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Large displacement Lagrangian mechanics Part I - Theory

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Abstract. In Lagrangian mechanics, attention is directed at the body as it moves through space. The region occupied by the body is called a configuration. All body points are labelled by the position they would have if the body were to occupy a chosen reference configuration. The reference configuration can be regarded as an extra fictional copy where notes are kept.

As the body moves and deforms, it is important to correctly observe the use of each configuration for computational purposes. The description of strain is particularly important. The present work establishes clearly the role of each configuration in total and in incremental forms. This work also details the differences between gradient and configurational calculus.

Key words: large displacement; strain Lagrangian; configurational calculus; gradient, equilibrium principle

1. Preliminaries

In Lagrangian, or referential, mechanics attention is directed at the body as it moves through space. A body is represented as a gross distribution of mass over its configuration, where the configuration is the region of Euclidean space occupied by the body. Mass is considered an appropriate scalar valued measure of matter. All quantities pertaining to the body and the body-points themselves are labelled by the position of the body-points when the body is in some reference configuration or placement. Commonly, but not necessarily, this is an initial position or an unstrained configuration. The choice of reference configuration is quite arbitrary. Any possible configuration would serve the theoretical purpose of having a way to label body-points. The only theoretical restriction is that the placement chosen for reference must be one which the body could occupy. The analyst is at liberty to choose any configuration deemed suitable for use. The body does not have to ever actually occupy or pass through the chosen placement. This placement is needed only to keep our definitions and naming conventions well organized. As such an artefice, the ability to refer a quantity to a configuration is crucial to our analysis

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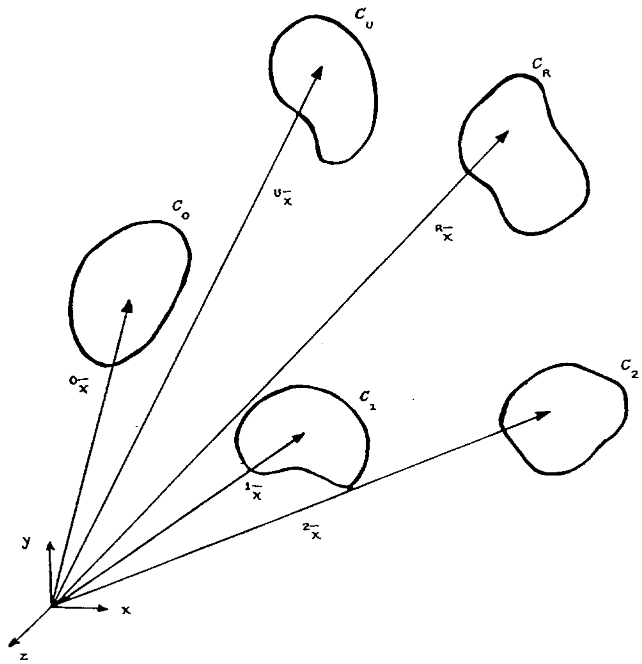

Fig. 1 Positions of a body-point X invarious configurations.

and computation. An informal way to think of a reference configuration is as an extra copy where notes can be made.

There is no restriction to use only one reference configuration. It is often useful or convenient to use several different placements as reference configurations. The configuration in which a quantity is measured has no inherent theoretical relationship to the configuration in which all body-points are labelled. So, the value of measured quantities is invariant with respect to choice of reference configuration. Throughout the changes and motions a body undergoes, it is always possible to identify a body-point by where it would be, if the body in a reference configuration. For example, particles in a vibrating body are often labelled by their position in an undistrurbed, equilibrium configuration.

In incremental calculations the choice of which configuration to use as a reference configuration is commonly a recently calculated one and the choice may be changed from time to time. Such a practice leads to the so-called Updated Lagrangian Formulation. In theory, this is all equally well, but, in practice this can lead to complication.

There are several configurations that will be used in developing the relations among strain tensors: (consult Fig. 1):

C_U Unstrained configuration. The only important feature of this configuration is that it is unstrained. Here the body is relaxed and has no stresses or forces acting on it. The location and orientation of this placement are unimportant. For bodies that are strained *ab initio*, this configuration is never actually occupied.

C_0 Initial configuration. This is the region of space occupied by the body at the beginning of the period of consideration. All other configurations that the bodies actually occupies at any point in its motion form a sequence that starts with this configuration. Usually problems are stated in such a way that all needed quantities are known *a priori* in C_0.

In the case of the strain tensor, we need to know it in some configuration. In the event that the placement in which strain is known is not the initial placement, we can use known relations to find the strain in C_0.

C_1 Present configuration. This configuration may be displaced and/or deformed from C_0.

C_2 Next configuration. This may be a configuration occupied by the body at some stage in its motion (or transplacement) later in sequence than C_1 and possibly incrementally close to C_1.

C_R Reference configuration. This represents any possible placement of the body in any location and having any orientation. It may be deformed. It is not necessarily in the sequence of configurations that the body actually occupies.

All calculations are always carried out in a reference configuration. Configurations with other names are used from occasionally in the argument presented. These are called C_A, C_B and so on.

1.1. Notation

Left superscripts

Left superscripts denote the configuration(s) in which a quantity is measured. For example, ${}^2\bar{x}(X)$ is the position vector of body-point X when the body is in configuration C_2. For displacements and other quantities which require more than one configuration for their definition, there are more than one left superscripts. To define a displacement requires two configurations. So, the first (leftmost) superscript is used for the "to" configuration. The second (rightmost) superscript is used for the "from" configuration. For example, the displacement of X from configuration C_A to configuration C_B is ${}^{AB}\bar{u}(X)$. Note that for displacements, ${}^{AA}\bar{u}=\bar{0}$.

Left subscript

The left subscript denotes the configuration of reference. Lack of a left subscript indicates the use of a so-called material formulation. So,

$$ {}^{AB}\bar{u}(X)={}^{AB}_{R}\bar{u}({}^{R}\bar{x}) \tag{1} $$

The configuration of reference can be any configuration and this has no effect on the value of the measured quantity. For a body property, T, when the body is in configuration C_A, it must be that ${}^{A}T$, ${}^{A}_{R}T$ and ${}^{A}_{S}T$ all describe the same quantity and must have the same value, though the formulae may look different. Though not necessary, it is common practice to keep all terms on each side of an equation referred to a single configuration. Special care must be exercised with spatial derivatives.

1.2. Spatial derivatives

The meaning of the calculus is clear so long as we are differentiating with respect to a well understood scalar, say time. For differentiation with respect to position there are at least two quite different derivatives that can arise. For differentiation with respect to any tensor, $\overset{m}{\bar{F}}$, we have the usual

$$\frac{\partial \overset{n}{\overline{P}}}{\partial \overset{m}{\overline{F}}}=\bar{j}^{m}\bar{j}^{l}\cdots\bar{j}^{c}\bar{j}^{b}\bar{j}^{a}\frac{\partial \overset{n}{\overline{P}}}{\partial \overset{m}{\overline{F}}{}^{abc\cdots lm}} \tag{2}$$

where $\bar{j}^{i}$ are the local basis vectors of $\overset{m}{\overline{F}}$. This can be developed completely without any restriction. Take the case of derivatives with respect to position. The independent tensor is position, $\bar{x}$. A differential change in position is

$$d\bar{x}=dx^{r}\bar{g}_{r} \tag{3}$$

The parametric values are the spatial coordinates. The basis vectors are the local tangent basis of space. So, the derivative of some quantity with respect to position in space is

$$\frac{\partial \overset{n}{\overline{P}}}{\partial \bar{x}}=\bar{g}^{r}\frac{\partial \overset{n}{\overline{P}}}{\partial x^{r}} \tag{4}$$

So far we have shown what mathematics we use for a derivative with respect to position. Now, let us put this into the context of Lagrangian continuum mechanics. There are at least two different reasons why we might wish to examine values to be found at nearby points in space.

One possibility is that the two locations under consideration are two distinct body-points within a single configuration. This is the familiar gradient calculus. It is important to distinguish it from other comparisons. Sometimes, these distinctions are not easy to notice in formulae for scalar components of tensor derivatives. We could also consider the same body-point as the body occupies two distinct configurations. This is an entirely different comparison from the gradient calculus. Call this configurational calculus; it is a particular case of the more general calculus of variations. In an effort to avoid any possible confusion between the two types of derivatives as will use $\frac{\partial \overset{n}{\overline{P}}}{\partial \bar{x}}$ for gradients and $\frac{\delta \overset{n}{\overline{P}}}{\delta \bar{x}}$ forconfigurational derivatives.

There is still the matter of which placements are to be used for measurement and reference. First look at the gradient calculus. Since the above arguments are true for tensors of any order we will simplify the notation here by considering the example of the dependent tensor being always of second order, although order can be used. There are several possible configurations of which to keep track. Let $\overline{\overline{Q}}$ be the gradient of $\overline{\overline{P}}$. Then

$$^{ABCD}_{\quad R}\overline{\overline{Q}}=\frac{\partial^{A}_{B}\overline{\overline{P}}}{\partial^{C}_{D}\bar{x}} \tag{5}$$

The material property, $\overline{\overline{P}}$, can be measured in any placement, say C_A. The configuration of reference of $^{A}\overline{\overline{P}}$ could be C_B. That is, when measuring the value of $^{A}\overline{\overline{P}}$, we named the body-points by the position they would have, if the body were in configuration C_B. Informally, we can imagine writing the measured values on a copy of the body in C_B. In this case we have $^{A}_{B}\overline{\overline{P}}$. This is a function of position in C_B. That is

$$^{A}_{B}\overline{\overline{P}}={}^{A}_{B}\overline{\overline{P}}(^{B}\bar{x}) \tag{6}$$

It can be differentiated with respect to ${}^{B}\bar{x}$. Differentiation is with respect to position as measured in configuration C_C. So, the configuration of measurement for position in the gradient must be the same as the configuration of reference for the property. Then C_B and C_C must be the same. This position may itself be referred to any configuration. The reference placement for the position vector is the configuration of reference for the gradient. So in Eq. (5) C_D is the same as C_R. With this clearly understood we may now make a convention for the notation of gradients. Where $\bar{\bar{\bar{Q}}}$ is the gradient of $\bar{\bar{P}}$, $\bar{\bar{\bar{Q}}}$ has the same left superscripts as $\bar{\bar{P}}$ plus one more left superscript. This extra left superscript is preceded by a comma and indicates the placement of measurement for the position vector with respect to which $\bar{\bar{P}}$ has been differentiated. The left subscript of $\bar{\bar{\bar{Q}}}$ denotes the reference configuration for the gradient, which is the reference configuration of the position vector with respect to which $\bar{\bar{P}}$ has been differentiated. So, for gradients we have

$$ {}^{A,C}_{R}\bar{\bar{\bar{Q}}} = \frac{\partial^{A}_{C}\bar{\bar{P}}}{\partial^{C}_{R}\bar{x}} \tag{7}$$

In a similar examination of the configurational calculus, we start with the five configurations. Let $\bar{\bar{\bar{Q}}}$ be the configurational derivative of $\bar{\bar{P}}$, then

$$ {}^{ABCD}_{R}\bar{\bar{\bar{Q}}} = \frac{\delta^{A}_{B}\bar{\bar{P}}}{\delta^{C}_{D}\bar{x}} \tag{8}$$

$\bar{\bar{P}}$ is measured in C_A and referred to C_B. $\bar{x}$ is measured in C_C and referred to C_D. What is being varied is C_C. Only one body-point is being watched as we make different choices of configuration in which to measure position. We observe changes in the value of ${}^{A}\bar{\bar{P}}$ as various placements near C_C are chosen for $\bar{x}$. Since this is the position with respect to which we wish to differentiate $\bar{\bar{P}}$, $\bar{\bar{P}}$ must be a function of this position. That is $\bar{\bar{P}}$ must be referred to the same configuration as the one in which position is measured. So C_B must be the same configuration as C_C. The choice of C_D, the reference placement for the position with respect to which we differentiate $\bar{\bar{P}}$, is independent. A notation like that of the gradient calculus can be used. So

$$ {}^{A;C}_{R}\bar{\bar{\bar{Q}}} = \frac{\delta^{A}_{C}\bar{\bar{P}}}{\delta^{C}_{R}\bar{x}} \tag{9}$$

is used for the configurational calculus. The punctuation among the left superscripts to denote the configuration of measurement of the position with respect to which $\bar{\bar{P}}$ is differentiated is changed from a comma to a semicolon.

1.3. Change of reference configuration

At any time it may be convenient or necessary for an analyst to change the reference configuration. In principle this is a simple operation. The referring of a quantity to a configuration can be thought of as writing the measured value of the quantity on a fictional copy of the configuration chosen for reference. This allows calculations to be carried out in a consistent fashion. Access to all body-points and their properties is easily achieved through the location of that body-point in the reference configuration. However, in the case of spatial derivatives this is not always the most useful method. It is appropriate in many calculations to transform the derivative so that the differentiation is with respect to position in the reference configuration,

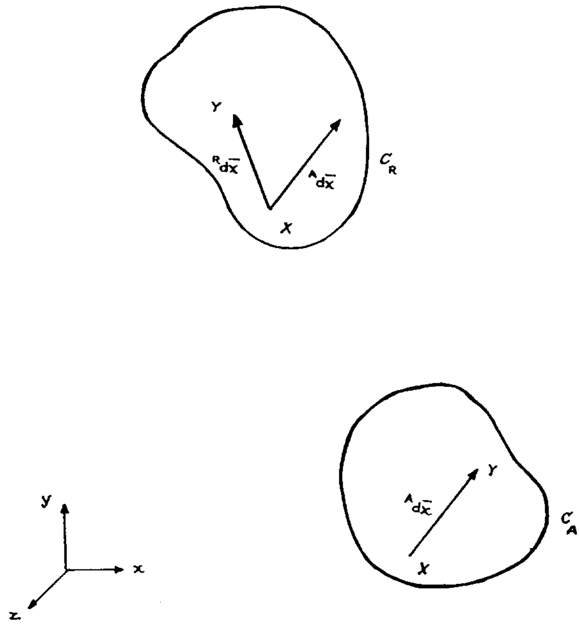

Fig. 2 Different position vectors ${}^{A}d\bar{x}$ and ${}^{R}d\bar{x}$

instead of with respect to position in the configuration of measurement.

Consider the transformation for gradients. To do this use three configurations, C_A, C_B and C_C and two distinct but nearby body-points, X and Y. Then some material property, $\overline{\overline{P}}$, is measured in coinfiguration C_C. So it is ${}^{C}\overline{\overline{P}}$. The values of this property at body-points X and Y are:

$$
{}^{C}\overline{\overline{P}}(X)={}^{C}_{C}\overline{\overline{P}}({}^{C}\bar{x}) \tag{10}
$$

and

$$
{}^{C}\overline{\overline{P}}(Y)={}^{C}_{C}\overline{\overline{P}}({}^{C}\bar{x}+{}^{C}d\bar{x}) \tag{11}
$$

Eq. (11) can be expressed as

$$
{}^{C}\overline{\overline{P}}(Y)={}^{C}_{C}\overline{\overline{P}}(\bar{x})+{}^{C}_{C}d\bar{x}\cdot\frac{\partial^{C}_{C}\overline{\overline{P}}}{\partial^{C}_{C}\bar{x}} \tag{12}
$$

Since body-points X and Y are very close to each other the difference in ${}^{C}\overline{\overline{P}}$, ${}^{C}d\overline{\overline{P}}$, can be expressed as

$$
{}^{C}\overline{\overline{P}}(Y)={}^{C}\overline{\overline{P}}(X)+{}^{C}d\overline{\overline{P}}(X,Y) \tag{13}
$$

Through the rest of this argument differentials refer to differential changes in values of properties between X and Y. So further references to, for example, ${}^{C}d\overline{\overline{P}}(X)$, are taken to mean ${}^{C}d\overline{\overline{P}}(X,Y)$. Take advantage of the invariance of ${}^{C}\overline{\overline{P}}(Y)$ to compare Eq. (12) and Eq. (13). The difference of these gives the usual form for a gradient

$$
{}^{C}d\overline{\overline{P}}={}^{C}_{R}d\overline{\overline{P}}={}^{C}_{C}d\bar{x}\cdot\frac{\partial^{C}_{C}\overline{\overline{P}}}{\partial^{C}_{C}\bar{x}} \tag{14}
$$

The value of ${}^{C}\overline{\overline{P}}$ is invariant with respect to configuration of reference and can be referred to

C_R.

$$ {}^{C}d\overline{\overline{P}} = {}^{C}_{R}d\overline{\overline{P}} = {}^{C}_{R}d\overline{x} \cdot \frac{\partial^{C}_{C}\overline{\overline{P}}}{\partial^{C}_{R}\overline{x}} \tag{15} $$

As an alternative, we could refere ${}^{C}\overline{\overline{P}}$ to C_A at the beginning of the argument. We find

$$ {}^{C}\overline{\overline{P}}(X) = {}^{C}_{C}\overline{\overline{P}}({}^{C}\overline{x}) = {}^{C}_{A}\overline{\overline{P}}({}^{A}\overline{x}) \tag{16} $$

$$ {}^{C}\overline{\overline{P}}(Y) = {}^{C}_{A}\overline{\overline{P}}({}^{A}\overline{x} + {}^{A}d\overline{x}) \tag{17} $$

$$ {}^{C}\overline{\overline{P}}\,(Y) = {}^{C}_{A}\overline{\overline{P}} + {}^{A}_{A}d\overline{x} \cdot \frac{\partial^{C}_{A}\overline{\overline{P}}}{\partial^{A}_{A}\overline{x}} \tag{18} $$

Using the same approach as for Eq. (14) one finds

$$ {}^{C}d\overline{\overline{P}} = {}^{C}_{A}d\overline{\overline{P}} = {}^{A}_{A}d\overline{x} \cdot \frac{\partial^{C}_{A}\overline{\overline{P}}}{\partial^{A}_{A}\overline{x}} \tag{19} $$

or

$$ {}^{C}d\overline{\overline{P}} = {}^{C}_{R}d\overline{\overline{P}} = {}^{A}_{R}d\overline{x} \cdot \frac{\partial^{C}_{A}\overline{\overline{P}}}{\partial^{A}_{R}\overline{x}} \tag{20} $$

The differential ${}^{A}_{R}d\overline{x}$ should be noted. As shown in Fig. 2, ${}^{A}d\overline{x}$ locates body-point Y with respect to body-point X when then body occupies C_A but not when the body occupies any other configuration.

This same process can be repeated to refer ${}^{C}\overline{\overline{P}}$ to C_B. We find

$$ {}^{C}d\overline{\overline{P}} = {}^{C}_{B}d\overline{\overline{P}} = {}^{B}_{B}d\overline{x} \cdot \frac{\partial^{C}_{B}\overline{\overline{P}}}{\partial^{B}_{B}\overline{x}} \tag{21} $$

or

$$ {}^{C}d\overline{\overline{P}} = {}^{C}_{R}d\overline{\overline{P}} = {}^{B}_{R}d\overline{x} \cdot \frac{\partial^{C}_{B}\overline{\overline{P}}}{\partial^{B}_{R}\overline{x}} \tag{22} $$

The differential position vector ${}^{B}d\overline{x}$ locates body-point Y with respect to body-point X, only when the body occupies configuration C_B. The argument so far can be applied immediately. Suppose that the differential of interest is ${}^{A}d\overline{x}$. Replace ${}^{C}d\overline{\overline{P}}$ with ${}^{A}d\overline{x}$. Then Eq. (22) becomes

$$ {}^{A}d\overline{x} = {}^{A}_{R}d\overline{x} = {}^{B}_{R}d\overline{x} \cdot \frac{\partial^{A}_{B}\overline{x}}{\partial^{B}_{R}\overline{x}} \tag{23} $$

This in turn may be substituted into Eq. (20) to give

$$ {}^{C}_{R}d\overline{\overline{P}} = {}^{B}_{R}d\overline{x} \cdot \frac{\partial^{A}_{B}\overline{x}}{\partial^{B}_{R}\overline{x}} \cdot \frac{\partial^{C}_{A}\overline{\overline{P}}}{\partial^{A}_{R}\overline{x}} \tag{24} $$

This leads use to see that

$$ \frac{\partial^{C}_{B}\overline{\overline{P}}}{\partial^{B}\overline{x}} = \frac{\partial^{A}_{B}\overline{x}}{\partial^{B}\overline{x}} \cdot \frac{\partial^{C}_{A}\overline{\overline{P}}}{\partial^{A}\overline{x}} \tag{25} $$

Now we have a useful transformation. At our convenience we may leave the gradient untransformed and merely copy values during a change of configuration of reference or we may take the gradient with respect to the new reference configuration.

This is applied in several important cases. Among the most famous are the First and Second Piola-Kirchhoff pseudo-stress-tensors. The term "pseudo" is very descriptive. These tensors are not identical to the Cauchy stress except under special conditions. Instead some transformation has been applied and a different quantity, a pseudo-stress, has been found. These can be very useful. Another very important "pseudo-" quantity is the Lagrange pseudo-strain tensor.

2. Strain tensors

A careful derivation of the Lagrangian (referential) strain tensor and of incremental forms of the Lagrangian strain tensor is useful here. It will clarify the later use of these important measures of material distortion. This derivation follows from the classical argument concerning the separation of two distinct body-points in various configurations. The separation of two body-points may change from an unstrained configuration to a deformed configuration. Consider the quadratic form for deformation:

$$ {}^1D = {}^1d\bar{x}\cdot{}^1d\bar{x} - {}^Ud\bar{x}\cdot{}^Ud\bar{x} \tag{26}$$

Note that ${}^1D$ as a symbol does not mention C_U. This is because any unstrained configuration will serve. Now, refer this to the deformed configuration, C_1. The differential ${}^Ud\bar{x}$ can be replaced by

$$ {}^U_1d\bar{x} = {}^1_1d\bar{x}\cdot\frac{\partial^U_1\bar{x}}{\partial^1_1\bar{x}} \tag{27}$$

We can maintain generality and facilitate the discussion of the use of an arbitrary reference placement if we keep the differential ${}^1d\bar{x}$ in the form

$$ {}^1_1d\bar{x} = {}^1_1d\bar{x}\cdot\frac{\partial^1_1\bar{x}}{\partial^1_1\bar{x}} \tag{28}$$

Introduce these forms to find

$$ {}^1_1D = {}^1_1d\bar{x}\cdot\left(\frac{\partial^1_1\bar{x}}{\partial^1_1\bar{x}}\right)\cdot{}^1_1d\bar{x}\cdot\left(\frac{\partial^1_1\bar{x}}{\partial^1_1\bar{x}}\right) - {}^1_1d\bar{x}\cdot\left(\frac{\partial^U_1\bar{x}}{\partial^1_1\bar{x}}\right)\cdot{}^1_1d\bar{x}\cdot\left(\frac{\partial^U_1\bar{x}}{\partial^1_1\bar{x}}\right) \tag{29}$$

or

$$ {}^1_1D = {}^1_1d\bar{x}\,{}^1_1d\bar{x} : \left[\left(\frac{\partial^1_1\bar{x}}{\partial^1_1\bar{x}}\right)\cdot\left(\frac{\partial^1_1\bar{x}}{\partial^1_1\bar{x}}\right)^T - \left(\frac{\partial^U_1\bar{x}}{\partial^1_1\bar{x}}\right)\cdot\left(\frac{\partial^U_1\bar{x}}{\partial^1_1\bar{x}}\right)^T\right] \tag{30}$$

or we can gather all of the information about distortion of the material into a single factor as

$$ {}^1_1D = {}^1_1d\bar{x}\ {}^1_1d\bar{x} : 2\ {}^1_1\bar{\bar{\varepsilon}}({}^1\bar{x}) \tag{31}$$

where ${}^1\bar{\bar{\varepsilon}}$ is the true strain tensor of the body in configuration C_1. ${}^1D$ can be referred to any configuration, say C_A, as

$$ {}^1_AD = {}^1_Ad\bar{x}\ {}^1_Ad\bar{x} : 2\ {}^1_A\bar{\bar{\varepsilon}}({}^A\bar{x}) \tag{32}$$

That is

$$ {}^1_AD = {}^1_Ad\bar{x}\ {}^1_Ad\bar{x} : \left[\left(\frac{\partial^1_1\bar{x}}{\partial^1_A\bar{x}}\right)\cdot\left(\frac{\partial^1_1\bar{x}}{\partial^1_A\bar{x}}\right)^T - \left(\frac{\partial^U_1\bar{x}}{\partial^1_A\bar{x}}\right)\cdot\left(\frac{\partial^U_1\bar{x}}{\partial^1_A\bar{x}}\right)^T\right] \tag{33}$$

In calculation the use of these differentials and gradients can be extremely inconvenient or impossible unless the chosen reference configuration is the present configuration. For any other case the analyst may prefer to transform the quantities so that the differentials are of, and derivatives are with respect to, position in the reference configuration. To do this retrace the steps in deriving ${}^{1}\bar{\bar{\varepsilon}}$ strarting by substitution for the differentials in Eq. (26) with

$$ {}^{1}_{A}d\bar{x}={}^{A}_{A}d\bar{x}\cdot\frac{\partial^{1}_{A}\bar{x}}{\partial^{A}_{A}\bar{x}} \tag{34}$$

and

$$ {}^{U}_{A}d\bar{x}={}^{A}_{A}d\bar{x}\cdot\frac{\partial^{U}_{A}\bar{x}}{\partial^{A}_{A}\bar{x}} \tag{35}$$

The same arithmetic steps lead us to

$$ {}^{1}_{A}D={}^{A}_{A}d\bar{x}\ {}^{A}_{A}d\bar{x}\ :\ \left[\left(\frac{\partial^{1}_{A}\bar{x}}{\partial^{A}_{A}\bar{x}}\right)\cdot\left(\frac{\partial^{1}_{A}\bar{x}}{\partial^{A}_{A}\bar{x}}\right)^{T}-\left(\frac{\partial^{U}_{A}\bar{x}}{\partial^{A}_{A}\bar{x}}\right)\cdot\left(\frac{\partial^{U}_{A}\bar{x}}{\partial^{A}_{A}\bar{x}}\right)^{T}\right] \tag{36}$$

$$ {}^{1}_{A}D={}^{A}_{A}d\bar{x}\ {}^{A}_{A}d\bar{x}\ :\ 2\,{}^{1A}_{A}\bar{\bar{E}}({}^{A}\bar{x}) \tag{37}$$

In Eq. (36) the expression in brackets has an important difference from the one in Eq. (30). This time all of the derivatives are with respect to position in the arbitrarily chosen refernce placement, C_A, instead of the present configuration, C_1. Likewise the differential position vectors that are the leading dyad of Eq. (36 and 37) are measured in C_A instead of C_1. This shows that ${}^{1A}\bar{\bar{E}}$ is not the same as ${}^{1}\bar{\bar{\varepsilon}}$. ${}^{1A}\bar{\bar{E}}$ contains the same information to describe how the material is strained in C_1, but ${}^{1A}\bar{\bar{E}}$ is rooted in C_A. We call ${}^{1A}\bar{\bar{E}}$ the Lagrange pseudo-strain of a body in C_1 transformed to C_A. It is possible to refer this to yet another configuration, C_R. One finds

$$ {}^{1}_{R}D={}^{A}_{R}d\bar{x}\ {}^{A}_{R}d\bar{x}\ :\ 2\,{}^{1A}_{R}\bar{\bar{E}}\ ({}^{R}\bar{x}) \tag{38}$$

$$ {}^{1}_{R}D={}^{A}_{R}d\bar{x}\ {}^{A}_{R}d\bar{x}\ :\ \left[\left(\frac{\partial^{1}_{A}\bar{x}}{\partial^{A}_{R}\bar{x}}\right)\cdot\left(\frac{\partial^{1}_{A}\bar{x}}{\partial^{A}_{R}\bar{x}}\right)^{T}-\left(\frac{\partial^{U}_{A}\bar{x}}{\partial^{A}_{R}\bar{x}}\right)\cdot\left(\frac{\partial^{U}_{A}\bar{x}}{\partial^{A}_{R}\bar{x}}\right)^{T}\right] \tag{39}$$

This shows how each configuration is used in the pseudo-strain. The use of different configurations for C_A and C_R is possible but it is not a common practice. This is because of the implied use of differentials of form ${}^{A}_{R}d\bar{x}$, where the measurement and reference configurations are not the same. This immediately leads to difficulties, illustrated in Fig. 2, that we were trying to avoid by transforming out of C_1.

The transformation between ${}^{1}\bar{\bar{\varepsilon}}$ and ${}^{1A}\bar{\bar{E}}$ can be found by referring Eq. (29) to C_R to get

$$ {}^{1}_{R}D={}^{A}_{R}d\bar{x}\ {}^{A}_{R}d\bar{x}\ :\ 2\left(\frac{\partial^{1}_{A}\bar{x}}{\partial^{A}_{R}\bar{x}}\right)\cdot{}^{1}_{R}\bar{\bar{\varepsilon}}\cdot\left(\frac{\partial^{1}_{A}\bar{x}}{\partial^{A}_{R}\bar{x}}\right)^{T} \tag{40}$$

Also, by the definition of ${}^{1A}\bar{\bar{E}}$, we find

$$ \left(\frac{\partial^{1}_{A}\bar{x}}{\partial^{A}_{R}\bar{x}}\right)\cdot{}^{1}\bar{\bar{\varepsilon}}\cdot\left(\frac{\partial^{1}_{A}\bar{x}}{\partial^{A}_{R}\bar{x}}\right)^{T}={}^{1A}_{R}\bar{\bar{E}} \tag{41}$$

Now we can transform true strain to pseudo-strain and vice versa. This allows us to use either

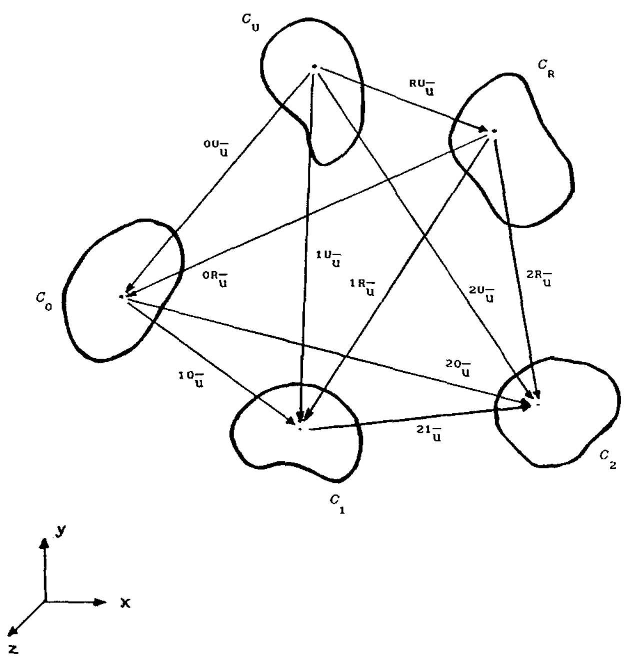

Fig. 3 Configurations of a body and displacements of body-point X.

form in further development knowing that which ever form is more convenient can be obtained at need. This also shows the condition required for the pseudo-strain, ${}^{1A}\bar{\bar{E}}$, to be the same as the true strain, ${}^{1}\bar{\bar{\varepsilon}}$. The deformation gradient, ${}^{1A}\bar{\bar{F}}$, must be an identity tensor. This only occurs when configurations C_1 and C_A are indeed the same configuration or are different only by a rigid body translation. This is the same condition as is required for the First or Second Piola-Kirchhoff psedudo-stress tensors to be identical with the Cauchy true stress tensor.

3. Displacement forms for the strain tensor

In most work done using a referential description, it is the Lagrange pseudo-strain that is used. So, further investigation is developed for that tensor. In practical analysis the position of a body-point is often tracked by the use of displacement rather than directly. So next we develop forms of the pseudo-strain tensor based on the use of displacement. It is usually the case that the transformation configuration, C_A, is the same as the reference configuration, C_R. However, for the sake of generality and clarity we will assume that C_A is not necessarily the same as C_R.

We may express ${}^{U}\bar{x}$ and ${}^{1}\bar{x}$ as displaced from ${}^{A}\bar{x}$. From Fig. 3 it can be seen that

$$ {}^{U}\bar{x} = {}^{A}\bar{x} - {}^{AU}\bar{u} \tag{42} $$

The deformation gradient can be written as

$$\frac{\partial^U_A \bar{x}}{\partial^A_R \bar{x}} = \bar{\bar{1}} - {}^{AU,A}_{R}\bar{\bar{u}} \tag{43}$$

where $\bar{\bar{1}}$ represents the identity tensor and

$${}^{AU,A}_{R}\bar{\bar{u}} = \frac{\partial^{AU}_{A}\bar{u}}{\partial^A_R \bar{x}} \tag{44}$$

is the displacement gradient. Similarly,

$${}^1\bar{x} = {}^A\bar{x} + {}^{1A}\bar{u}, \quad \frac{\partial^1_A \bar{x}({}^A\bar{x})}{\partial^A_R \bar{x}} = \frac{\partial^A_A \bar{x}({}^A\bar{x})}{\partial^A_R \bar{x}} + \frac{\partial^{1A}_A u({}^A\bar{x})}{\partial^A_R \bar{x}}$$

$$\frac{\partial^1_A \bar{x}}{\partial^A_R \bar{x}} = \bar{\bar{1}} + {}^{1A,A}_{R}\bar{\bar{u}} \tag{45}$$

Now these expressions are substituted into ${}^1_R D$. This gives

$${}^1_R D = {}^A_R d\bar{x} \; {}^A_R d\bar{x} \; : \; 2 \; {}^{1A}_{R}\bar{\bar{E}} \tag{46}$$

where the pseudo-strain tensor

$${}^{1A}_{R}\bar{\bar{E}} = \frac{1}{2}\left[\left\{ {}^{1A,A}_{R}\bar{\bar{u}}^T + {}^{1A,A}_{R}\bar{\bar{u}} + \left({}^{1A,A}_{R}\bar{\bar{u}} \cdot {}^{1A,A}_{R}\bar{\bar{u}}^T \right) \right\} + \left\{ {}^{AU,A}_{R}\bar{\bar{u}}^T + {}^{AU,A}_{R}\bar{\bar{u}} - \left({}^{AU,A}_{R}\bar{\bar{u}} \cdot {}^{AU,A}_{R}\bar{\bar{u}}^T \right) \right\}\right] \tag{47}$$

The braces in Eq. (46) emphasize a grouping of tensors in the final expression in terms of the original two gradients as thier source. It can be seen that the first term in braces depends on C_1 and on C_A but not on C_U. So a different choice of C_U will not affect this term. The second term in braces depends on C_A and on C_U. So a change in C_1 will not affect this term. This property reappears later. It is easily shown that ${}^1D$ is invariant with respect to the choice of an unstrained configuration; so is ${}^{1A}\bar{\bar{E}}$. In general, all six terms are nonzero and all six must be evaluated.

3.1. Special cases

If the transformation configuration is unstrained, then

$${}^A d\bar{x} = {}^U d\bar{x}, \quad {}^{1A,A}_{R}\bar{\bar{u}} = {}^{1U,U}_{R}\bar{\bar{u}} \text{ and } {}^{AU,A}_{R}\bar{\bar{u}} = {}^{UU,U}_{R}\bar{\bar{u}} = \bar{\bar{0}} \tag{48}$$

This shows the quadratic measure of deformations:

$${}^1_R D = {}^U_R d\bar{x} \, {}^U_R d\bar{x} \; : \; [{}^{1U,U}_{R}\bar{\bar{u}} + {}^{1U,U}_{R}\bar{\bar{u}}^T + {}^{1U,U}_{R}\bar{\bar{u}} \cdot {}^{1U,U}_{R}\bar{\bar{u}}^T] \tag{49}$$

$${}^1_R D = {}^U_R d\bar{x} \, {}^U_R d\bar{x} \; : \; 2 \, {}^{1U}_{R}\bar{\bar{E}}({}^R\bar{x}) \tag{50}$$

We recognise ${}^{1U}_{R}\bar{\bar{E}}$ is the Green-Lagrange (pseudo-) strain tensor. If we chose C_U as the reference placement, then this is the so-called Total Lagrangian Formulation.

If the transformation configuration is the present configuration, then

$${}^A d\bar{x} = {}^1 d\bar{x}, \quad {}^{1A,A}_{R}\bar{\bar{u}} = {}^{11,1}_{R}\bar{\bar{u}} = \bar{\bar{0}} \text{ and } {}^{AU,A}_{R}\bar{\bar{u}} = {}^{1U,I}_{R}\bar{\bar{u}} \tag{51}$$

which yields

$${}^1_R D = {}^1_R d\bar{x} \; {}^1_R d\bar{x} \; : \; [{}^{1U,U}_{R}\bar{\bar{u}} + {}^{1U,U}_{R}\bar{\bar{u}}^T - {}^{1U,U}_{R}\bar{\bar{u}} \cdot {}^{1U,U}_{R}\bar{\bar{u}}^T] \tag{52}$$

$$ {}^1_RD = {}^1_R d\bar{x}\ {}^1_R d\bar{x}\ :\ 2\,{}^{11}_R\bar{\bar{E}}({}^R\bar{x}) \tag{53}$$

where ${}^{11}\bar{\bar{E}}$, the Lagrange pseudo-strain, is identical with the Lagrange true strain, ${}^1\bar{\bar{\varepsilon}}$. This must be, since each time the configuration of transformation matches the configuratioin of measurement the transformation is pre- and post multiplication by the identity tensor. If we use C_1 as a reference configuration, then this is an example of an updated Lagrangian formulation. Therefore the last (nonlinear) term is not always added to strain tensors in referential (Lagrangian) formulations, but will always be subtracted. The only exception to this is where the final term can be neglected altogether because it vanishes. This will happen if the placement of transformation is unstrained. That is where

$$ {}^{AU}\bar{\bar{u}} = \bar{\bar{0}} \tag{54}$$

${}^{11}\bar{\bar{E}}$ is recognized as being coincident with the Euler strain tensor, ${}^1\bar{\bar{\epsilon}}$. This is so even though the derivative and philosophy of the two measures is quite different. In using the spatial (Eulerian) strain tensor the analyst keeps attention directed at a location in space as the material passes by. In this particular updated Lagrangian description, the analyst has attention focused on the body-point and keeps track of the body-point as it moves through the current configuration. By choosing to update the placements of transformation and reference to the current configuration all vectors measured, or that are the basis of a measurement, are coincidentally the same as those that would be employed in an Eulerian style of investigation. For that particular case the two tensors must have the same values.

If the present configuration is an unstrained configuration, then

$$ {}^{1A,A}_{\ R}\bar{\bar{u}} = {}^{U,A}_{\ R}\bar{\bar{u}} = -{}^{AU,A}_{\ R}\bar{\bar{u}} \tag{55}$$

This choice yields

$$ {}^1_RD = {}^U_RD = {}^A_R d\bar{x}\ {}^A_R d\bar{x}\ :\ 2\,{}^{UA}_R\bar{\bar{E}} = {}^A_R d\bar{x}\ {}^A_R d\bar{x}\ :\ 2\,{}_R\bar{\bar{0}} = 0 \tag{56}$$

This could have been arrived at more directly from the definition of ${}^UD$ without involving displacements. The result that ${}^UD$ is zero is expected and gives that ${}^{UA}\bar{\bar{E}}$, ${}^U\bar{\bar{\varepsilon}}$ and ${}^U\bar{\bar{\epsilon}}$ are all zero for any unstrained placement, but gives us no further insight into the nature of being unstrained.

4. Increments of strain tensors

The same quadratic measure of deformation in C_2, when transformed to C_A and referred to C_R, is

$$ \begin{aligned} {}^2_RD = {}^A_R d\bar{x}\ {}^A_R d\bar{x}\ :\ &\Big[\Big\{{}^{2A,A}_{\ R}\bar{\bar{u}}^T + {}^{2A,A}_{\ R}\bar{\bar{u}} + \big({}^{2A,A}_{\ R}\bar{\bar{u}} \cdot {}^{2A,A}_{\ R}\bar{\bar{u}}^T\big)\Big\} \\ &+ \Big\{{}^{AU,A}_{\ R}\bar{\bar{u}}^T + {}^{AU,A}_{\ R}\bar{\bar{u}} - \big({}^{AU,A}_{\ R}\bar{\bar{u}} \cdot {}^{AU,A}_{\ R}\bar{\bar{u}}^T\big)\Big\}\Big] \end{aligned} \tag{57}$$

$$ {}^2_RD = {}^A_R d\bar{x}\ {}^A_R d\bar{x}\ :\ 2\,{}^{2A}_R\bar{\bar{E}}({}^R\bar{x}) \tag{58}$$

It should be noted that the terms involving ${}^{AU}_{\ R}\bar{\bar{u}}$ (the second group in braces) have not changed.

It is possible to think of the displacement from C_A to C_2 as a superposition of a displacement field from C_1 to C_1 and a further increment of displacement from C_1 to C_2. That is

$$^{2}\bar{x} = {}^{A}\bar{x} + {}^{2A}\bar{u} = {}^{A}\bar{x} + {}^{1A}\bar{u} + {}^{21}\bar{u} \tag{59}$$

In that case, the displacement gradient can be expresed as

$$\frac{\partial^{2A}_{A}\bar{u}}{\partial^{A}_{R}\bar{x}} = {}^{2A,A}_{R}\bar{\bar{u}} = \frac{\partial^{1A}_{A}\bar{u}}{\partial^{A}_{R}\bar{x}} + \frac{\partial^{21}_{A}\bar{u}}{\partial^{A}_{R}\bar{u}} = {}^{1A,A}_{R}\bar{\bar{u}} + {}^{21,A}_{R}\bar{\bar{u}} \tag{60}$$

Substituting these gradients into the expression for the quadratic measure of deformation gives

$$\begin{aligned} {}^{2}_{R}D = {}^{A}_{R}d\bar{x}\ {}^{A}_{R}d\bar{x} : \Big[&\left\{ {}^{1A,A}_{R}\bar{\bar{u}} + {}^{1A,A}_{R}\bar{\bar{u}}^{T} + {}^{1A,A}_{R}\bar{\bar{u}} \cdot {}^{1A,A}_{R}\bar{\bar{u}}^{T} \right\} \\ &+ \left\{ {}^{21,A}_{R}\bar{\bar{u}} + {}^{21,A}_{R}\bar{\bar{u}}^{T} + {}^{1A,A}_{R}\bar{\bar{u}} \cdot {}^{21,A}_{R}\bar{\bar{u}}^{T} + {}^{21,A}_{R}\bar{\bar{u}} \cdot {}^{1A,A}_{R}\bar{\bar{u}}^{T} \right\} \\ &+ \left\{ {}^{21,A}_{R}\bar{\bar{u}} \cdot {}^{21,A}_{R}\bar{\bar{u}}^{T} \right\} \\ &+ \left\{ {}^{AU,A}_{R}\bar{\bar{u}}^{T} + {}^{AU,A}_{R}\bar{\bar{u}} - \left({}^{AU,A}_{R}\bar{\bar{u}} \cdot {}^{AU,A}_{R}\bar{\bar{u}}^{T} \right) \right\} \Big] \end{aligned} \tag{61}$$

$${}^{2}_{R}D = {}^{A}_{R}d\bar{x}\ {}^{A}_{R}d\bar{x} : 2\,{}^{2A}_{R}\bar{\bar{E}} \tag{62}$$

It is illustrative in this discussion to keep these terms in separate groups in ${}^{2A}_{R}\bar{\bar{E}}$ such that

$${}^{2A}_{R}\bar{\bar{E}} = {}^{2A}_{R}\bar{\bar{E}}^{1} + {}^{2A}_{R}\bar{\bar{E}}^{2} + {}^{2A}_{R}\bar{\bar{E}}^{3} + {}^{2A}_{R}\bar{\bar{E}}^{4} \tag{63}$$

where

$${}^{2A}_{R}\bar{\bar{E}}^{1} = \frac{1}{2}\left\{ {}^{1A,A}_{R}\bar{\bar{u}} + {}^{1A,A}_{R}\bar{\bar{u}}^{T} + {}^{1A,A}_{R}\bar{\bar{u}} \cdot {}^{1A,A}_{R}\bar{\bar{u}}^{T} \right\} \tag{64}$$

$${}^{2A}_{R}\bar{\bar{E}}^{2} = \frac{1}{2}\left\{ {}^{21,A}_{R}\bar{\bar{u}} + {}^{21,A}_{R}\bar{\bar{u}}^{T} + {}^{1A,A}_{R}\bar{\bar{u}} \cdot {}^{21,A}_{R}\bar{\bar{u}}^{T} + {}^{21,A}_{R}\bar{\bar{u}} \cdot {}^{1A,A}_{R}\bar{\bar{u}}^{T} \right\} \tag{65}$$

$${}^{2A}_{R}\bar{\bar{E}}^{3} = \frac{1}{2}\left\{ {}^{21,A}_{R}\bar{\bar{u}} \cdot {}^{21,A}_{R}\bar{\bar{u}}^{T} \right\} \tag{66}$$

$${}^{2A}_{R}\bar{\bar{E}}^{4} = \frac{1}{2}\left\{ {}^{AU,A}_{R}\bar{\bar{u}}^{T} + {}^{AU,A}_{R}\bar{\bar{u}} - {}^{AU,A}_{R}\bar{\bar{u}} \cdot {}^{AU,A}_{R}\bar{\bar{u}}^{T} \right\} \tag{67}$$

If the body achieves configuration C_2 as a result of an incremental change from C_1, then ${}^{2A}_{R}\bar{\bar{E}}$ can be thought of as incrementally changed from ${}^{1A}_{R}\bar{\bar{E}}$. It is recognized that

$${}^{R}_{1}D = {}^{A}_{R}d\bar{x}\ {}^{A}_{R}d\bar{x} : 2\left({}^{2A}_{R}\bar{\bar{E}}^{1} + {}^{2A}_{R}\bar{\bar{E}}^{4}\right) \tag{68}$$

The effect of an increment of motion must appear in ${}^{21}_{R}\bar{\bar{u}}$ in terms ${}^{2A}_{R}\bar{\bar{E}}^{2}$ and ${}^{2A}_{R}\bar{\bar{E}}^{3}$. ${}^{2A}_{R}\bar{\bar{E}}^{2}$ contains terms linear in ${}^{21}_{R}\bar{u}$ and may be called the linear part of the increment to ${}^{1A}_{R}\bar{\bar{E}}$. ${}^{2A}_{R}\bar{\bar{E}}^{3}$ contains the nonlinear (quadratic) term and may be called the nonlinear part of the increment to ${}^{1A}_{R}\bar{\bar{E}}$. This is mentioned in Underhill (1992), underhill, *et al.* (1989), Gadala (1980), Gadala, *et al.* (1982), Abo-Elkhier (1985) and Abo-Elkhier, *et al.* (1985).

4.1. Special cases

C_A is C_U. If C_R were also C_U, and the initial configuration were unstrained, then a Total

Lagrangian formulation would be in use.

$$ {}^{A}_{R}d\bar{x}={}^{U}_{R}d\bar{x},\quad {}^{1A,A}_{R}\bar{\bar{u}}={}^{1U,U}_{R}\bar{\bar{u}},\quad {}^{21,A}_{R}\bar{\bar{u}}={}^{21,U}_{R}\bar{\bar{u}},\quad {}^{AU,A}_{R}\bar{\bar{u}}={}^{UU,U}_{R}\bar{\bar{u}}=\bar{\bar{0}} \tag{69} $$

We find

$$ {}^{2U}_{R}\bar{\bar{E}}^{1}=\frac{1}{2}\left\{{}^{1U,U}_{R}\bar{\bar{u}}+{}^{1U,U}_{R}\bar{\bar{u}}^{T}+{}^{1U,U}_{R}\bar{\bar{u}}\cdot{}^{1U,U}_{R}\bar{\bar{u}}^{T}\right\} $$

$$ {}^{2U}_{R}\bar{\bar{E}}^{2}=\frac{1}{2}\left\{{}^{21,U}_{R}\bar{\bar{u}}+{}^{21,U}_{R}\bar{\bar{u}}^{T}+{}^{1U,U}_{R}\bar{\bar{u}}\cdot{}^{2U,U}_{R}\bar{\bar{u}}^{T}+{}^{2U,U}_{R}\bar{\bar{u}}\cdot{}^{1U,U}_{R}\bar{\bar{u}}^{T}\right\} $$

$$ {}^{2U}_{R}\bar{\bar{E}}^{3}=\frac{1}{2}\left\{{}^{2U,U}_{R}\bar{\bar{u}}\cdot{}^{2U,U}_{R}\bar{\bar{u}}^{T}\right\} \tag{70} $$

and ${}^{2U}_{R}\bar{\bar{E}}^{4}$ is $\bar{\bar{0}}$ as always in a Total Lagrangian formulation.

If the transformation configuration is the next coinfiguration, C_2, then

$$ {}^{A}_{R}d\bar{x}={}^{2}_{R}d\bar{x},\quad {}^{1A,A}_{R}\bar{\bar{u}}={}^{12,2}_{R}\bar{\bar{u}}=-{}^{21,2}_{R}\bar{\bar{u}}\quad {}^{21,A}_{R}\bar{\bar{u}}={}^{21,2}_{R}\bar{\bar{u}},\quad {}^{AU,A}_{R}\bar{\bar{u}}={}^{2U,2}_{R}\bar{\bar{u}} \tag{71} $$

We find

$$ {}^{22}_{R}\bar{\bar{E}}^{1}=\frac{1}{2}\left\{-{}^{21,2}_{R}\bar{\bar{u}}-{}^{21,2}_{R}\bar{\bar{u}}^{T}+(-{}^{21,2}_{R}\bar{\bar{u}})\cdot(-{}^{21,2}_{R}\bar{\bar{u}}^{T})\right\} $$

$$ {}^{22}_{R}\bar{\bar{E}}^{2}=\frac{1}{2}\left\{{}^{21,2}_{R}\bar{\bar{u}}+{}^{21,2}_{R}\bar{\bar{u}}^{T}+(-{}^{21,2}_{R}\bar{\bar{u}})\cdot(-{}^{21,2}_{R}\bar{\bar{u}}^{T})+({}^{21,2}_{R}\bar{\bar{u}})\cdot(-{}^{21,2}_{R}\bar{\bar{u}}^{T})\right\} $$

$$ {}^{22}_{R}\bar{\bar{E}}^{3}=\frac{1}{2}\left\{{}^{21,2}_{R}\bar{\bar{u}}\cdot{}^{21,2}_{R}\bar{\bar{u}}^{T}\right\} $$

$$ {}^{22}_{R}\bar{\bar{E}}^{4}=\frac{1}{2}\left\{{}^{2U,2}_{R}\bar{\bar{u}}+{}^{2U,2}_{R}\bar{\bar{u}}^{T}-{}^{2U,2}_{R}\bar{\bar{u}}\cdot{}^{2U,2}_{R}\bar{\bar{u}}^{T}\right\} \tag{72} $$

Compare the expression for ${}^{22}_{R}\bar{\bar{E}}^{4}$ to the expression for ${}^{11}_{R}\bar{\bar{E}}$. In both cases the placement where strain is measured is the same one whither gradients have been transformed. So, these two expressions must have the same form; just change the superscript 1 to 2. This also brings about coincidence with the Eulerian strain tensor, ${}^{2}\bar{\bar{\epsilon}}$. So it is comforting to find that

$$ {}^{22}_{R}\bar{\bar{E}}^{1}+{}^{22}_{R}\bar{\bar{E}}^{2}+{}^{22}_{R}\bar{\bar{E}}^{3}=\bar{\bar{0}} \tag{73} $$

If C_A is C_1, then this is a case of the referential formulation when the configuration of transformation has been "updated" to C_1. Here

$$ {}^{A}_{R}d\bar{x}={}^{1}_{R}d\bar{x},\quad {}^{1A,A}_{R}\bar{\bar{u}}={}^{11,1}_{R}\bar{\bar{u}}=\bar{\bar{0}}\qquad {}^{21,A}_{R}\bar{\bar{u}}={}^{21,1}_{R}\bar{\bar{u}},\quad {}^{AU,A}_{R}\bar{\bar{u}}={}^{1U,1}_{R}\bar{\bar{u}} \tag{74} $$

We find

$$ {}^{21}\bar{\bar{E}}^{1}=\bar{\bar{0}}^{1} $$

$$ {}^{21}_{1}\bar{\bar{E}}^{2}=\frac{1}{2}\left\{{}^{21,1}_{1}\bar{\bar{u}}+{}^{21,1}_{1}\bar{\bar{u}}^{T}\right\} $$

$$ {}^{21}_{1}\bar{\bar{E}}^{3}=\frac{1}{2}\left\{{}^{21,1}_{1}\bar{\bar{u}}\cdot{}^{21,1}_{1}\bar{\bar{u}}^{T}\right\} $$

$$ {}^{21}_{1}\bar{\bar{E}}^{4}=\frac{1}{2}\left\{ {}^{1U,1}_{\quad 1}\bar{\bar{u}}+{}^{1U,1}_{\quad 1}\bar{\bar{u}}^{T}-{}^{1U,1}_{\quad 1}\bar{\bar{u}}\cdot{}^{1U,1}_{\quad 1}\bar{\bar{u}}^{T}\right\} \tag{75}$$

Two comments should be made about these terms. The first term involves only gradients of displacement from C_1 to C_1 and so ${}^{21}\bar{\bar{E}}^{1}$ is zero. The other point of interest is the form of the term, ${}^{21}\bar{\bar{E}}^{4}$. This is identical to ${}^{11}\bar{\bar{E}}$. This shows the pseudo-strain tensor changing by a linear increment plus a nonlinear increment from ${}^{11}_{1}\bar{\bar{E}}$. As the body passes through configuration C_1 the referential and spatial formulations coincide. As the body proceeds to deform an incremental change in strain occurs.

The difference between ${}^{21}\bar{\bar{E}}^{2}$ and ${}^{2U}\bar{\bar{E}}^{2}$, the linear parts of the increments, should be noticed. Compare the last two terms in each

$$ {}^{2U}_{U}\bar{\bar{E}}^{2}=\frac{1}{2}\left\{ {}^{21,U}_{\quad U}\bar{\bar{u}}+{}^{21,U}_{\quad U}\bar{\bar{u}}^{T}+{}^{1U,U}_{\quad U}\bar{\bar{u}}\cdot{}^{21,U}_{\quad U}\bar{\bar{u}}^{T}+{}^{21,U}_{\quad U}\bar{\bar{u}}\cdot{}^{1U,U}_{\quad U}\bar{\bar{u}}^{T}\right\}$$

$$ {}^{21}_{1}\bar{\bar{E}}^{2}=\frac{1}{2}\left\{ {}^{21,1}_{\quad 1}\bar{\bar{u}}+{}^{21,1}_{\quad 1}\bar{\bar{u}}^{T}+{}^{11,1}_{\quad 1}\bar{\bar{u}}\cdot{}^{21,1}_{\quad 1}\bar{\bar{u}}^{T}+{}^{21,1}_{\quad 1}\bar{\bar{u}}\cdot{}^{11,1}_{\quad 1}\bar{\bar{u}}^{T}\right\} \tag{76}$$

Since ${}^{11,1}_{\quad 1}\bar{\bar{u}}=\bar{\bar{0}}$, the last two terms of ${}^{21}\bar{\bar{E}}^{2}$ vanish identically. This is a distinctive mark of Updated Lagrangian formulations.

5. Implementation of change of configuration of transformation

Updated Lagrangian calculations occasionally "update" the configuration of transformation to a recently calculated configuration. This is accomplished easily by the use of the deformation gradient. This change can be found by the following argument. The Lagrangian pseudo strain in configuration C_A transformed to C_P and referred to C_R is:

$$ {}^{AP}_{R}\bar{\bar{E}}=\frac{1}{2}\left[\left(\frac{\partial\,{}^{A}_{P}\bar{x}}{\partial\,{}^{P}_{R}\bar{x}}\right)\cdot\left(\frac{\partial\,{}^{A}_{P}\bar{x}}{\partial\,{}^{P}_{R}\bar{x}}\right)^{T}-\left(\frac{\partial\,{}^{U}_{P}\bar{x}}{\partial\,{}^{P}_{R}\bar{x}}\right)\cdot\left(\frac{\partial\,{}^{U}_{P}\bar{x}}{\partial\,{}^{P}_{R}\bar{x}}\right)^{T}\right] \tag{77}$$

The pseudo strain in C_A transformed to C_Q and referred to C_R is:

$$ {}^{AQ}_{R}\bar{\bar{E}}=\frac{1}{2}\left[\left(\frac{\partial\,{}^{A}_{Q}\bar{x}}{\partial\,{}^{Q}_{R}\bar{x}}\right)\cdot\left(\frac{\partial\,{}^{A}_{Q}\bar{x}}{\partial\,{}^{Q}_{R}\bar{x}}\right)^{T}-\left(\frac{\partial\,{}^{U}_{Q}\bar{x}}{\partial\,{}^{Q}_{R}\bar{x}}\right)\cdot\left(\frac{\partial\,{}^{U}_{Q}\bar{x}}{\partial\,{}^{Q}_{R}\bar{x}}\right)^{T}\right] \tag{78}$$

The deformation gradient is used to find

$$\frac{\partial\,{}^{A}_{Q}\bar{x}}{\partial\,{}^{Q}\bar{x}}=\left(\frac{\partial\,{}^{P}_{Q}\bar{x}}{\partial\,{}^{Q}\bar{x}}\right)\cdot\left(\frac{\partial\,{}^{A}_{P}\bar{x}}{\partial\,{}^{P}\bar{x}}\right),\quad \frac{\partial\,{}^{U}_{Q}\bar{x}}{\partial\,{}^{Q}\bar{x}}=\left(\frac{\partial\,{}^{P}_{Q}\bar{x}}{\partial\,{}^{Q}\bar{x}}\right)\cdot\left(\frac{\partial\,{}^{U}_{P}\bar{x}}{\partial\,{}^{P}\bar{x}}\right)$$

so that we arrive at

$$ {}^{AQ}_{R}\bar{\bar{E}}\left(\frac{\partial\,{}^{P}_{Q}\bar{x}}{\partial\,{}^{Q}_{R}\bar{x}}\right)\cdot{}^{AP}_{R}\bar{\bar{E}}\cdot\left(\frac{\partial\,{}^{P}_{Q}\bar{x}}{\partial\,{}^{Q}_{R}\bar{x}}\right)^{T} \tag{80}$$

In practice, a difficulty can arise since, typically, the position of a body-point in the new transformation placement, C_Q, is known as a function of position in the old transformation configuration, C_P, rather than the other way around. So, the deformation gradient in Eq. (80) is not known explicitly. Fortunately, this difficulty is avoidable since all motions are required to be unique invertible functions of position. It can be shown from the invariance of ${}^{P}d\bar{x}$ that

$$\frac{\partial^{P}_{Q}\bar{x}}{\partial^{Q}\bar{x}}=\left(\frac{\partial^{Q}_{P}\bar{x}}{\partial^{P}\bar{x}}\right)^{-1} \tag{81}$$

The updated pseudo strain tensor becomes

$${}^{AQ}_{R}\bar{\bar{E}}\left(\frac{\partial^{Q}_{P}\bar{x}}{\partial^{P}_{R}\bar{x}}\right)^{-1}\cdot{}^{AP}_{R}\bar{\bar{E}}\cdot\left(\frac{\partial^{Q}_{P}\bar{x}}{\partial^{P}_{R}\bar{x}}\right)^{-T} \tag{82}$$

5.1. Example

Suppose a calculation follows the motion of a body from an initial, unstrained configuration, C_0. The body moves and deforms and the pseudo strain in C_1 is calculated, with C_0 as the transformation and reference configurations, to be ${}^{10}_{0}\bar{\bar{E}}$. The pseudo strain tensor is updated so that C_1 is the transformation and reference configurations by the use of

$${}^{11}_{1}\bar{\bar{E}}\left(\frac{\partial^{1}_{0}\bar{x}}{\partial^{0}_{0}\bar{x}}\right)^{-1}\cdot{}^{10}_{0}\bar{\bar{E}}\cdot\left(\frac{\partial^{1}_{0}\bar{x}}{\partial^{0}_{0}\bar{x}}\right)^{-T} \tag{83}$$

Let the calculation continue and the pseudo strain be required in a subsequent coinfiguration, C_2, where the displacement from C_1 to C_2 is known explicitly. Then the pseudo strain in configuration C_2 can be founds as incrementally changed from the pseudo strain in configuration C_1. Take advantage of the fact that ${}^{21}\bar{\bar{E}}^{1}$ vanishes. Also, use the identity between ${}^{21}\bar{\bar{E}}^{4}$ and ${}^{11}E$. We express the pseudo strain in the new placement with the previous configuration as the transformation placement, as the strain in the previous placement plus a linear increment, ${}^{21}\bar{\bar{E}}^{2}$, plus a nonlinear increment, ${}^{21}\bar{\bar{E}}^{3}$.

$${}^{21}_{1}\bar{\bar{E}}={}^{11}_{1}\bar{\bar{E}}+{}^{21}_{1}\bar{\bar{E}}^{2}+{}^{21}_{1}\bar{\bar{E}}^{3} \tag{84}$$

$${}^{21}_{1}\bar{\bar{E}}={}^{11}_{1}\bar{\bar{E}}+\frac{1}{2}\left\{{}^{21,1}_{1}\bar{\bar{u}}+{}^{21,1}_{1}\bar{\bar{u}}^{T}+{}^{21,1}_{1}\bar{\bar{u}}\cdot{}^{21,1}_{1}\bar{\bar{u}}^{T}\right\} \tag{85}$$

Since all of the displacements are well known, there is no problem. C_2 can then be made the transformation and reference configurations by the same kind of manipulation as before

$${}^{22}_{2}\bar{\bar{E}}\left(\frac{\partial^{2}_{1}\bar{x}}{\partial^{1}_{1}\bar{x}}\right)^{-1}\cdot{}^{21}_{1}\bar{\bar{E}}\cdot\left(\frac{\partial^{2}_{1}\bar{x}}{\partial^{1}_{1}\bar{x}}\right)^{-T} \tag{86}$$

The strain is now expressed as the true strain in C_2. The calculation is ready to proceed with the next incremental step.

6. Conclusions

This work discusses the roles of various configurations involved in the analysis of large displacements using a referential description of matter. In particular, the details of how placements are used for reference and transformation are given. This specifies many features of the Updated Lagrangian Formulations and includes an example of how the updating can be done.

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