2. Boundary element formulation and the difficulties in error analysis

Consider the following potential problem governed by the Laplace equation in the two-dimensional domain Ω bounded by the boundary Γ,

$$\nabla^2 u = 0 \tag{1}$$

with the boundary condition

$$q = \alpha u + \beta \tag{2}$$

where u, $q(\equiv \partial u/\partial n)$ are the potential and the flux, derivative of u in the normal direction n on the boundary, and α, β are the constants. Eq. (2) is ordinarily the mixed condition but can represent the Dirichlet and Neumann conditions by selecting appropriately two constants α and β.

Introducing the fundamental solutions for the Laplace equation:

$$u^* = \frac{1}{2\pi}\log\frac{1}{r}, \qquad q^* = \frac{\partial u^*}{\partial n} \tag{3}$$

Eq. (1) is transformed to the following boundary integral equation:

$$cu = \int_\Gamma (u^* q - q^* u)\, d\Gamma \tag{4}$$

where r in Eq. (3) and c in Eq. (4) are respectively the distance between the source and the field points, and the constant depending on the place where the source point is taken.

Eq. (4) is discretized by boundary elements such as

$$c^i u^i = \sum_{j=1}^{N} \int_{\Gamma_j} (u^* q - q^* u) d\Gamma \tag{5}$$

where i denotes the selected point on the boundary and N indicates the number of boundary elements. Using assumed interpolations for u and q on the boundary elements, Eq. (5) is reduced to

$$\boldsymbol{Hu} = \boldsymbol{Gq} \tag{6}$$

where $\boldsymbol{u}$ and $\boldsymbol{q}$ are the vectors of u and q on the boundary nodes and the coefficient matrices $\boldsymbol{H}$ and $\boldsymbol{G}$ are computed on each boundary element by integrating the product of the fundamental solution with the interpolation function. Eq. (6) is solved, for the approximation of u and q, simultaneously with the mixed boundary condition Eq. (2).

In order to estimate the errors of the solution obtained by Eq. (6) for the construction of adaptive elements, the principal difficulties are as follows:

(1) When the Dirichlet condition is given on one part of the boundary and the Neumann condition on the other, the solution is q on the first part and u on the other. Therefore, their dimensions are different and cannot be compared directly. Conventional methods use appropriate constants for nondimensionalization of these variables. It is evident that the result will depend on such selection.

(2) In the finite element methods, however, the global equation is constructed by assembling the equations holding locally with the aid of the finite element discretization. For the displacement method, the unknowns are the displacement components alone. On the other hand, the discretized Eq. (5) reduced from Eq. (3) is related to a whole boundary elements. This fact indicates that the error on one element affects the accuracy of the entire solution, and that the modification of the element by the criterion of the magnitude of its error seems not sufficient. If the mixed boundary condition is specified, the solution is obtainable by eliminating either u or q by using Eq. (2). However, the error estimation will be different depending on the way of elimination.

Consequently, a unified strategy is required for the appropriate error estimation rule over the whole problem irrespective of the Dirichlet, Neumann and mixed boundary conditions.

3. Sample point error analysis

Since the details of the original sample point error analysis is given in Kamiya and Kawaguchi (1992a, b), here we briefly summarize it for convenience. The scheme is based on the fact that the boundary integral equation in the conventional boundary element analysis holds, only on the selected collocation points on the boundary, and does not on other boundary points than those indicated. Therefore, on the latter boundary points called "sample points", the solution inevitably yields some inconsistency (or residual) of the boundary integral equation, which is thought to appear owing to the discretization error of the boundary elements (other errors such as numerical integration should be removed by sufficiently precise and careful computation). We suppose that the prescribed value for the boundary condition does not have any error.

And therefore, the errors on the boundary for the Dirichlet and Neumann conditions are those of u and q, respectively, which have different dimensions. This is one difficulty arising in the direct boundary formulation as mentioned above.

The inconsistency due to the approximate solutions $\hat{u}$, $\hat{q}$ is defined as

$$r \equiv c\hat{u} - \int_{\Gamma} (u^* \hat{q} - q^* \hat{u}) d\Gamma \neq 0 \tag{7}$$

Since the errors e_u and e_q are thought to required amendments to the approximate solution for fulfillment of the integral equation on the sample point, the following equation is derived

$$c(\hat{u} + e_u) = \int_{\Gamma} [u^*(\hat{q} + e_q) - q^*(\hat{u} + e_u)] d\Gamma \tag{8}$$

which leads to

$$r = \int_{\Gamma} (u^* e_q - q^* e_u) d\Gamma - c e_u \tag{9}$$

It should be mentioned that the solution inconsistency r at the sample point can be decided when the approximate solution is obtained for the Dirichlet or Neumann boundary condition. Distribution of e_u and e_q is not known in advance and is assumed, on the element j, approximately by

$$e_u = e_u^j \Psi, \quad e_q = e_q^j \Psi \tag{10}$$

where Ψ is supposed to be a triangular interpolation function (approximation to the quadratic distribution diminishing at the extreme points; the error will be higher function than the assumed distribution of the variable) in terms of the local coordinate $(-1 < \xi < 1)$

$$\Psi = 1 + \xi(-1 \leq \xi \leq 0), \qquad 1 - \xi(0 \leq \xi \leq 1) \tag{11}$$

for the sample point taken on the middle of the boundary element ($\xi = 0$) with linear interpolation. In general, it is possible to take some sample points on one element but the adaptive process requires iterative computation and thus, the indicated selection is for the convenience to reduce computation time for the adaptation and further corresponds to the way of element modification; h-version bisection at the middle point of the element in this paper.

From the discretized equation of Eq. (9) for each sample point taken on the middle of every boundary element, e_u^j, e_q^j can be determined because r is already known. Formally Eq. (9) is written as

$$\boldsymbol{r} = \boldsymbol{B}\boldsymbol{e} \tag{12}$$

where $\boldsymbol{e}$ is the vector of either e_u^j or e_q^j on each element. It must be mentioned that for the Dirichlet condition $e_u = 0$ and $e_q \neq 0$, and for the Neumann condition $e_q = 0$ and $e_u \neq 0$.

In the case of the mixed boundary condition, owing to Eq. (2)

$$e_q = \alpha e_u \tag{13}$$

holds and is employed in Eq. (9) to eliminate either e_u or e_q; and thus for instance, after elimination of e_q

$$\boldsymbol{r} = \boldsymbol{B}' \boldsymbol{e}_u \tag{14}$$

where $\boldsymbol{e}_u$ is the vector of e_u^j on each element.

We should recognize, from Eq. (12) or (14), that the influence of the element-wise error on the solution inconsistency is not by only its magnitude but by the product with the coefficient matrix $\boldsymbol{B}$ or $\boldsymbol{B}'$, composed of the integral over the boundary element of the fundamental solution. For the total N boundary elements, Eq. (14), in component, is

$$r_j = \sum_{k=1}^{N} B'_{jk} e_u^k \tag{15}$$

Each term of the right-hand side of the above equation is expressed by

$$\eta_{jk} = B'_{jk} e_u^k \tag{16}$$

which is defined as the extended error indicator.

η_{jk} represents *the magnitude of the influence* of error $(e_u)^k$ of the element k on the solution inconsistency at the sample point taken on the element j. As indicated above, for the Dirichlet condition $e_u = 0$ and $e_q \neq 0$, and for the Neumann condition $e_q = 0$ and $e_u \neq 0$, but for the mixed boundary condition both e_u and e_q are not vanishing. However, as mentioned in the derivation of Eqs. (13) to (16), e_u and e_q are related by Eq. (13) and formally Eq. (16) is formulated by eliminating one of them. The required modification for the problem with the mixed boundary condition is contained in Eqs. (13) to (16). η_{jk} is employed to choose the elements to be refined for the h-version adaptive process: in comparison with the specified criterion η_{tol}, if

$$\eta_{jk} \geq \eta_{tol} \tag{17}$$

the "element k" is divided into two at the sample point (middle point on the element). *This means that the error on the element k multiplied by the corresponding fundamental solution has relatively large amount of contribution to the solution inconsistency at the sample point on the element j, and then the element j and then the element k is refined.* η itself does represent neither the absolute error nor the relative percentage error of the solution, which is thought as the measure for the mesh refinement. Refinement iteration is repeated until a sufficiently accurate solution is obtained.

4. Examples and discussion

Two numerical examples are shown for the two-dimensional problem to verify the potentiality of the extension of the original adaptive scheme. The employed boundary elements are straight line segments with linear interpolation (linear elements, with double modes on the corner points; selection of the linear elements is closely connected to the error model employed). The sample points are taken at the middle of each element. Careful computation of the numerical integral by the adaptive integration and of the simultaneous linear equation is performed to avoid additional errors. A whole adaptive process starts with the least number of boundary elements sufficient for the given problem: indication of the geometry and the prescribed boundary condition. As a criterion of Eq. (17), η_{tol}, the average of η_{jk} is employed, because it is one of the value easily obtainable and worked well in the previous studies.

Fig. 1 shows the first example of the L-shaped two-dimensional domain. The side FA is in the condition of heat transfer with the parameter (a) $\alpha = -30$ (1/m) and (b) $\alpha = -0.01$ (1/m).

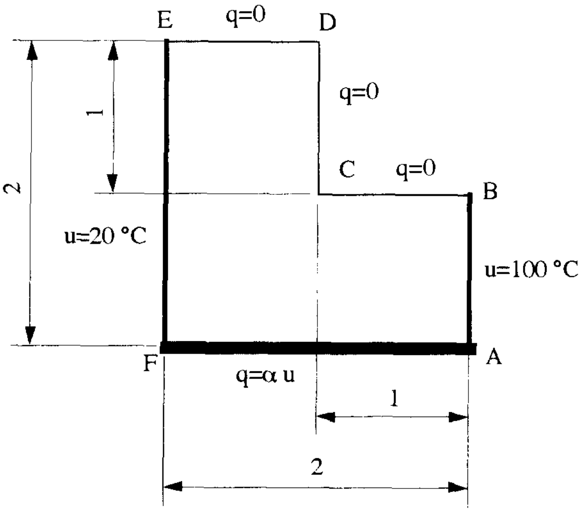
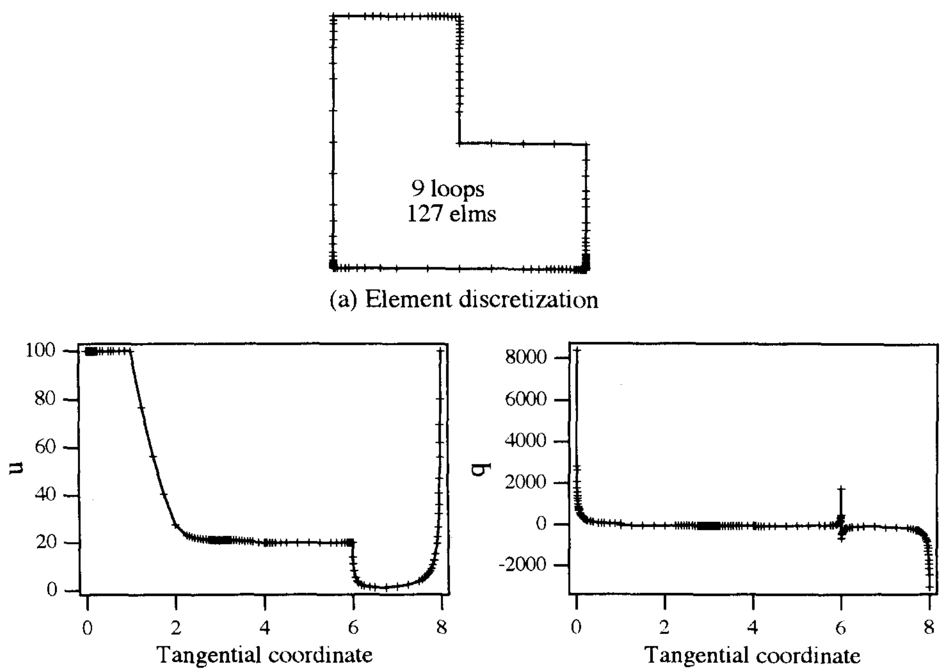


Fig. 1 Example 1 (heat conduction problem)



(b) Potential

(c) Flux

Fig. 2 Computation results (after 9 loops, $\alpha = -30$)

The latter corresponds nearly to the adiabatic condition $q = 0$. Figs. 2 to 4 are the obtained results after a few adaptive iterations: the adaptive meshes and the distributions of u and q along the boundary measured from the point A counter-clockwise. Accuracy of the results is partially verified by comparing them with sufficiently fine, uniform boundary meshes, say, 200 elements. These results indicate that the desired adaptive boundary element distributions are

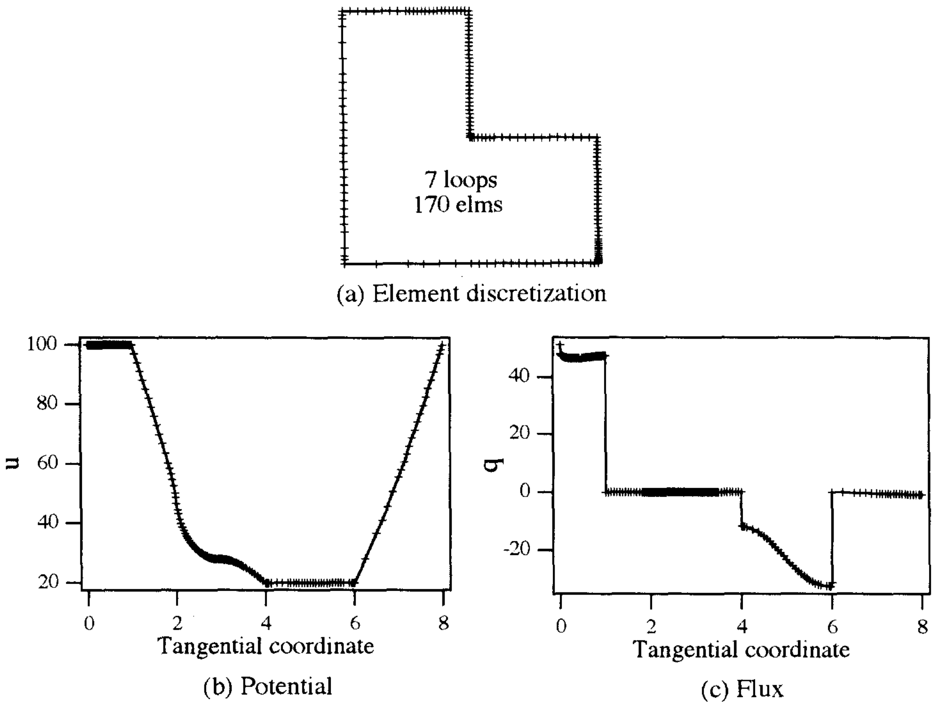


Fig. 3 Computation results (after 7 loops, $\alpha = -0.01$)

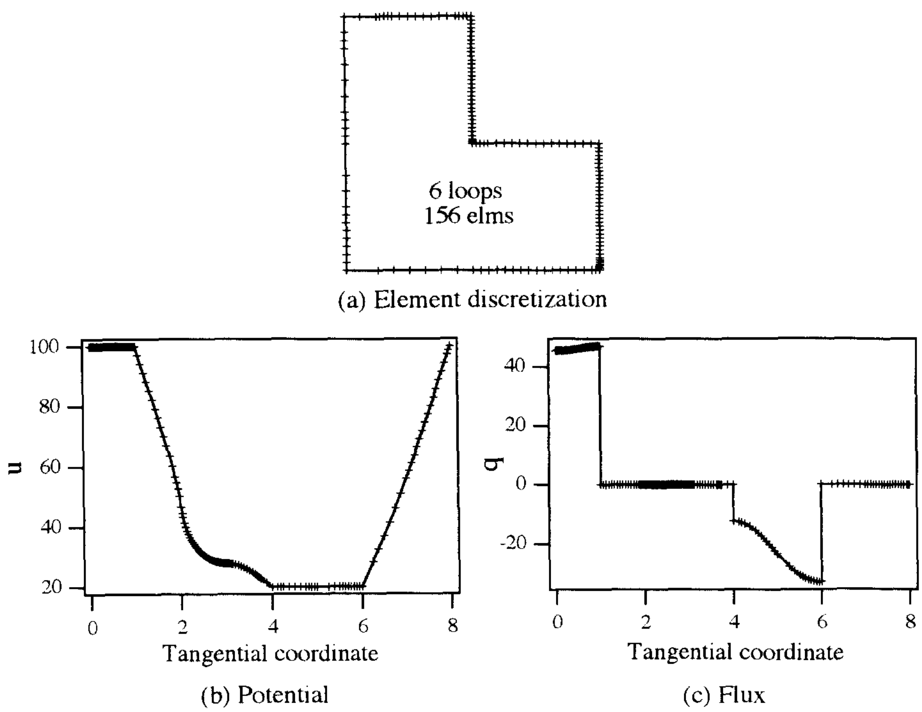


Fig. 4 Adaptive boundary elements for the adiabatic case (after 6 loops)

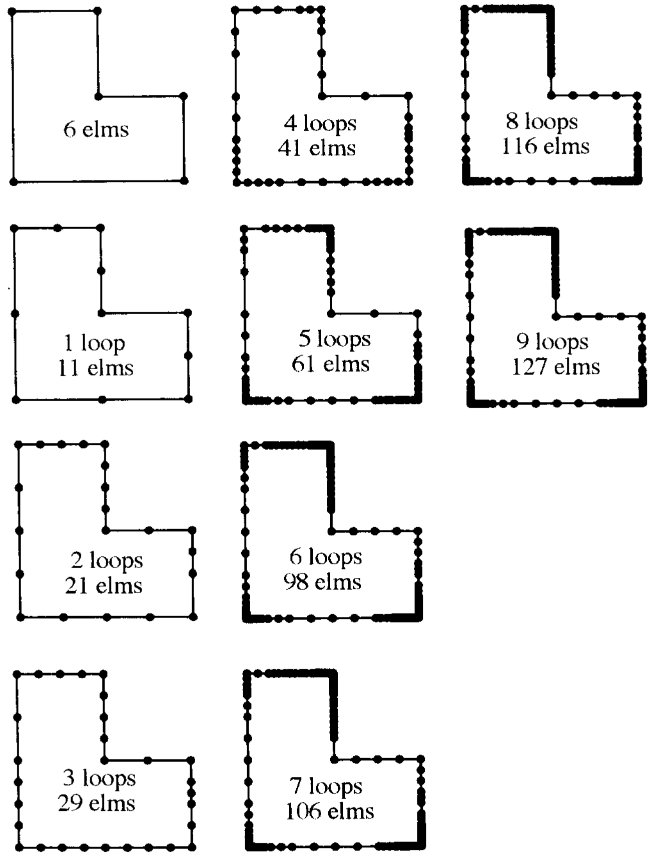


Fig. 5 Adaptive process ($\alpha=-30$)

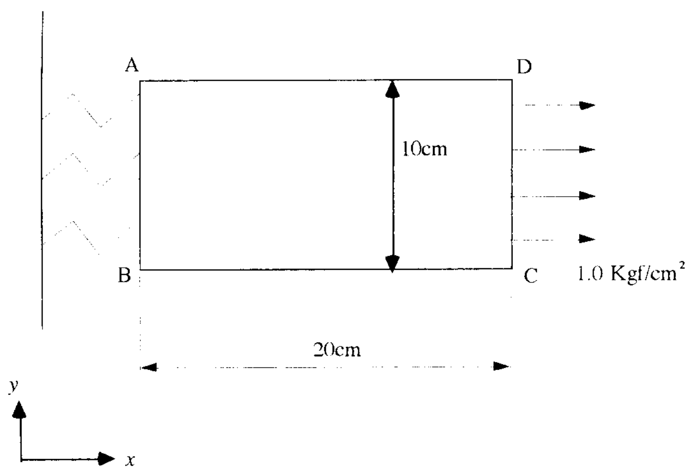


Fig. 6 Example 2 (two-dimensional elastostatics)

different for each distinct problem with distinct boundary conditions, and cannot be predicted in advance. The final meshes for the heat transfer condition with $\alpha=-0.01$ (Fig. 3) is almost

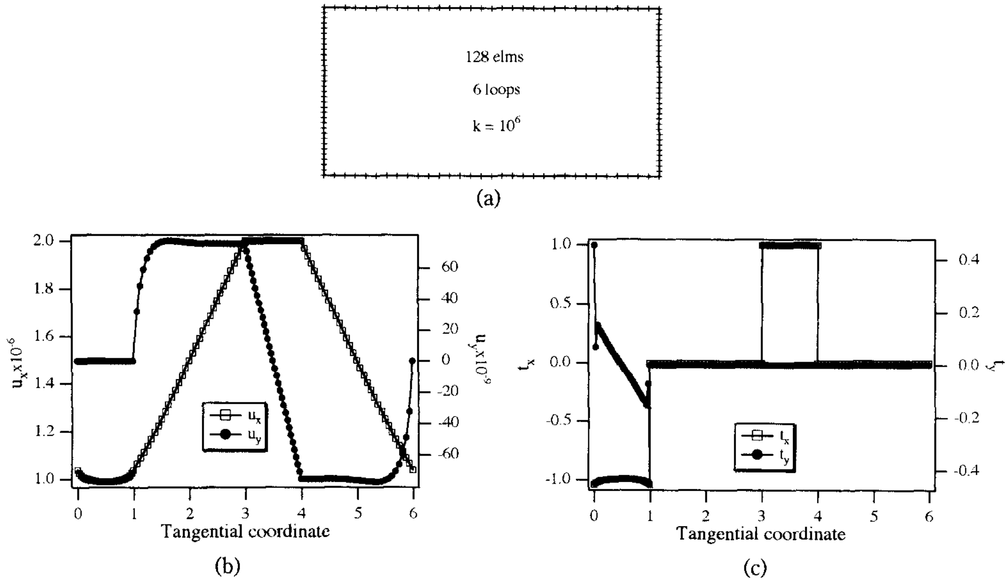


Fig. 7 Computation results (after 6 loops, $k = 10^6$)

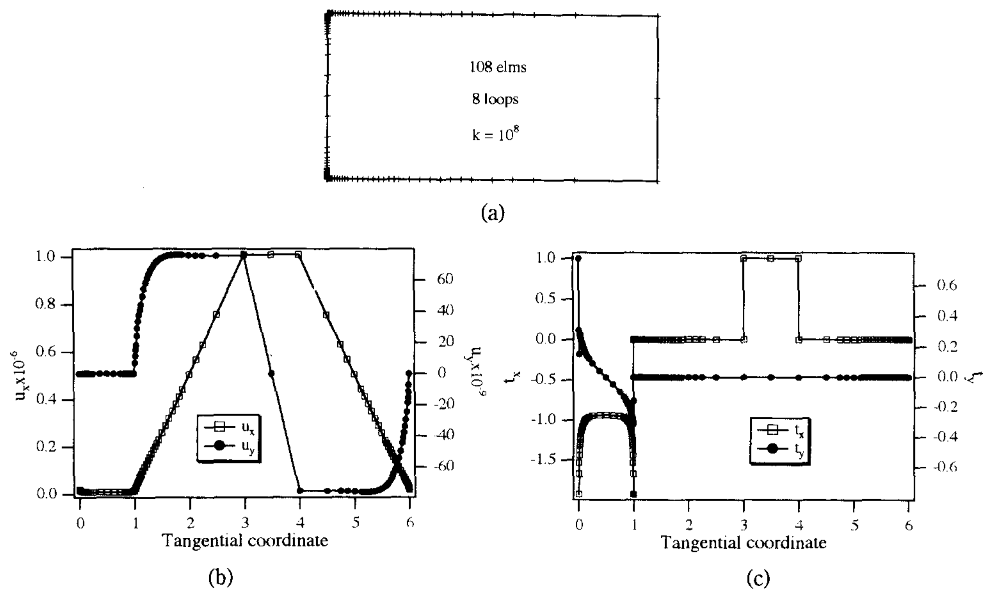


Fig. 8 Computation results (after 8 loops, $k = 10^8$)

identical to the adiabatic case (Fig. 4). The detailed adaptive process is shown for $\alpha = -30$ in Fig. 5, from the initial 6 elements to the ninth iterative loop.

Fig. 6 is the second example in a two-dimensional elastostatic rectangular plate problem under the plane stress state. The left side of the plate is constrained in the y-direction but the displacement in the x-direction is proportional to the force thereon: the boundary condition is $u_y = 0$, $t_x = -ku_x$ (k is constant; t and u are the traction and the displacement, respectively). The formulation and adaptive scheme for the elastostatic problems were shown in Kamiya and Kawaguchi

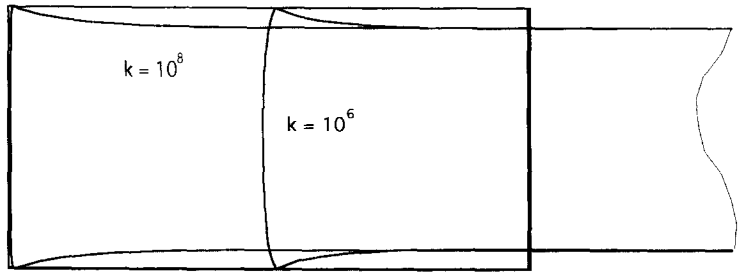


Fig. 9 Deformation of the plate near the constrained side

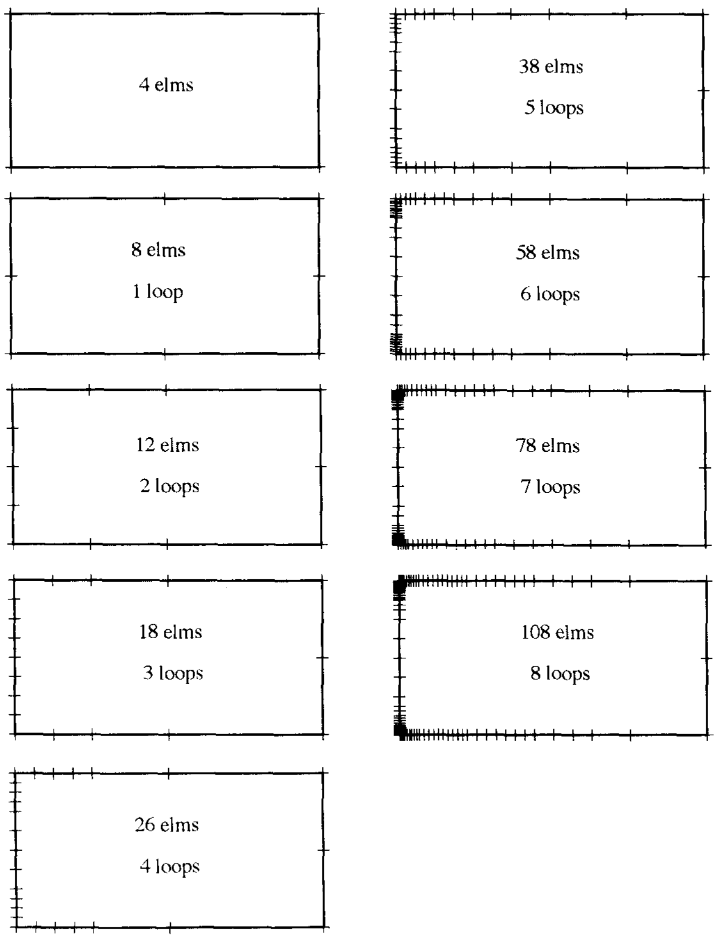


Fig. 10 Adaptive process ($k=10^8$)

(1992b) and are applied here (Young's modulus$=2.1\times10^6$ Kgf/cm^2, Poisson's ratio$=0.3$). The magnitude of k is taken as 10^6, 10^7, 10^8, 10^{13} Kgf/cm^3. Only four initial elements, one on each side, are sufficient for this problem. Figs. 7 and 8 are the results of the distributions of the displacement and traction components along the boundary measured from the point A counter-clockwise. Fig. 9 is the schematic enlarged view of the deformation of the plate in the vicinity

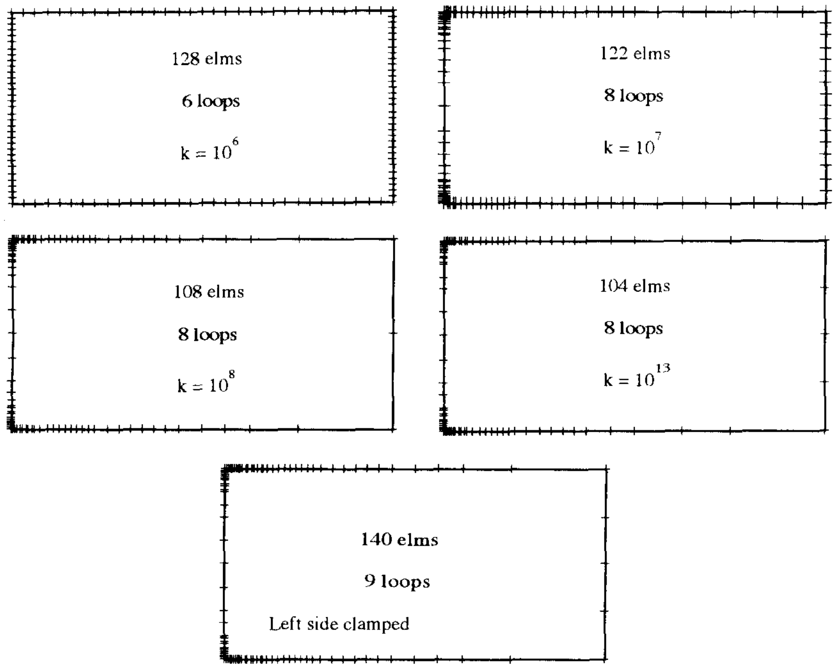


Fig. 11 Comparison of adaptive element distributions

of the left side. It can be noticed from Figs. 7 to 9 that for larger k the condition on the left side approaches the clamped condition. Fig. 10 shows the adaptive iteration process from the initial four elements to the eighth loop. For different values of k, Fig. 11 declares the clear difference of the adaptive boundary element distributions.

Further to the first example for the scaler-valued problem, the second example shows the potentiality of the proposed adaptive boundary element scheme for the case of vector-valued problem (nodal values have each two components in x- and y-directions in the problem considered here), even for the problem with the mixed boundary condition.

5. Conclusion

The adaptive boundary element h-scheme using the sample point error analysis and the mesh refinement by the extended error indicator was extended to include the mixed boundary condition for the potential and elastostatic problems. Only slight modifications of the original scheme and of the computer code are necessary to give acceptable results for the above-mentioned boundary conditions on the numerical examples. The scheme is also applicable to a class of symmetric condition, e.g., the displacements in the x- and y-components are identical, $u_x = u_y$ on the line $x=y$. By using the indicated adaptive scheme, without *a priori* proper prediction of the boundary element distribution, one can obtain accurate solution.

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