Structural Engineering and Mechanics, Vol. 4, No. 2 (1996) 177-190
DOI: http://dx.doi.org/10.12989/sem.1996.4.2.177

A numerical model for externally prestressed beams

M.A. Pisani†

Department of Structural Engineering. Politecnico di Milano, P.zza L. da Vinci 32, 20133 Milan, Italy

Abstract. A method to numerically evaluate the behaviour of single span beams, prestressed with external tendons and symmetrically loaded is presented. This algorithm, based on the Finite Difference Method, includes second order effects and large displacements in an attempt to more fully understand the behaviour of the beam up to collapse. The numerical technique discussed is particularly appropriate for the analysis of R.C. and P.C. beams rehabilitated or strengthened by means of external prestressing but it is reliable for the analysis of new beams as well.

Key words: external prestressing; numerical method; simple span beams.

1. Introduction

An initial distinction is helpful when dealing with precompression: frequently the prestressing tendons are placed inside the concrete cross section and bonded by filling the tendon ducts with cement grout; this is called conventional prestressing. When, on the contrary, the bond between concrete and prestressing tendons is eliminated, friction inside the ducts is artificially reduced to minimum values and the tendons transfer their loads to concrete through end anchorages and deviators, the terms "unbonded" and "external" prestressing are adopted.

The term "external prestressing" is used if the tendons are placed outside the concrete cross section, otherwise the name "unbonded prestressing" is adopted.

Unbonded tendons are mainly used in slab systems and cylindrical or conic vessels, while external prestressing is being increasingly adopted in the construction of new bridges (mainly precast segmental bridges) and large roofs, or the rehabilitation and the strengthening of existing structures.

Referring to the design of bridge superstructures it is interesting to remember that the first post-tensioned bridge, designed by Dischinger and constructed in 1936-37 at Aue, Saxony, is externally prestressed. Nevertheless, this prestressing technique was substantially abandoned until the eighties.

The arguments used to promote external prestressing are related both to durability and to workmanship during construction or rehabilitation operations.

With regard to durability, external tendons allow inspection of the corrosion protection (very difficult in conventional prestressing), control and correction of the prestressing forces and replacement of tendons, when needed. If external post-tensioning is adopted in the casting of new structures, no duct is placed inside the concrete section, but this means easier pouring of concrete and no obstruction of the ducts (as sometimes happens in conventional post-tensioning). When

† Assistant Professor

this prestressing technique is adopted to strengthen an existing structure only anchorages and deviators need be added to the old structure and no crib is required.

If, then, external prestressing has interesting advantages compared to conventional prestressing and is old technology, nevertheless the behaviour of beams prestressed with external tendons at the ultimate limit state is a topic which has yet to be examined in any depth. Usually the analysis of these beams is carried out adopting the hypothesis that the tensile stress in the tendons is constant, or independent of the level of the loads (see for instance E.C.2), although Naaman (1990), (1991) demonstrated that even before the beam cracks, the increase of the tensile strain in the tendons is more than one half of that measurable in the cables of a similar beam conventionally prestressed.

The absence of bond between tendons and concrete implies the solution of a structural problem, whereas the evaluation of the stress distribution in a conventional P.C. beam is a problem related to its sections: the changes in the shape of the concrete beam involve changes in the relative position of the tendons, so that their influence on the equilibrium conditions of the deformed beam may be significant. Many works neglect this effect though its extent is not well known, as the hypothesis of small displacements is usually adopted even though in a condition near collapse the displacement of the beam may be considerable.

To more fully understand the beam behaviour, a numerical algorithm, based on the Finite Difference Method, was developed.

The present work aims to describe this algorithm that includes second order effects, large displacements and the change in length of the beam due to compression, neglects shear deformation both before and after cracking, and adopts the hypothesis that bending of the beam is a continuous function. This assumption is particularly advisable in the analysis of simple span R.C. or P.C. beams (i.e. existing bridges) repaired or strengthened by means of external posttensioning, but demonstrated to be reliable for the analysis of new beams prestressed with unbonded tendons as well.

2. Formulation of the problem

Let's adopt the global coordinates x, y, z. The shape of the undeformed beam, or when no external load and no prestressing are applied, will be described by means of these coordinates, set with plane yz coincident with the plane of symmetry of the cross section, origin O placed in the center of rotation of the left bearing (a hinge, see Fig. 1) and the z axis parallel to the beam axis (or usually coincident with the lower edge of the beam). Notice that mathematically speaking, origin O could be placed anywhere; nevertheless the previous assumption will allow us to explain more clearly the physical meaning of some terms and to avoid difficulties related to the definition of the boundary conditions.

From a general point of view a beam prestressed with external tendons may be considered as the coupling of two substructures: the concrete beam and the external tendons.

Fig. 1 shows that the interaction between the two substructures is restricted to the points (with coordinates y_{pi}, z_{pi} in the undeformed shape of the concrete beam) where anchorages and deviators are placed.

To describe the beam behaviour, we will evaluate the displacements of the trace of the z axis on the concrete substructure that, through the hypothesis that plane sections remain plane

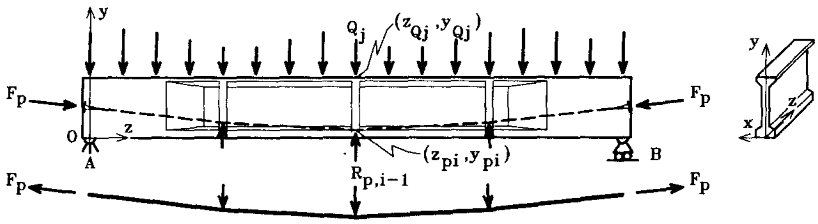

Fig. 1 Undeformed shape of the beam

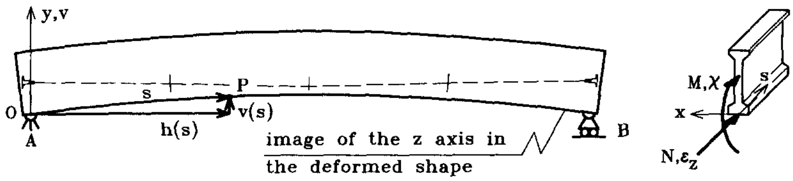

Fig. 2 Deformed shape of the beam

and orthogonal to this trace (shear deformation is neglected) after bending, will allow us to determine the displacement of every point of the beam and then the response of the tendons too.

2.1 The defomed shape of the beam

Let us call the curvilinear coordinate s, measured on the image of the z axis in the deformed shape (or usually the lower edge of the bent concrete substructure), the displacement of a point P of this image in the y direction $v(s)$, while $h(s)$ is the position of point P on the z axis (or the sum of its abscissa before bending plus its horizontal displacement). The curvature χ of the concrete substructure in the yz plane can be written as (see Fig. 2):

$$\chi(s)=\sqrt{\left(\frac{d^2h(s)}{ds^2}\right)^2+\left(\frac{d^2v(s)}{ds^2}\right)^2}=\sqrt{h''^2(s)+v''^2(s)} \tag{1}$$

Moreover, $h(s)$ and $v(s)$ are related by the equation:

$$\left(\frac{dh(s)}{ds}\right)^2+\left(\frac{dv(s)}{ds}\right)^2=h'^2(s)+v'^2(s)=1 \tag{2}$$

Extracting $h''(s)$ from the first derivative of Eq. (2) and introducing it in Eq. (1) we get:

$$\frac{v''^2(s)}{\chi^2(s)}+v'^2(s)=1 \tag{3}$$

$$h'(s)=\frac{v''(s)}{\chi(s)} \tag{4}$$

so that $v(s)$ is now independent of $h(s)$ and, if $\chi(s)$ is known, can be computed assigning two boundary conditions, while $h(s)$ will be determined after $v(s)$, making use of the last boundary condition (i.e. $v(0)=v(s_{max})=0$, $h(0)=0$).

Eq. (3) is non linear and gives two solutions. They can be written as:

$$\chi(s)=(-1)^{\delta}\frac{v''(s)}{\sqrt{1-v'^2(s)}} \qquad (\delta=1,\ 2) \tag{5}$$

but stating that for $\delta=1$ Eq. (5) violates the arbitrary assumptions adopted in Figs. 1, 2 regarding the positive sign of $v(s)$ and $\chi(s)$, we get:

$$\chi(s)=\frac{v''(s)}{\sqrt{1-v'^2(s)}} \tag{6}$$

or:

$$\frac{\chi(s)}{|\chi(s)|}=\frac{v''(s)}{|v''(s)|} \tag{7}$$

so that $\chi(s)$ and $v''(s)$ must have the same sign. (Note that owing to the change in length of the beam, h and v are functions of s, not z, and therefore Eq. (6) is quite different from the one usually adopted).

2.2 Correlation between the concrete beam axis and the cross section behaviour

In the preceding paragraph we implicitly adopted the hypothesis that the image of the z axis in the deformed shape is a continuous function of s. Consequently precast segmental beams, especially if cast with dry joints, are excluded from this analysis that aims to numerically simulate the behaviour of beams with continuous reinforcement.

If the conditions we have already fixed are fulfilled, curvature $\chi(s)$ can be set equal to the curvature of each section of the concrete substructure and if the hypothesis that plane sections remain plane after deformation is adopted, the computation of the concrete cross section respons is well established (Rotter 1985, Mirabella 1986).

From the analysis of the concrete cross section we get both curvature and strain ε_z (tensile strain is negative) on the image of the z axis in the deformed shape of the concrete beam. The curvilinear coordinate s is related to ε_z by means of Eq. (8):

$$s=s(z)-\int_0^z (1-\varepsilon_z)\,dz \tag{8}$$

where z is the coordinate of the cross section before bending.

2.3 Evaluation of the tensile stress in the tendon

Once $h(s)$ and $v(s)$ are known, it is possible to evaluate the increase in strain $\Delta\varepsilon_p$ in the tendons (tensile stresses and strains in the tendons are positive).

Named L_i the length of a straight segment of the tendon in the undeformed shape we have:

$$L_i=\sqrt{(y_{pi+1}-y_{pi})^2+(z_{pi+1}-z_{pi})^2} \tag{9}$$

while if l_i is the length of the same segment of the deformed shape we get:

$$\delta_{zi}=h(s_{pi+1})-y_{pi+1}v'(s_{pi+1})-[h(s_{pi})-y_{pi}v'(s_{pi})] \tag{10}$$

$$\delta_{yi} = v(s_{pi+1}) + y_{pi+1} h'(s_{pi+1}) - [v(s_{pi}) + y_{pi} h'(s_{pi})] \tag{11}$$

$$1_i = \sqrt{\delta_{zi}^2 + \delta_{yi}^2} \tag{12}$$

where s_{pi+1} is the curvilinear coordinate of the concrete cross section where the i-th deviator is placed. Note that δ_{zi} and δ_{yi} are the lengths of the projection on z and y axes respectively of the i-th segment of the tendon after bending.

If the tendons are not protected with cement grout (or they are protected with grease, so that friction is usually negligible), $\Delta\varepsilon_p$ is uniform over the entire tendon and is equal to:

$$\Delta\varepsilon_p = \frac{\sum_i 1_i}{\sum_i L_i} - 1 \tag{13}$$

otherwise each straight segment will have a different increase in strain $\Delta\varepsilon_{pi}$

$$\Delta\varepsilon_{pi} = \frac{1_i}{L_i} - 1 \tag{14}$$

Once strain $\Delta\varepsilon_{pi}$ in the deformed shape is known it is easy to determine the stress resultant F_{pi} in the tendon by means of its constitutive law.

2.4 The internal forces in the concrete substructure

Named Q_j the j-th concentrated external load acting on the beam in the y direction in Fig. 1, the vertical reaction V_A of bearing A is:

$$V_A = \sum_j Q_j \left[1 - \frac{h(s_{Qj})}{h(s_{max})} \right] \tag{15}$$

where s_{Q_j} is determined setting $z = z_{Q_j}$ in Eq. (8), while $h(s_{max})$ is evaluted in the section where bearing B acts.

By means of Eqs. (10) and (11) we can now evaluate the slope β_i of the i-th segment of the tendon after bending Fig. 3:

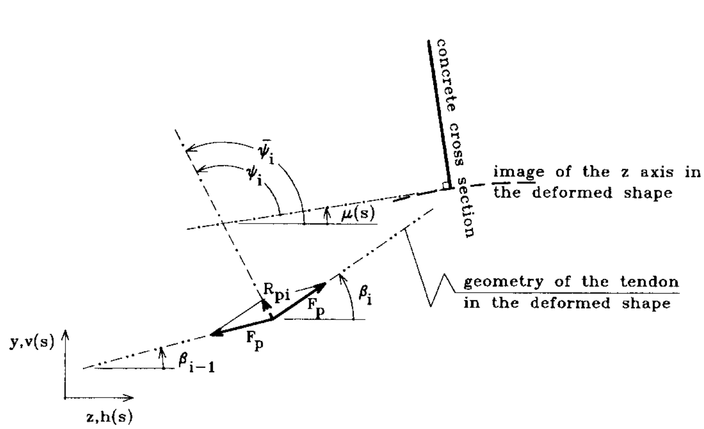

Fig. 3 Orientation of R_{pi}

$$\beta_i = \text{arctg}\ \frac{\delta_{yi+1}}{\delta_{zi+1}} \tag{16}$$

If F_p is constant along the tendon, the direction $\overline{\psi}_i$ (with respect to the z axis) of the stress resultant R_{pi} transferred by the i-th deviator to the concrete substructure is:

$$\overline{\psi}_i = \frac{\pi + \beta_i + \beta_{i-1}}{2} \tag{17}$$

and then:

$$R_{pi} = 2F_p \cos(\overline{\psi}_i - \beta_i) \tag{18}$$

By naming the slope of the concrete cross section $\mu(s)$, we get:

$$\mu(s) = \arcsin\ v'(s) \tag{19}$$

and the direction of R_{pi} with respect to the beam axis in the deformed shape (or the image of the z axis) is:

$$\psi_i = \overline{\psi}_i - \mu(s) \tag{20}$$

Finally, the interanl forces in the concrete cross section are:

$$\begin{aligned} N(s) = {} & [V_A - \textstyle\sum_j Q_j \Omega_j] \cdot v'(s) + F_p \cos(\mu(s) - \beta_0) + \\ & + \textstyle\sum_i R_{pi} \Omega_i \cos\psi_i \end{aligned} \tag{21}$$

$$\begin{aligned} M(s) = {} & V_A [h(s) + y_A \cdot v'(s)] + \\ & - \textstyle\sum_j Q_j \Omega_j [h(s) - h(s_{Qj}) + y_{Qj} \cdot v'(s_{Qj})] + \\ & - F_p [v(s) - v(s_{p1}) - y_{p1} \cdot h'(s_{p1})] \cos\beta_0 + \\ & + F_p [h(s) - h(s_{p1}) + y_{p1} \cdot v'(s_{p1})] \sin\beta_0 + \\ & - \textstyle\sum_i R_{pi} \Omega_i [v(s) - v(s_{pi+1}) - y_{pi+1} \cdot h'(s_{pi+1})] \cos\overline{\psi}_i + \\ & + \textstyle\sum_i R_{pi} \Omega_i [h(s) - h(s_{pi+1}) + y_{pi+1} \cdot v'(s_{pi+1})] \sin\overline{\psi}_i \end{aligned} \tag{22}$$

where

$$\begin{aligned} & \Omega_i = 1 \ \text{ if } s_{pi+1} \leq s \\ & \Omega_i = 0 \ \text{ if } s_{pi+1} > s \\ & \Omega_j = 1 \ \text{ if } s_{Qj} \leq s \\ & \Omega_j = 0 \ \text{ if } s_{Qj} > s \end{aligned} \tag{23}$$

2.5. The iterative process

Because of the coupling of two distinct substructures interacting only in a discrete set of points, the evaluation of the deformed shape of a concrete beam prestressed with external tendons leads to the solution of a hyperstatic problem, even if we are dealing with a single span beam.

This observation, together with the assumptions of geometrically nonlinear behaviour of the structure and mechanical nonlinear response of the materials, implies an iterative process to determine the deformed shape. In other words $v(s)$ can be evaluated, by means of Eq. (3), once $\chi(s)$ is known, but $\chi(s)$ and s themselves have to be determined once the internal forces, that depend on $h(s)$ and $v(s)$ are known.

Stating that we aim to evaluate the beam behaviour under increasing loads, a step by step method is preferable.

The first step of this analysis has to compute the deformed shape of the beam subjected to prestressing and loaded by permanent loads. As a matter of fact, during prestressing operations we get the final tensile stain ε_{p1} in the tendons (index 1 means first step of loading), measured on the beam deformed by prestressing and permanent loads, while the tensile stress in the undeformed shape is unknown.

Although at this stage no crack is admissible and the stress level is usually small, so that displacements are small and second order effects are negligible, it is preferable to adopt the same equations applied to evaluate the ultimate limit state of the beam, unless creep effects are considered.

Dealing with long term loading (this analysis is independent of time), the time evolution of the tensile force in the tendons is usually included by modifying the initial stain ε_{p1}.

Mola *et al.* (1993) discuss some simple operational techniques, suitable for practical applications, capable of rectifying ε_{p1}. These formulas adopt the hypotheses of linear viscoelasticity and small displacements.

3. Solution by means of a numerical algorithm

$\chi(s)$ is an implicit function of s, so Eqs. (3) and (4) have to be solved by means of a numerical

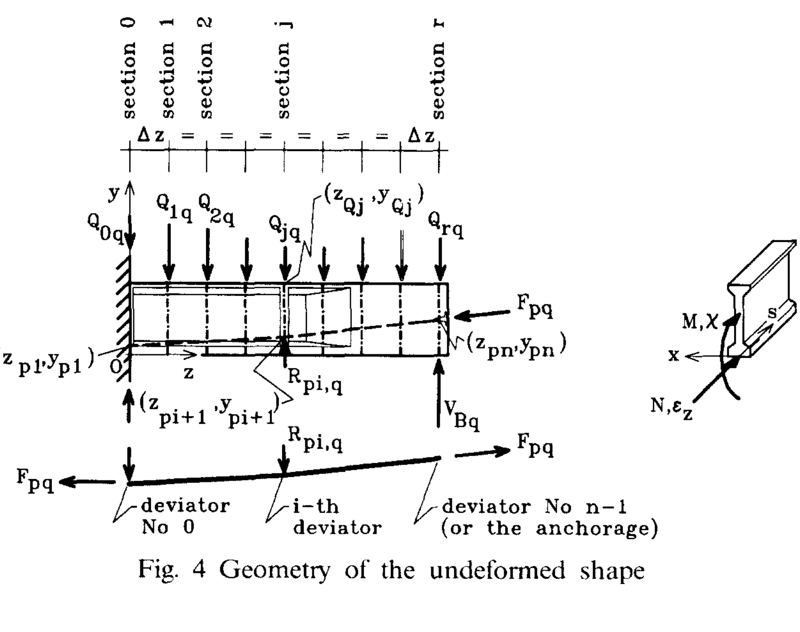

Fig. 4 Geometry of the undeformed shape

algorithm.

We will adopt the well known Finite Difference Method.

This algorithm is useful whether we are dealing with a boundary value problem, or with an initial value problem. Nevertheless, if we study a boundary value problem Eq. (3) becomes a system of quadratic algebraic equations whose solution is unduly cumbersome.

We will transform our problem into an initial value problem by simply adopting the hypothesis that the beam and the loads are symmetric with reference to midspan.

The concrete substructure is then subdivided in r segments of equal length Δz in Fig. 4, where external concentrated vertical loads and deviator reactions act.

Point O is placed in the midspan section.

3.1. The deformed shape of the beam

Having named all the variables related to the k-th concrete cross section with index k, all the terms related to the q-th loading step with index q and referring to the notation drawn in Fig.4, Eq. (8) holds:

$$\begin{aligned}\Delta s_{k,q}=s_{k,q}-s_{k-1,q}=\int_0^{k\Delta z}(1-\varepsilon_{zq})\,dz-\int_0^{(k-1)\Delta z}(1-\varepsilon_{zq})\,dz=\\=\Delta z\left(1-\frac{\varepsilon_{zk,q}+\varepsilon_{zk-1,q}}{2}\right)\end{aligned} \tag{24}$$

and Eq. (3) becomes:

$$\begin{aligned}g_1&=\frac{2}{\Delta s_{k+1,q}(\Delta s_{k,q}+\Delta s_{k+1,q})}\\ g_2&=\frac{2}{\Delta s_{k+1,q}\cdot\Delta s_{k,q}}\\ g_3&=\frac{2}{\Delta s_{k,q}(\Delta s_{k,q}+\Delta s_{k+1,q})}\\ g_4&=g_1\frac{\Delta s_{k,q}}{2}\\ g_5&=g_2\frac{\Delta s_{k+1,q}-\Delta s_{k,q}}{2}\\ g_6&=g_3\frac{\Delta s_{k+1,q}}{2}\end{aligned} \tag{25}$$

$$\begin{aligned}\frac{1}{\chi^2_{k,q}}\,(g_1 v_{k+1,q}-g_2 v_{k,q}+g_3 v_{k-1,q})^2+\\+(g_4 v_{k+1,q}+g_5 v_{k,q}-g_6 v_{k-1,q})^2=1\end{aligned} \tag{26}$$

Setting:

$$a=\frac{g_1^2}{\chi^2_{k,q}}+g_4^2$$

$$b = \frac{g_1}{\chi^2_{k,q}}(g_2 v_{k,q} - g_3 v_{k-1,q}) - g_4(g_5 v_{k,q} - g_6 v_{k-1,q})$$

$$c = \frac{(g_2 v_{k,q} - g_3 v_{k-1,q})^2}{\chi^2_{k,q}} + (g_5 v_{k,q} - g_6 v_{k-1,q})^2 - 1 \tag{27}$$

we get:

$$v_{k+1,q} = \frac{b \pm \sqrt{b^2 - ac}}{a} \tag{28}$$

and the true solution satisfies Eq. (7):

$$\frac{g_1 v_{k+1,q} - g_2 v_{k,q} + g_3 v_{k-1,q}}{|g_1 v_{k+1,q} - g_2 v_{k,q} + g_3 v_{k-1,q}|} = \frac{\chi_{k,q}}{|\chi_{k,q}|} \tag{29}$$

(or $\chi_{k,q}$ and $g_1 v_{k+1,q} - g_2 v_{k,q} + g_3 v_{k-1,q}$ hold the same sign).

Moreover, the boundary conditions:

$$v(0) = 0 \qquad \text{i.e. } v_{0,q} = 0 \tag{30}$$

$$v'(0) = 0 \qquad \text{i.e. } v_{-1,q} = v_{1,q} \tag{31}$$

allow us to specialize Eq. (26) for $k=1$;

$$v_{1,q} = \frac{\chi_{0,q} \Delta s^2_{1,q}}{2} \tag{32}$$

Once $v_{0,q}$ and $v_{1,q}$ are known, $v_{2,q}$ can be computed by means of Eq. (28), and so on until $k+1=r$. In the same way the boundary conditions lead to:

$$h_{0,q} = 0 \tag{33}$$

while Eq. (4) becomes:

$$h_{k+1,q} = \frac{g_6}{g_4} h_{k-1,q} - \frac{g_5}{g_4} h_{k,q} + \frac{1}{g_4 \chi_{k,q}}(g_1 v_{k+1,q} - g_2 v_{k,q} + g_3 v_{k-1,q}) \tag{34}$$

that, because of symmetry, if $k=0$ leads to the approximate solution:

$$h_{1,q} = \Delta s_{1,q} \tag{35}$$

3.2. The tensile stress in the tendons at the q-th loading step

By naming the abscissa and the ordinate of the i-th deviator after bending $\omega_{i+1,q}$ and $\varphi_{i+1,q}$, Eqs. (9)-(12) become:

$$\begin{aligned} h_q(s_{pi}) &= h_{m,q} \\ v_q(s_{pi}) &= v_{m,q} \end{aligned} \qquad m\colon\ m\Delta z = z_{pi} \tag{36}$$

$$\begin{aligned} \omega_{i,q} &= h_{m,q} - y_{pi}(g_4 v_{m+1,q} + g_5 v_{m,q} - g_6 v_{m-1,q}) \\ \varphi_{i,q} &= v_{m,q} + y_{pi}(g_4 h_{m+1,q} + g_5 h_{m,q} - g_6 h_{m-1,q}) \end{aligned} \tag{37}$$

(the evaluation of g_4, g_5 and g_6 is still performed by means of Eq. (25), setting $k=m$)

$$\begin{aligned}\delta_{zi,q}&=\omega_{i+1,q}-\omega_{i,q}\\ \delta_{yi,q}&=\varphi_{i+1,q}-\varphi_{i,q}\end{aligned} \tag{38}$$

$$1_{i,q}=\sqrt{\delta^2_{zi,q}+\delta^2_{yi,q}} \tag{39}$$

Eq.(37) fails when $m=0$ or $m=r$ in Fig. 4.

If a deviator is placed in the midspan section, setting $m=0$ ($i=1$) it is:

$$\begin{aligned}\omega_{1,q}&=0\\ \varphi_{1,q}&=y_{p1}\end{aligned} \tag{40a}$$

while if no deviator is placed in the midspan section we get:

$$\begin{aligned}\omega_{1,q}&=0\\ \varphi_{1,q}&=\varphi_{2,q}\end{aligned} \tag{40b}$$

When $m=r$ ($i+1=n$), if we approximate the first derivative of $h_{r,q}$ and $v_{r,q}$ with:

$$\begin{aligned}h'_q(s_r)&=\frac{h_{r,q}-h_{r-1,q}}{\Delta s_r}\\ v'_q(s_r)&=\frac{v_{r,q}-v_{r-1,q}}{\Delta s_r}\end{aligned} \tag{41}$$

we get:

$$\begin{aligned}\omega_{n,q}&=h_{r,q}-y_{pn}\frac{v_{r,q}-v_{r-1,q}}{\Delta s_r}\\ \varphi_{n,q}&=v_{r,q}+y_{pn}\frac{h_{r,q}-h_{r-1,q}}{\Delta s_r}\end{aligned} \tag{42}$$

Having named the tensile strain in the tendon at the end of the prestressing operation ε_{p1}, its stress resultant F_{p1} and its length (when undeformed) L, we have:

$$\begin{aligned}\sum_{i=1}^{n-1} 1_{i1}&=L(1+\varepsilon_{p1})\\ \sum_{i=1}^{n-1} 1_{iq}&=L(1+\varepsilon_{pq})\end{aligned} \tag{43}$$

and removing L from the preceding equation we get:

$$\varepsilon_{pq}=\frac{\sum_{i=1}^{n-1} 1_{iq}}{\sum_{i=1}^{n-1} 1_{i1}}(1+\varepsilon_{p1})-1 \tag{44}$$

Eq. (44) gives the strain ε_{pq} in the tendon at load step q if the tendon is not protected with cement grout (i.e. ε_{pq} is constant along the tendon).

Once ε_{pq} in known, F_{pq} is easily calculable by means of the constitutive law of the tendon.

3.3. The internal forces

Because of symetry Eq. (15) becomes:

$$V_{Bq}=\frac{Q_{oq}}{2}+\sum_{j=1}^{r} Q_{jq} \tag{45}$$

and Eqs. (16)-(20) hold:

$$\beta_{i,q}=\operatorname{arctg}\frac{\delta_{yi+1,q}}{\delta_{zi+1,q}} \tag{46}$$

[where $\delta_{yi+1,q}$ and δ_{zi+1q} are still evaluated by means of Eqs. (36)-(42)].

$$\overline{\psi}_{i,q}=\frac{\pi+\beta_{i,q}+\beta_{i-1,q}}{2} \tag{47}$$

$$R_{pi,q}=2F_{pq}\cos(\overline{\psi}_{i,q}-\beta_{i,q}) \tag{48}$$

$$v'_q(s_k)=g_4 v_{k+1,q}+g_5 v_{k,q}-g_6 v_{k-1,q} \tag{49}$$

[where g_4, g_5 and g_6 are defined by Eqs. (25), (30), (31) and (41)].

$$\mu_{k,q}=\arcsin v'_q(s_k) \tag{50}$$

$$\psi_{i,q}=\overline{\psi}_{i,q}-\mu_{k,q} \tag{51}$$

$$\begin{aligned}N_q(s_k)=&-\left[V_{Bq}-\sum_{j=1}^{r} Q_{jq}\Omega_j\right]\cdot v'_q(s_k)+F_{pq}\cos(\mu_{k,q}+\beta_{n-2,q})+\\&-\sum_{i=1}^{n-2} R_{pi,q}\Omega_i\cos\psi_{i,q}\end{aligned} \tag{52}$$

$$\begin{aligned}M_q(s_k)=&V_{Bq}[h_{r,q}-h_{k,q}-y_B\cdot v'_q(s_k)]+\\&-\sum_{j=1}^{r} Q_{jq}\Omega_j[h_{j,q}-h_{k,q}-y_{Qj}\cdot v'_q(s_j)]+\\&+F_{pq}[-v_{kq}+\varphi_{n,q}]\cos\beta_{n-2,q}+\\&-F_{pq}[-h_{kq}+\omega_{n,q}]\sin\beta_{n-2,q}+\\&-\sum_{i=1}^{n-2} R_{pi,q}\Omega_i[-v_{k,q}+\varphi_{i+1,q}]\cos\overline{\psi}_{i,q}+\\&+\sum_{i=1}^{n-2} R_{pi,q}\Omega_i[-h_{k,q}+\omega_{i+1,q}]\sin\overline{\psi}_{i,q}\end{aligned} \tag{53}$$

where $\omega_{i,q}$ and $\varphi_{i,q}$ are still evaluated by means of Eqs. (36), (37), (40), (42).

Stating that inequalities Eq. (23) can be written referring to the undeformed shape, terms Ω_i and Ω_j become:

$$\begin{aligned}&\Omega_i=1 \quad \text{if} \quad z_{pi+1}\geq z_k=k\cdot\Delta z\\&\Omega_i=0 \quad \text{if} \quad z_{pi+1}<z_k \qquad (1\leq i\leq n-2)\end{aligned}$$

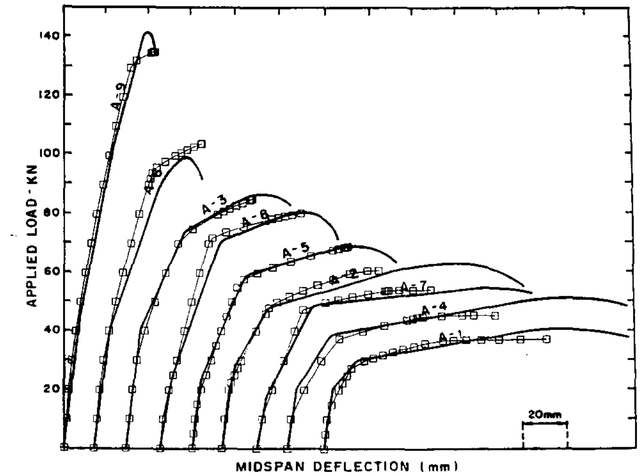

Fig. 5 Tests by Gongchen Du and Xuekang Tao (1985)

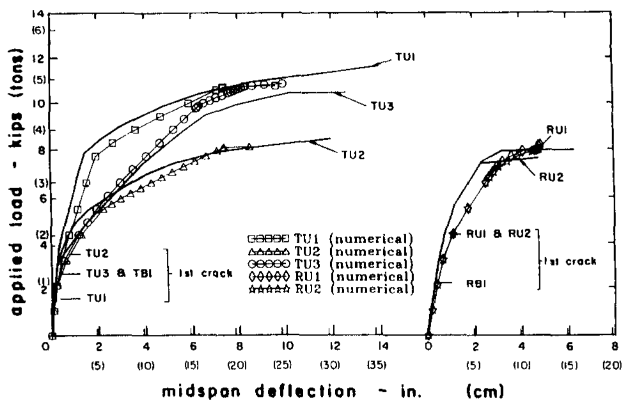

Fig. 6 Tests by Mattock *et al.* (1971)

$$\begin{aligned} &\Omega_j = 1 \quad \text{if} \quad j \geq k \\ &\Omega_j = 0 \quad \text{if} \quad j < k \qquad (1 \leq j \leq r) \end{aligned} \tag{54}$$

3.4. The deformed shape of the beam after prestressing

As already observed, we are interested in the beam behaviour both before cracking and after it, until collapse occurs. To achieve this target a step by step analysis with increasing loads was chosen.

At each q-th step an iterative solution of Eqs. (3) and (4) is required: if the final values $h_{k,q-1}$ and $v_{k,q-1}$ obtained at the end of the preceding step are adopted as a first approximation of the present step, by means of Eqs. (36)-(44) the tensile stress in the tendons is computed, while Eqs. (45)-(54) give new internal actions. The concrete cross section analysis gives new curvatures $\chi_{k,q}$ and strains $\varepsilon_{zk,q}$ that allow us to update $h_{k,q}$ and $v_{k,q}$ and so on until $h_{k,q}$ and $v_{k,q}$ do not vary any more.

The first step corresponds to the evaluation of the deformed shape of the beam after prestressing. This step is performed neglecting Eqs. (39), (43) and (44) (the tensile stress F_{p1} is the final value measured at the end of this step referring to the deformed shape of the beam), or setting F_{p1}=known constant value and adopting the undeformed shape of the concrete beam (i.e. $h_{k,1}=z_k=k\cdot\Delta z$, $v_{k,1}=0$) to start the iterative process.

4. Comparisons with the experimental data

The numerical model presented was devised to replace experimental tests, or to enable a reliable numerical research on the behaviour of R.C. beams repaired or strengthened by means of external post-tensioning.

To achieve this goal no simplifying assumption is set in dealing with second order effects and large displacements.

The method brings with it essentially two limitations:

(1) cracking is spread over a segment of finite length in the concrete beam;

(2) friction between the tendon and the deviators is neglected.

Verification on the reasonableness of these assumptions and on the reliability of the numerical model is still in progress.

Among 28 experimental tests already numerically reproduced, Fig. 5 shows those by Gongchen Du *et al.* (1985) (thick curves). These tests refer to beams prestressed with unbonded tendons, that do not exactly meet the specifications of the method but can be modelled (thin lines) by adopting a dense subdivision of the beam and by placing deviators in each cross section.

The good agreement between the numerical output and the experimental data does not change if the stress evolution in the tendons is considered. These results show that friction is not significant in the analysis, at least if monostrands are adopted.

Similar results are shown in Fig. 6 where the experimental deflections, measured by Mattok *et al.* (1971) on five specimens, are drawn (thick curves). Referring to these tests it should be noted that the displacements measured under constant loads were discarded because this delayed behaviour is not included in the numerical method.

Another interesting aspect concerns cracks distribution: some of the specimens tested were prestressed, but no reinforcing steel was added. Some of them, when loaded, exhibited one single crack, where collapse occurred because of concrete crushing. Nevertheless the precision of the numerical results is still surprisingly good. This unexpected outcome may be accounted for by observing that midspan deflection and tensile strains in the tendons (the experimental data) depend on the behaviour of the entire beam, whereas the stress distribution around the crack should be considerably different from the one evaluated numerically.

5. Conclusions

A numerical model that aims to describe the behaviour, up to collapse, of R.C. and P.C. beams rehabilitated or strengthened by means of external prestressing, but demonstrated to be reliable for the analysis of new beams prestressed with unbonded tendons as well, is presented.

The algorithm, based on the Finite Difference Method, includes second order effects, large displacements and the change in length of the beam due to compression, neglects shear deformation both before and after cracking, friction between the tendons and the deviators, and adopts the hypothesis that the bending of the beam is a continuous function. Span to depth ratio is then taken into account through the relative displacements of deviators and anchorages.

The numerical outputs match the experimental data available in scientific literature.

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